

Uzbek Mathematical Journal
 2026, Volume 70, Issue 3, pp. 67–78
 DOI: 10.29229/uzmj.2026-3-8

Optimal quadrature formula for approximate solution of a singular integral equation with the Hilbert kernel in the space $L_2^{(3)}(0, 2\pi)$

Jabborov Kh.Kh

Abstract. In this article, an optimal quadrature formula for the approximate calculation of the solution of a Fredholm-type singular integral equation of the second kind with the Hilbert kernel in the space $L_2^{(3)}(0, 2\pi)$ is constructed. In the proposed quadrature formula, the Sard problem arises, where the interval $[0, 2\pi]$ is evenly divided into any N points, and this problem is solved using the Sobolev method. For this purpose, an analytical representation of the optimal coefficients of the quadrature formula is obtained by minimizing the norm of the error functional in the conjugate space $L_2^{(3)*}(0, 2\pi)$.

Keywords: Hilbert space; singular integral; error functional; optimal quadrature formula; singular integral equations; Hilbert kernel

MSC (2020): 65D32, 65R20

1. INTRODUCTION

Singular integral equations arise in the mathematical formulation of problems in physics, mechanics, engineering, and numerous other fields. One such equation is the following Fredholm-type singular integral equation of the second kind:

$$a \cdot g(t) + \frac{b}{2\pi} \int_0^{2\pi} g(x) \cot \frac{x-t}{2} dx = \varphi(t), \quad (1.1)$$

where $g(t)$ is an unknown function, and $\varphi(t)$ is a known function that satisfies the Lipschitz condition $[1, 2]$. a and b are arbitrary constants, which in general can be complex numbers.

This kernel

$$\cot \frac{x-t}{2}$$

is known as the Hilbert kernel. It is defined in the sense of the Cauchy principal value with properties similar to a singular Cauchy-type integral, having a singularity when $x = t$ in the square where $0 < x < 2\pi$ and $0 < t < 2\pi$.

Using Hilbert transformations, the integral equation (1.1), if $a \neq 0$, has the following solution (see [3], p. 144, [4], p. 115):

$$g(t) = \frac{a}{a^2 + b^2} \varphi(t) - \frac{b}{2\pi(a^2 + b^2)} \int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx + \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} \varphi(x) dx. \quad (1.2)$$

The main problem encountered in the solution of the integral equation (1.1) in the form (1.2) is the calculation of Cauchy's principal value integral

$$\int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx. \quad (1.3)$$

This problem persists even when the expression (1.2) is regularized.

Articles dedicated to the numerical solution of the integral equation (1.1) are scarce. In this work, we develop a numerical method for solving the singular integral (1.3) in the sense of Cauchy's principal value, which appears in the solution of the integral equation (1.1) in the form (1.2). This method is based on a functional approach. Since this method considers functions belonging to the Hilbert space $L_2^{(3)}(0, 2\pi)$, it is considered highly effective and accurate.

2. STATEMENT OF THE PROBLEM

We consider the following quadrature formula

$$\int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx \cong \sum_{\beta=0}^N C_\beta \varphi(x_\beta), \quad (2.1)$$

where C_β are coefficients, x_β are nodal points, and $x_\beta \in [0, 2\pi]$. The functions $\varphi(x)$ belong to the Hilbert space $L_2^{(3)}(0, 2\pi)$, which is defined as

$$L_2^{(3)}(0, 2\pi) = \{\varphi(x) : [0, 2\pi] \rightarrow \mathbb{R}, \varphi''(x) \text{ is absolutely continuous, } \varphi'''(x) \in L_2(0, 2\pi)\}.$$

In this $L_2^{(3)}(0, 2\pi)$ space, the inner product of functions φ and ψ is defined as follows

$$\langle \varphi, \psi \rangle = \int_0^{2\pi} \varphi'''(x) \psi'''(x) dx, \quad (2.2)$$

and the norm of the function φ in this space is calculated using the inner product (2.2) as follows

$$\|\varphi\| = \sqrt{\int_0^{2\pi} \left(\varphi'''(x)\right)^2 dx}. \quad (2.3)$$

The difference between the integral and the quadrature sum in the considered quadrature formula (2.1)

$$\begin{aligned} \int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx - \sum_{\beta=0}^N C_\beta \varphi(x_\beta) &= \int_{-\infty}^{\infty} (\varepsilon_{[0, 2\pi]}(x) \cot \frac{x-t}{2} - \sum_{\beta=0}^N C_\beta \delta(x - x_\beta)) \varphi(x) dx = \\ &= \int_{-\infty}^{\infty} \ell(x) \varphi(x) dx = (\ell, \varphi) \end{aligned} \quad (2.4)$$

is called the *error* of the quadrature formula.

For the derived expression (2.4), the following *error functional* corresponding to the conjugate space $L_2^{(3)*}(0, 2\pi)$ is obtained

$$\ell(x) = \varepsilon_{[0, 2\pi]}(x) \cot \frac{x-t}{2} - \sum_{\beta=0}^N C_\beta \delta(x - x_\beta), \quad (2.5)$$

where $\varepsilon_{[0, 2\pi]}(x)$ is the characteristic function of the interval $[0, 2\pi]$, and $\delta(x)$ is Dirac's delta-function.

According to the definition of the norm of a linear continuous functional, the following is known (see [5], p. 82):

$$\|\ell\|_{L_2^{(3)*}} = \sup_{\|\varphi\| \neq 0} \frac{|(\ell, \varphi)|}{\|\varphi\|}.$$

From this equality, the Cauchy-Schwarz inequality follows

$$|(\ell, \varphi)| \leq \|\ell\|_{L_2^{(3)*}} \|\varphi\|_{L_2^{(3)}}. \quad (2.6)$$

According to the Cauchy-Schwarz inequality, the following conditions hold for the norm (2.3) of the function $\varphi(x)$

$$(\ell, x^\alpha) = 0, \quad \alpha = 0, 1, 2. \quad (2.7)$$

According to the conditions (2.7), the quadrature formula (2.1) is exact for the function $d_2x^2 + d_1x + d_0$. These conditions ensure that the error functional is bounded.

Thus, the upper bound of the error (2.4) of the quadrature formula (2.1) is estimated using the norm of the error functional $\ell(x)$. From this, the following problem arises.

Problem 1. Find the form of the norm of the error functional $\ell(x)$ belonging to the conjugate space $L_2^{(3)*}(0, 2\pi)$.

It can be observed that the error functional depends on the coefficients C_β and the node points x_β . Minimizing the norm of the error functional with respect to both the coefficients C_β and the node points x_β is called the *Nikolsky problem*, and this problem was first studied in Nikolsky's works [6, 7]. The problem of minimizing the norm of the error functional only with respect to the coefficients C_β , while fixing the node points x_β in advance, is called the *Sard problem*, and this problem was first studied in the works of Sard [8, 9].

In this work, we solve the Sard problem. For this, we construct a quadrature formula at nodal points x_β where $x_\beta = h\beta$ ($h = \frac{2\pi}{N}, N \in \mathbb{N}$) are equidistantly distributed points.

The coefficients C_β that minimize the norm of the error functional $\ell(x)$, that is

$$\inf_{C_\beta} \|\ell\|_{L_2^{(3)*}(0, 2\pi)} \quad (2.8)$$

are called *optimal coefficients* and are denoted by $\mathring{C}_\beta$. The quadrature formula constructed using these optimal coefficients is called an *optimal quadrature formula*. Therefore, to construct an optimal quadrature formula in the sense of Sard in the space $L_2^{(3)}(0, 2\pi)$, it is necessary to solve the following problem.

Problem 2. Find the optimal coefficients $\mathring{C}_\beta$ that achieve the value in (2.8).

The main purpose of this article is to construct an optimal quadrature formula that calculates the integral equation (1.1) with high accuracy in the space $L_2^{(3)}(0, 2\pi)$. Using the functional method in Hilbert spaces, optimal quadrature formulas were constructed for singular integrals with Cauchy kernels in works [10, 11, 12, 13], and for singular integrals with Hilbert kernels in works [14, 15]. Furthermore, optimal quadrature formulas for regular integrals in various other spaces were constructed using the Sobolev method in works [16, 17].

The remainder of this article is structured as follows. In the third section, the extremal function corresponding to the error functional $\ell(x)$ is determined, and using this function, the expression for the norm of the error functional (2.5) is derived, thus solving Problem 1. In the fourth section, to find the minimum of the quantity $\|\ell\|^2$ with respect to the coefficients C_β , a system of linear equations for the coefficients of the optimal quadrature formula in the Sard sense in the space $L_2^{(3)}(0, 2\pi)$ is formulated and solved using the Sobolev method. In the fifth section, the analytical form of the optimal coefficients for the quadrature formula (2.1) is obtained, thereby solving Problem 2.

3. FINDING THE EXTREMAL FUNCTION AND EXPRESSING THE NORM OF THE ERROR FUNCTIONAL

To solve problem 1, that is, to find the form of the norm of the error functional (2.5), we use the concept of an extremal function.

The function $\psi_\ell(x)$ is called an *extremal function* corresponding to the error functional $\ell(x)$ if the following equality holds (see, [18, 19, 20])

$$(\ell, \psi_\ell) = \|\ell\|_{L_2^{(3)*}} \cdot \|\psi_\ell\|_{L_2^{(3)}}. \quad (3.1)$$

The concept of an extremal function was first introduced by S.L. Sobolev and is defined as follows

$$\psi_\ell = -\ell(x) * G_3(x) + P_2(x),$$

where $G_3(x) = \frac{|x|^5}{2 \cdot 5!}$, and $P_2(x)$ is a second-degree polynomial.

We note that $*$ is the convolution and the convolution of two functions

$$\varphi(x) * \psi(x) = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)dy = \int_{-\infty}^{\infty} \varphi(y)\psi(x-y)dy$$

is defined by the formula.

Since the space $L_2^3(0, 2\pi)$ we are considering is a Hilbert space, we present the Riesz theorem on the general form of a linear continuous functional.

Theorem 3.1. (*Riesz theorem*) *Let H be a Hilbert space. Then for each linear continuous functional $\ell \in H^*$, there exists a unique function ψ_ℓ in the space H itself, called the Riesz element, which satisfies the condition $(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle$ corresponding to this functional, and the following equality holds for this function*

$$\|\ell\|_{H^*} = \|\psi_\ell\|_H$$

and conversely.

For any $\psi_\ell \in H$, there exists a unique functional $\ell \in H^*$ that satisfies the conditions $(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle$ and $\|\ell\|_{H^*} = \|\psi_\ell\|_H$.

According to the theorem, the extremal function $\psi_\ell(x)$ is a Riesz element and

$$(\ell, \psi_\ell) = \|\ell\|^2 = \|\psi_\ell\|^2.$$

Based on the theorem, we calculate the square of the norm of the error functional

$$\begin{aligned} \|\ell\|^2 = & - \int_0^{2\pi} \int_0^{2\pi} \frac{|x-y|^5}{2 \cdot 5!} \cot \frac{x-t}{2} \cot \frac{y-t}{2} dx dy + \\ & + 2 \sum_{\beta=0}^N C_\beta \int_0^{2\pi} \frac{|x-h\beta|^5}{2 \cdot 5!} \cot \frac{x-t}{2} dx - \sum_{\beta=0}^N \sum_{\gamma=0}^N C_\beta C_\gamma \frac{|h\beta-h\gamma|^5}{2 \cdot 5!}. \end{aligned} \quad (3.2)$$

Thus, the square of the norm of the error functional $\ell(x)$ has been determined, and problem 1 has been solved.

4. FINDING THE CONDITIONAL MINIMUM OF THE ERROR FUNCTIONAL

Now we will tackle the second problem. The square of the norm of the error functional obtained in (3.2) is a multi-variable function with respect to the coefficients C_β . According to the rule for finding the conditional extremum of a multi-variable function, we will determine its local minimum in conjunction with the conditions (2.7). To achieve this, we construct the Lagrange function:

$$\Phi(C_\beta, d_2, d_1, d_0) = \|\ell\|^2 + 2 \sum_{\alpha=0}^2 d_\alpha (\ell, x^\alpha), \quad \beta = 0, 1, \dots, N.$$

According to (2.7), the following holds true here

$$(\ell, x^\alpha) = \int_0^{2\pi} x^\alpha \cot \frac{x-t}{2} dx - \sum_{\beta=0}^N C_\beta (h\beta)^\alpha = 0, \quad \alpha = 0, 1, 2.$$

By setting the partial derivatives of the resulting function Φ with respect to the coefficients C_β and the unknowns d_2, d_1, d_0 to zero, we obtain the following system of linear equations

$$\sum_{\gamma=0}^N C_\gamma \frac{|h\beta-h\gamma|^5}{2 \cdot 5!} + d_2 (h\beta)^2 + d_1 (h\beta) + d_0 = f_3(h\beta), \quad \beta = 0, 1, \dots, N, \quad (4.1)$$

$$\sum_{\gamma=0}^N C_\gamma (h\gamma)^2 = g_2, \quad (4.2)$$

$$\sum_{\gamma=0}^N C_\gamma (h\gamma) = g_1, \quad (4.3)$$

$$\sum_{\gamma=0}^N C_{\gamma} = g_0, \quad (4.4)$$

where

$$\begin{aligned} f_3(h\beta) &= \int_0^{2\pi} \frac{|x - h\beta|^5}{2 \cdot 5!} \cot \frac{x - t}{2} dx = \\ &= \frac{1}{5!} \left[\sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+5)(2k)!} \left((2\pi - t)^{2k+5} - 2(h\beta - t)^{2k+5} - t^{2k+5} \right) + \right. \\ &\quad + 5(t - h\beta) \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+4)(2k)!} \left((2\pi - t)^{2k+4} - 2(h\beta - t)^{2k+4} + t^{2k+4} \right) + \\ &\quad + 10(t - h\beta)^2 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+3)(2k)!} \left((2\pi - t)^{2k+3} - 2(h\beta - t)^{2k+3} - t^{2k+3} \right) + \\ &\quad + 10(t - h\beta)^3 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+2)(2k)!} \left((2\pi - t)^{2k+2} - 2(h\beta - t)^{2k+2} + t^{2k+2} \right) + \\ &\quad + 5(t - h\beta)^4 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+1)(2k)!} \left((2\pi - t)^{2k+1} - 2(h\beta - t)^{2k+1} - t^{2k+1} \right) + \\ &\quad \left. + 2(t - h\beta)^5 \ln \left| \frac{\sin \frac{t}{2}}{\sin \frac{h\beta - t}{2}} \right| \right], \\ g_2 &= 2 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+2)(2k)!} \left((2\pi - t)^{2k+2} - t^{2k+2} \right) + \\ &\quad + 4t \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+1)(2k)!} \left((2\pi - t)^{2k+1} + t^{2k+1} \right), \\ g_1 &= 2 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+1)(2k)!} \left((2\pi - t)^{2k+1} + t^{2k+1} \right), \\ g_0 &= \int_0^{2\pi} \cot \frac{x - t}{2} dx = 0, \end{aligned}$$

here, B_{2k} are the Bernoulli numbers, and the following formula was used (see [21], p. 224)

$$\int x^p \cot x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k+p)(2k)!} x^{2k+p}, \quad [p \geq 1, |x| < \pi].$$

The system (4.1)-(4.4) consists of $(N+4)$ unknowns and $(N+4)$ linear equations with respect to $C_{\gamma}, \gamma = 0, 1, \dots, N, d_0, d_1, d_2$. To solve this system, we will employ the functional method proposed by Sobolev. For this method, it is crucial to use a discrete analogue of the differential operator d^6/dx^6 . In [22], discrete analogue of the differential operator d^6/dx^6 has been constructed.

Theorem 4.1. *The discrete analog of the differential operator d^6/dx^6 satisfying the equality $D_3(h\beta) * G_3(h\beta) = \delta(h\beta)$ has the following form*

$$D_3(h\beta) = \frac{5!}{h^6} \begin{cases} \sum_{t=1}^2 K_t \cdot q_t^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + \sum_{t=1}^2 K_t, & |\beta| = 1, \\ -2^5 + \sum_{t=1}^2 \frac{K_t}{q_t}, & \beta = 0, \end{cases} \quad (4.5)$$

where $\delta(h\beta)$ is a discrete delta function, $K_1 = \frac{(1-q_1)^7}{E_5(q_1)}$, $K_2 = \frac{(1-q_2)^7}{E_5(q_2)}$, $E_5(x)$ is a fifth-degree Euler-Frobenius polynomial, and q_1 and q_2 are roots of the Euler-Frobenius polynomial $E_4(q)$ satisfying the inequality $|q_t| < 1$, ($t = 1, 2$).

Theorem 4.2. *The discrete operator $D_3(h\beta)$ has the following properties:*

$$\begin{aligned} 1) \quad & D_3(h\beta) * 1 = 0 \\ 2) \quad & D_3(h\beta) * (h\beta) = 0 \\ 3) \quad & D_3(h\beta) * (h\beta)^2 = 0 \\ 4) \quad & hD_3(h\beta) * G_3(h\beta) = \delta(h\beta) \end{aligned} \quad (4.6)$$

The proofs of the theorems are provided in [22].

Since we will be using functions with discrete arguments and operations on them from this point forward, we will introduce some concepts related to these.

Definition 4.1. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β .

Definition 4.2. The *inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 4.3. The *convolution* of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

We denote the left side of equation (4.1) as

$$u(h\beta) = \sum_{\gamma=0}^N C_\gamma \frac{|h\beta - h\gamma|^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0. \quad (4.7)$$

We transform this expression (4.7) into a convolution form using the concept of discrete argument function mentioned above

$$u(h\beta) = C_\beta * G_3(h\beta) + d_2(h\beta)^2 + d_1(h\beta) + d_0. \quad (4.8)$$

By applying the discrete operator $D_3(h\beta)$ to this expression and using its properties, we obtain

$$C_\beta = hD_3(h\beta) * u(h\beta). \quad (4.9)$$

Thus, to find the coefficients of the optimal quadrature formula, it is necessary to determine the general form of the function $u(h\beta)$ at all integer points. From equation (4.1), we see that $u(h\beta) =$

$f_3(h\beta)$ for $\beta = 0, 1, \dots, N$. Now, assuming $C_\beta = 0$, we need to find the form of the function $u(h\beta)$ from expression (4.8) when $\beta < 0$ and $\beta > N$.

When $\beta < 0$,

$$\begin{aligned} u(h\beta) &= - \sum_{\gamma=0}^N C_\gamma \frac{(h\beta - h\gamma)^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0 = \\ &= - \frac{g_0}{240}(h\beta)^5 + \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + p_2(h\beta)^2 - p_1(h\beta) + p_0 + d_2(h\beta)^2 + d_1(h\beta) + d_0, \end{aligned}$$

here, p_2, p_1, p_0 are coefficients that are equal to the following

$$p_2 = \frac{1}{24} \sum_{\gamma=0}^N C_\gamma (h\gamma)^3, \quad p_1 = \frac{1}{48} \sum_{\gamma=0}^N C_\gamma (h\gamma)^4, \quad p_0 = \frac{1}{240} \sum_{\gamma=0}^N C_\gamma (h\gamma)^5.$$

When $\beta > N$,

$$\begin{aligned} u(h\beta) &= \sum_{\gamma=0}^N C_\gamma \frac{(h\beta - h\gamma)^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0 = \\ &= \frac{g_0}{240}(h\beta)^5 - \frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 - p_2(h\beta)^2 + p_1(h\beta) - p_0 + d_2(h\beta)^2 + d_1(h\beta) + d_0, \end{aligned}$$

Thus, the general form of $u(h\beta)$ became as follows:

$$u(h\beta) = \begin{cases} -\frac{g_0}{240}(h\beta)^5 + \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + (d_2 + p_2)(h\beta)^2 + (d_1 - p_1)(h\beta) + (d_0 + p_0), & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ \frac{g_0}{240}(h\beta)^5 - \frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 + (d_2 - p_2)(h\beta)^2 + (d_1 + p_1)(h\beta) + (d_0 - p_0), & \beta > N. \end{cases}$$

Here, to simplify the form of $u(h\beta)$, we introduce the following notations

$$\begin{aligned} d_2 + p_2 &= a_2^-, \quad d_1 - p_1 = a_1^-, \quad d_0 + p_0 = a_0^-, \\ d_2 - p_2 &= a_2^+, \quad d_1 + p_1 = a_1^+, \quad d_0 - p_0 = a_0^+. \end{aligned}$$

From this, we can determine the following

$$\begin{aligned} d_2 &= \frac{a_2^- + a_2^+}{2}, \quad d_1 = \frac{a_1^- + a_1^+}{2}, \quad d_0 = \frac{a_0^- + a_0^+}{2}, \\ p_2 &= \frac{a_2^- - a_2^+}{2}, \quad p_1 = \frac{a_1^- - a_1^+}{2}, \quad p_0 = \frac{a_0^- - a_0^+}{2}. \end{aligned}$$

Then, taking into account that $g_0 = 0$, we write the expression for $u(h\beta)$ as follows

$$u(h\beta) = \begin{cases} \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + a_2^-(h\beta)^2 + a_1^-(h\beta) + a_0^-, & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ -\frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 + a_2^+(h\beta)^2 + a_1^+(h\beta) + a_0^+, & \beta > N. \end{cases} \quad (4.10)$$

Here, $a_2^-, a_1^-, a_0^-, a_2^+, a_1^+, a_0^+$ are unknown quantities. If these unknowns are determined, the general form of $u(h\beta)$ will be established, and the optimal coefficients C_β , $\beta = 0, 1, \dots, N$ can be found using formula (4.9). Now let's find these unknowns.

First, we will determine a_0^- and a_0^+ from expression (4.10).

At $\beta = 0$,

$$a_0^- = f_3(0),$$

and at $\beta = N$,

$$a_0^+ = f_3(hN) + \frac{g_1}{48}(hN)^4 - \frac{g_2}{24}(hN)^3 - a_2^+(hN)^2 - a_1^+(hN)$$

we determine that it is equal to. Then $u(h\beta)$ takes the following form

$$u(h\beta) = \begin{cases} \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + a_2^-(h\beta)^2 + a_1^-(h\beta) + f_3(0), & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ -\frac{g_1}{48}((hN)^4 - (h\beta)^4) + \frac{g_2}{24}((h\beta)^3 - (hN)^3) + a_2^+((h\beta)^2 - (hN)^2) + a_1^+(h\beta - hN) + f_3(hN), & \beta > N. \end{cases} \quad (4.11)$$

Now we will determine the remaining unknowns $a_2^-, a_1^-, a_2^+, a_1^+$. According to our assumption, since $C_\beta = 0$ when $\beta < 0$ and $\beta > N$, therefore based on (4.9)

$$C_\beta = hD_3(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0, 2\pi]$$

$$D_3(h\beta) * u(h\beta) = 0,$$

$$\sum_{\gamma=-\infty}^{\infty} D_3(h\beta - h\gamma)u(h\gamma) = 0,$$

Taking into account expression (4.11), we can rewrite it as follows

$$\begin{aligned} \sum_{\gamma=-\infty}^{-1} D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_3(h\beta - h\gamma)u(h\gamma) = 0, \\ \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)u(-h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))u(h(\gamma + N)) = 0, \end{aligned}$$

By substituting the function $u(h\beta)$ into the last expression, we obtain an equation with respect to the unknowns $a_2^-, a_1^-, a_2^+, a_1^+$

$$\begin{aligned} & a_2^- \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)(h\gamma)^2 - a_1^- \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)(h\gamma) + \\ & + a_2^+ \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))h^2(\gamma^2 + 2N\gamma) + a_1^+ \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))(h\gamma) = \\ & = - \sum_{\gamma=0}^N D_3(h\beta - h\gamma)f_3(h\gamma) - \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) \left(\frac{g_1}{48}(h\gamma)^4 + \frac{g_2}{24}(h\gamma)^3 + f_3(0) \right) - \\ & - \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N)) \left(-\frac{g_1}{48}(h\gamma)^4 + \left(\frac{g_2 h^3}{24} - \frac{g_1 h^4 N}{12} \right) \gamma^3 + \left(\frac{g_2 h^3 N}{8} - \frac{g_1 h^4 N^2}{8} \right) \gamma^2 + \right. \\ & \quad \left. + \left(\frac{g_2 h^3 N^2}{8} - \frac{g_1 h^4 N^3}{12} \right) \gamma + f_3(hN) \right); \end{aligned} \quad (4.12)$$

To find the unknowns $a_2^-, a_1^-, a_2^+, a_1^+$, we calculate equation (4.12) for $\beta = -1, \beta = -2, \beta = N + 1$, and $\beta = N + 2$, thus obtaining the following system of linear equations

$$\begin{cases} A_{11}a_2^- + A_{12}a_1^- + A_{13}a_2^+ + A_{14}a_1^+ = B_1 \\ A_{21}a_2^- + A_{22}a_1^- + A_{23}a_2^+ + A_{24}a_1^+ = B_2 \\ A_{31}a_2^- + A_{32}a_1^- + A_{33}a_2^+ + A_{34}a_1^+ = B_3 \\ A_{41}a_2^- + A_{42}a_1^- + A_{43}a_2^+ + A_{44}a_1^+ = B_4 \end{cases} \quad (4.13)$$

where

$$A_{11} = \frac{5!}{h^4} \left(-28 + \sum_{i=1}^2 \frac{K_i D_{i2}}{q_i^2} \right);$$

$$\begin{aligned}
A_{12} &= -\frac{5!}{h^5} \left(-30 + \sum_{i=1}^2 \frac{K_i D_{i1}}{q_i^2} \right); \\
A_{13} &= \frac{5!}{h^4} \left(\sum_{i=1}^2 K_i q_i^N (D_{i2} + 2ND_{i1}) \right); \\
A_{14} &= \frac{5!}{h^5} \left(\sum_{i=1}^2 K_i D_{i1} q_i^N \right); \\
A_{21} &= \frac{5!}{h^4} \left(-118 + \sum_{i=1}^2 K_i \left(1 - \frac{1}{q_i^2} + \frac{D_{i2}}{q_i^3} \right) \right); \\
A_{22} &= -\frac{5!}{h^5} \left(-60 + \sum_{i=1}^2 K_i \left(1 - \frac{1}{q_i^2} + \frac{D_{i1}}{q_i^3} \right) \right); \\
A_{23} &= -\frac{5!}{h^5} \left(\sum_{i=1}^2 K_i q_i^{N+1} (D_{i2} + 2ND_{i1}) \right); \\
A_{24} &= \frac{5!}{h^5} \left(\sum_{i=1}^2 K_i D_{i1} q_i^{N+1} \right); \\
A_{31} &= \frac{5!}{h^4} \left(\sum_{i=1}^2 K_i D_{i2} q_i^N \right); \\
A_{32} &= -\frac{5!}{h^5} \left(\sum_{i=1}^2 K_i D_{i1} q_i^N \right); \\
A_{33} &= \frac{5!}{h^4} \left(-28 - 60N + \sum_{i=1}^2 \frac{K_i}{q_i^2} (D_{i2} + 2ND_{i1}) \right); \\
A_{34} &= \frac{5!}{h^5} \left(-30 + \sum_{i=1}^2 \frac{K_i D_{i1}}{q_i^2} \right); \\
A_{41} &= \frac{5!}{h^4} \left(\sum_{i=1}^2 K_i D_{i2} q_i^{N+1} \right); \\
A_{42} &= -\frac{5!}{h^5} \left(\sum_{i=1}^2 K_i D_{i1} q_i^{N+1} \right); \\
A_{43} &= \frac{5!}{h^4} \left(-118 - 120N + \sum_{i=1}^2 K_i \left(\left(1 - \frac{1}{q_i^2} \right) (1 + 2N) + \frac{D_{i2} + 2ND_{i1}}{q_i^3} \right) \right); \\
A_{44} &= \frac{5!}{h^5} \left(-60 + \sum_{i=1}^2 K_i \left(1 - \frac{1}{q_i^2} + \frac{D_{i1}}{q_i^3} \right) \right); \\
B_1 &= -\frac{5!}{h^6} \left(-\frac{g_1 h^4}{3} - g_2 h^3 - 30f_3(0) + \sum_{i=1}^2 K_i \left(F_i + \frac{1}{q_i^2} \left(\frac{D_{i4} g_1 h^4}{48} + \frac{D_{i3} g_2 h^3}{24} + D_{i0} f_3(0) \right) + q_i^N \left(-\frac{D_{i4} g_1 h^4}{48} + \right. \right. \right. \\
&\quad \left. \left. + D_{i3} h^3 \left(\frac{g_2}{24} - \frac{g_1 h N}{12} \right) + D_{i2} h^2 \left(\frac{g_2 h N - g_1 h^2 N^2}{8} \right) + D_{i1} h \left(\frac{g_2 h^2 N^2}{8} - \frac{g_1 h^3 N^3}{12} \right) + D_{i0} f_3(hN) \right) \right) \right); \\
B_2 &= -\frac{5!}{h^6} \left(-\frac{215g_1 h^4}{24} - \frac{19g_2 h^3}{2} - 30f_3(0) + \sum_{i=1}^2 K_i \left(F_i q_i + \left(1 - \frac{1}{q_i^2} \right) \left(\frac{g_1 h^4}{48} + \frac{g_2 h^3}{24} + f_3(0) \right) + \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{q_i^3} \left(\frac{D_{i4}g_1h^4}{48} + \frac{D_{i3}g_2h^3}{24} + D_{i0}f_3(0) \right) + q_i^{N+1} \left(-\frac{D_{i4}g_1h^4}{48} + D_{i3}h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + \right. \\
& \quad \left. + D_{i2}h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) + D_{i1}h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) + D_{i0}f_3(hN) \right) \Bigg); \\
B_3 = & -\frac{5!}{h^6} \left(\frac{g_1h^4}{3} - 24h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) - 28h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) - 30h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) - 30f_3(hN) + \right. \\
& + \sum_{i=1}^2 K_i \left(\sum_{\gamma=0}^N q_i^{N-\gamma} f_3(h\gamma) + q_i^N \left(\frac{D_{i4}g_1h^4}{48} + \frac{D_{i3}g_2h^3}{24} + D_{i0}f_3(0) \right) + \frac{1}{q_i^2} \left(-\frac{D_{i4}g_1h^4}{48} + \right. \\
& \quad \left. + D_{i3}h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + D_{i2}h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) + D_{i1}h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) + D_{i0}f_3(hN) \right) \Bigg); \\
B_4 = & -\frac{5!}{h^6} \left(\frac{215g_1h^4}{24} - 228h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) - 118h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) - 60h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) - \right. \\
& - 30f_3(hN) + \sum_{i=1}^2 K_i \left(\sum_{\gamma=0}^N q_i^{N+1-\gamma} f_3(h\gamma) + q_i^{N+1} \left(\frac{D_{i4}g_1h^4}{48} + \frac{D_{i3}g_2h^3}{24} + D_{i0}f_3(0) \right) + \left(1 - \frac{1}{q_i^2} \right) \left(-\frac{g_1h^4}{48} + \right. \\
& \quad \left. + h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) + h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) + f_3(hN) \right) + \frac{1}{q_i^3} \left(-\frac{D_{i4}g_1h^4}{48} + \right. \\
& \quad \left. + D_{i3}h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + D_{i2}h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) + D_{i1}h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) + D_{i0}f_3(hN) \right) \Bigg);
\end{aligned}$$

here

$$\begin{aligned}
D_{10} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}, & D_{11} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}\gamma, & D_{12} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}\gamma^2, & D_{13} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}\gamma^3, & D_{14} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}\gamma^4, \\
D_{20} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}, & D_{21} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}\gamma, & D_{22} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}\gamma^2, & D_{23} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}\gamma^3, & D_{24} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}\gamma^4, \\
F_1 &= \sum_{\gamma=0}^N q_1^{\gamma} f_3(\gamma), & F_2 &= \sum_{\gamma=0}^N q_2^{\gamma} f_3(\gamma).
\end{aligned}$$

We solve the system of linear equations (4.13) using Cramer's method to determine the unknowns $a_2^-, a_1^-, a_2^+, a_1^+$.

Thus, the general form of the function $u(h\beta)$ has been determined.

5. OPTIMAL COEFFICIENTS OF THE QUADRATURE FORMULA

In this section, we present the main result for the quadrature formula (2.1).

Theorem 5.1. *The optimal coefficients of the quadrature formula (2.1), with nodes equidistantly spaced in the space $L_2^{(3)}(0, 2\pi)$, have the following form*

$$\begin{aligned}
C_0 = & \frac{5!}{h^5} \left[\frac{g_1h^4}{48} + \frac{g_2h^3}{24} + a_2^- h^2 - a_1^- h - 31 \cdot f_3(0) + f_3(1) + \sum_{n=1}^2 \frac{K_n F_n}{q_n} + \right. \\
& + \sum_{n=1}^2 \frac{K_n}{q_n} \left(\frac{D_{n4}g_1h^4}{48} + \frac{D_{n3}g_2h^3}{24} + D_{n2}a_2^- h^2 - D_{n1}a_1^- h + D_{n0}f_3(0) \right) + \\
& + \sum_{n=1}^2 K_n q_n^{N-1} \left(-\frac{D_{n4}g_1h^4}{48} + D_{n3}h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + D_{n2}h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} + a_2^+ \right) + \right. \\
& \left. + D_{n1}h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} + 2a_2^+ hN + a_1^+ \right) + D_{n0}f_3(N) \right) \Bigg];
\end{aligned}$$

$$\begin{aligned}
C_\beta &= \frac{5!}{h^5} \left[\sum_{n=1}^2 K_n q_n^{\beta-1} \left(\frac{D_{n4} g_1 h^4}{48} + \frac{D_{n3} g_2 h^3}{24} + D_{n2} a_2^- h^2 - D_{n1} a_1^- h + D_{n0} f_3(0) \right) + \right. \\
&+ \sum_{\gamma=0}^N \sum_{n=1}^2 K_n q_n^{|\beta-\gamma|-1} f_3(\gamma) + f_3(\beta-1) - 32 f_3(\beta) + f_3(\beta+1) + \\
&+ \sum_{n=1}^2 K_n q_n^{N-\beta-1} \left(-\frac{D_{n4} g_1 h^4}{48} + D_{n3} h^3 \left(\frac{g_2}{24} - \frac{g_1 h N}{12} \right) + D_{n2} h^2 \left(\frac{g_2 h N}{8} - \frac{g_1 h^2 N^2}{8} + a_2^+ \right) + \right. \\
&+ \left. D_{n1} h \left(\frac{g_2 h^2 N^2}{8} - \frac{g_1 h^3 N^3}{12} + 2a_2^+ h N + a_1^+ \right) + D_{n0} f_3(N) \right) \Big], \quad \beta = 1, 2, \dots, N-1; \\
\\
C_N &= \frac{5!}{h^5} \left[-\frac{g_1 h^4}{48} + \frac{g_2 h^3}{24} - \frac{g_1 h^4 N}{12} + \frac{g_2 h^3 N}{8} - \frac{g_1 h^4 N^2}{8} + \frac{g_2 h^3 N^2}{8} - \right. \\
&- \frac{g_1 h^4 N^3}{12} + 2a_2^+ h^2 N + a_2^+ h^2 + a_1 h - 31 \cdot f_3(N) + f_3(N-1) + \\
&+ \sum_{n=1}^2 K_n q_n^{N-1} \left(\frac{D_{n4} g_1 h^4}{48} + \frac{D_{n3} g_2 h^3}{24} + D_{n2} a_2^- h^2 - D_{n1} a_1^- h + D_{n0} f_3(0) \right) + \\
&+ \sum_{n=1}^2 \sum_{\gamma=0}^N K_n q_n^{N-\gamma-1} f_3(\gamma) + \sum_{n=1}^2 \frac{K_n}{q_n} \left(-\frac{D_{n4} g_1 h^4}{48} + D_{n3} h^3 \left(\frac{g_2}{24} - \frac{g_1 h N}{12} \right) + \right. \\
&+ \left. D_{n2} h^2 \left(\frac{g_2 h N}{8} - \frac{g_1 h^2 N^2}{8} + a_2^+ \right) + D_{n1} h \left(\frac{g_2 h^2 N^2}{8} - \frac{g_1 h^3 N^3}{12} + 2a_2^+ h N + a_1^+ \right) + D_{n0} f_3(N) \right) \Big];
\end{aligned}$$

Proof: To prove the theorem, we use the expression (4.9). For this purpose, the discrete operator $D_3(h\beta)$ is defined by formula (4.5), while the function $u(h\beta)$ is determined using expression (4.11). Thus,

$$\begin{aligned}
C_\beta &= h D_3(h\beta) * u(h\beta) = h \sum_{\gamma=-\infty}^{\infty} D_3(h\beta - h\gamma) u(h\gamma) = \\
&= h \left(\sum_{\gamma=-\infty}^{-1} D_3(h\beta - h\gamma) u(h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma) u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_3(h\beta - h\gamma) u(h\gamma) \right) = \\
&= h \left(\sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) u(-h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma) u(h\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N)) u(h(\gamma + N)) \right) = \\
&= h \left(\sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) \left(\frac{g_1 h^4}{48} \gamma^4 + \frac{g_2 h^3}{24} \gamma^3 + a_2^- h^2 \gamma^2 - a_1^- h \gamma + f_3(0) \right) + \right. \\
&+ \sum_{\gamma=0}^N D_3(h\beta - h\gamma) f_3(\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N)) \left(-\frac{g_1 h^4}{48} \gamma^4 + \left(\frac{g_2 h^3}{24} - \frac{g_1 h^4 N}{12} \right) \gamma^3 + \right. \\
&+ \left. \left(\frac{g_2 h^3 N}{8} - \frac{g_1 h^4 N^2}{8} + a_2^+ h^2 \right) \gamma^2 + \left(\frac{g_2 h^3 N^2}{8} - \frac{g_1 h^4 N^3}{12} + 2a_2^+ h^2 N + a_1^+ h \right) \gamma + f_3(N) \right) \Big).
\end{aligned}$$

From the final expression, we obtain the coefficient C_0 when $\beta = 0$, the coefficients C_β when $\beta = 1, 2, \dots, N-1$, and the coefficient C_N when $\beta = N$.

The theorem is thus proven. $\square$

Problem 2 has been solved.

6. CONCLUSION

In this article, an optimal quadrature formula in the space $L_2^{(3)}(0, 2\pi)$ was developed using the Sobolev method for high-precision approximate calculation of singular integral equations with a Hilbert kernel, and an analytical expression for its optimal coefficients was derived.

REFERENCES

- [1] Gakhov F. D., Boundary Value Problems. Nauka, Moscow, (1977).
- [2] Muskhelishvili N. I., Singular Integral Equations. P. Noordhoff, Groningen, Netherlands, (1953).
- [3] Mikhlin S. G., Integral equations and their applications. Moscow, (1949).
- [4] Estrada R., Kanwal R. P., Singular Integral Equations. Springer Science+Business Media, LLC, (2000).
- [5] Han W., Atkinson K. E., Theoretical Numerical Analysis. Springer, New York, (2009).
- [6] Nikolskii S. M., To Question About Estimation of Approximation by Quadrature Formulas. Uspekhi Mat. Nauk, (1950) Vol. 5, P. 165–177.
- [7] Nikolskii S. M., Quadrature Formulas. Nauka, Moscow, (1988).
- [8] Sard A., Best Approximate Integration Formulas; Best Approximation Formulas. Amer. J. Math. (1949) Vol. 71, P. 80–91.
- [9] Sard A., Linear Approximation. AMS, (1963).
- [10] Akhmedov D. M., Abdikayumova G. A., Construction of optimal quadrature formulas with derivatives for Cauchy type singular integrals in the space $L_2^{(m)}(0, 1)$. Uzbek Mathematical Journal, (2023) Vol. 67, Iss. 2, P. 5–17.
- [11] Akhmedov D. M., Atamuradova B. M., Construction of optimal quadrature formulas for Cauchy type singular integrals in the $W_2^{(1,0)}(0, 1)$ space. Uzbek Mathematical Journal, (2022) Vol. 66, Iss. 2, P. 5–9.
- [12] Shadimetov K. M., Akhmedov D. M., Approximate Solution of a Singular Integral Equation Using the Sobolev Method. Lobachevskii J. Math. (2022) Vol. 43, P. 496–505.
- [13] Shadimetov K. M., Akhmedov D. M., Numerical Integration Formulas for Hypersingular Integrals. Numer. Math.: Theory Methods Appl. (2024) Vol. 17, P. 805–826.
- [14] Shadimetov K. M., Akhmedov D. M., Jabborov K. K., Numerical Solution of a Singular Integral Equation of the First Kind with Hilbert Kernel. Int. J. Anal. Appl. (2025) Vol. 23, 224.
- [15] Akhmedov D. M., Jabborov K. K., On an optimal method for the approximate solution of singular integral equations. Problems Of Computing And Applied Mathematics, (2022), Iss. 5/1(44), P. 14–22.
- [16] Hayotov A. R., Milovanović G. V., Shadimetov K. M., On an optimal quadrature formula in the sense of Sard. Numerical Algorithms, (2011) Vol. 57, Iss. 4, P. 487–510.
- [17] Hayotov A. R., Babaev S. S., Kurbonnazarov A. I., Optimization of approximate integrals of rapidly oscillating functions in the Hilbert space. Results in Applied Mathematics, (2025),
- [18] Sobolev S. L., Introduction to the Theory of Cubature Formulas. Nauka, (1974).
- [19] Sobolev S. L., The coefficients of optimal quadrature formulas. Selected works of S. L. Sobolev, (2006), P. 561–565.
- [20] Shadimetov K. M., Optimal lattice quadrature and cubature formulas in Sobolev spaces. Fan va texnologiya Publishing House, Tashkent, (2019).
- [21] Gradshteyn I. S., Ryzhik I. M., Table of Integrals, Series, and Products. Academic Press, (2007).
- [22] Shadimetov K. M., The Discrete Analogue of the Differential Operator d^{2m}/dx^{2m} and Its Construction. Questions of Computational and Applied Mathematics, (1985), P. 22–35.

Jabborov Kh.Kh.,
 V.I.Romanovskiy Institute of Mathematics,
 Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;
 University of World Economy and Diplomacy,
 Tashkent, Uzbekistan
 e-mail: jabborovx@bk.ru