Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.7-14
DOI: 10.29229/uzmj.2025-4-1

Exponential stability of a numerical solution of a hyperbolic system with negative nonlocal characteristic velocities and measurement error

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Abstract. In this work, the problem of stabilizing the equilibrium state for a hyperbolic system with negative nonlocal characteristic velocities and measurement error is investigated. A mixed problem is considered for such systems, when a limited perturbation of measurement errors is taken into account in the boundary conditions. The study is based on the use of the adequacy between the stability for a mixed problem for the original hyperbolic system of linear differential equations and the stability of the initial-boundary difference problem for it. When analyzing the initial-boundary difference problems constructed in this way, the properties of logarithmic norms are used. Algorithms are proposed that make it possible to obtain sufficient conditions for the exponential stability of a numerical solution of an initial-boundary difference problem with nonlocal coefficients and limited perturbation of measurement errors in boundary conditions. Sufficient conditions are presented in the form of matrix inequalities, which involve matrices of boundary conditions. The results are presented in the form of an a priori estimate of the numerical solution in the norm through the norms of the functions of the initial data and the norms of perturbation of measurement errors.

Keywords: Lyapunov's function, hyperbolic system, nonlocal characteristic velocity, Lyapunov stability.

MSC (2020): 65M12, 65M06, 35L40

1. Introduction

In [1], the Lyapunov stability of the equilibrium position of a mixed problem for a one-dimensional single hyperbolic equation with a positive nonlocal coefficient of the derivative with respect to x is studied. Exponential stability is tested using spectral analysis of the linearized hyperbolic equation. The Lyapunov function is constructed. Using the Lyapunov function, the exponential stability of the steady-state solution of the hyperbolic equation is proved. The case of positive characteristic velocities for the scalar case is considered.

In [2], a mixed problem is studied for a nonlinear equation with positive coefficients of the derivative with respect to x. The Lyapunov stability of an equilibrium state is investigated based on the Lyapunov stability theory. The Lyapunov exponential stability of the solution to the mixed problem is proven. Based on the construction of a discrete Lyapunov function, the authors were able to transfer the results obtained to the case of a discrete problem. The case of positive characteristic velocities for the scalar case is considered.

The work [3] is devoted to the analysis of exponential stability in the Lyapunov sense of steady-state numerical solutions of a hyperbolic differential equation with nonlocal coefficients in front of the derivatives with respect to x. They obtain sufficient and necessary conditions for the exponential stability of a numerical solution of an initial-boundary difference problem with nonlocal coefficients and limited perturbation of measurement errors in boundary conditions. The case of positive characteristic velocities for the scalar case is considered.

The Articles [4-11], is devoted to the analysis of exponential stability in the Lyapunov sense of steady-state numerical solutions of a classical hyperbolic system of linear differential equations in canonical form. A mixed problem is considered for such systems. The study is based on the use of the adequacy between the stability for a mixed problem for the original hyperbolic system of linear differential equations and the stability of the initial-boundary difference problem for it. When analyzing the initial-boundary difference problems constructed in this way, the properties of logarithmic norms are used. Algorithms are proposed that make it possible to obtain sufficient conditions for the exponential stability of a numerical solution of an initial-boundary difference problem for a classical hyperbolic system of linear differential equations in canonical form. Sufficient conditions are presented in the form algebraic inequalities, which involve coefficients of boundary conditions. The results are presented in the form of an a priori estimate of the numerical solution in the norm through the norms of the functions of the initial data. Nonlocal characteristic velocities are not considered.

2. DIFFERENTIAL MIXED PROBLEM

Consider the following symmetric t-hyperbolic system:

$$\frac{\partial U}{\partial t} - \mathbf{M}(\mathcal{A}(t))\frac{\partial U}{\partial x} = 0, \quad t \in [0, +\infty),\ x \in [0,1], \tag{2.1}$$

where

$$\begin{aligned} \mathbf{M}(\mathcal{A}(t)) &\triangleq \operatorname{diag}\left({}_1\mu({}_1a(t)), {}_2\mu({}_2a(t)), \ldots, {}_n\mu({}_na(t))\right), \\ U &\triangleq ({}_1u, {}_2u, \ldots, {}_nu)^T, \\ \mathcal{A}(t) &\triangleq ({}_1a(t), {}_2a(t), \ldots, {}_na(t))^T, \\ &{}_i\mu(s) \text{ are some specified functions.} \end{aligned}$$

Here the characteristic speeds $\mathbf{M}(\mathcal{A}(t))$ depend on the integral of the unknown vector function $U(t,x)$ over the entire region [0,1]

$$\mathcal{A}(t) = \int_0^1 U(t,x)dx, \quad t \in (0, +\infty) \tag{2.2}$$

or component by component

$${}_ia(t) = \int_0^1 {}_iu\, dx, \quad i = \overline{1,n}.$$

Initial conditions for system (2.1):

$$U(0,x) = \Phi(x), \quad x \in [0,1], \tag{2.3}$$

Here $\Phi(x) \triangleq \left({}_1\varphi(x), {}_2\varphi(x), \ldots, {}_n\varphi(x)\right)^T$ - is the given initial vector function.

In this work, we limit ourselves to the case when the characteristic velocity functions are negative, i.e. $\mathbf{M}(\mathcal{A}(t)) > 0$. In this case, it is known from the theory of hyperbolic systems that boundary conditions for system (2.1) are required only on the right boundary, at $x = 1$:

$$-\mathbf{M}(\mathcal{A}(t))U(t,1) = V(t), \tag{2.4}$$

where $\mathbf{V}(t) \triangleq \left({}_1\mathcal{V}(t), {}_2\mathcal{V}(t), \ldots, {}_n\mathcal{V}(t)\right)^T$ - is the vector function controller.

From works [1] and [2] it follows that with an appropriate choice of $\mathcal{M}(\mathcal{A}(t))$, $U(t,0)$, $\mathbf{V}(t)$ it is possible to prove the correctness of the formulation of the mixed problem (2.1)-(2.4).

In this work we will consider one special case of specifying boundary conditions.

$$-\mathbf{V}(t) + \mathbf{M}^*\dot{U}^* = \mathbf{R}\left\{-\mathbf{M}(\mathcal{A}(t))\left[U(t,0) + \Delta(t)\right] + \mathbf{M}^*\dot{U}^*\right\}, \quad t \in (0, +\infty), \tag{2.5}$$

where

$$\mathbf{M}^* \triangleq \mathbf{M}(U^*) = \operatorname{diag}\left({}_1\mu({}_1u^*), {}_2\mu({}_2u^*), \ldots, {}_n\mu({}_nu^*)\right),$$

$$U^* \triangleq ({}_1u^*, {}_2u^*, \ldots, {}_nu^*)^T, \quad \mathbf{R} \triangleq \operatorname{diag}({}_1r, {}_2r, \ldots, {}_nr),$$

$$\Delta(t) \triangleq \left({}_1\delta(t), {}_2\delta(t), \ldots, {}_n\delta(t)\right)^T.$$

and ${}_ir \in [0,1)$, $i = \overline{1,n}$ are given coefficients, and U^* is quadratic matrix with coefficients ${}_iu^*$, where ${}_iu^* > 0$, $i = \overline{1,n}$ are given equilibrium state and $\Delta(t)$ is constrained disturbance.

Note that for a given equilibrium state U^*, the value of the characteristic vector function is calculated as follows

$$-\mathbf{M}\left(\mathcal{A}(t)\right)\Big|_{\bar{U}=\bar{U}^*} = -\mathbf{M}\left(U^*\right)$$

In this work we limit ourselves to the following family of characteristic velocities of the type

$$ {}_i\mu(s) = \frac{{}_iP}{{}_iQ + s}, \quad s \in [0, +\infty), \quad i = \overline{1, n} \tag{2.6}$$

with ${}_iP > 0,\ {}_iQ > 0,\ \forall i \in \{m+1, m+2, \ldots, n\}$.

So, consider the following mixed control problem

$$\begin{cases} \dfrac{\partial U}{\partial t} - \mathbf{M}(\mathcal{A}(t))\dfrac{\partial U}{\partial x} = 0, & t \in [0, +\infty),\ x \in (0,1), \\ U(0,x) = \Phi(x), & x \in (0,1), \\ -\mathbf{V}(t) + \mathbf{M}^*\dot{U}^* = \mathbf{R}\left\{-\mathbf{M}(\mathcal{A}(t))[U(t,0) + \Delta(t)] + \mathbf{M}^*\dot{U}^*\right\}, & t \in (0, +\infty), \\ -\mathbf{M}(\mathcal{A}(t))U(t,1) = \mathbf{V}(t), & t \in [0, +\infty), \\ \mathcal{A}(t) = \displaystyle\int_0^1 U(t,x)dx, & t \in (0, +\infty), \end{cases} \tag{2.7}$$

where U - is the vector function to be determined.

Let's consider transformations regarding equilibrium U^*:

$$\widetilde{U}(t,x) = U(t,x) - U^*, \widetilde{A}(t) = \mathcal{A}(t) - U^*, \quad \widetilde{\Phi}(x) = \Phi(x) - U^*,$$

$$\widetilde{\mathbf{M}}_{\widetilde{A}}(t) = \mathbf{M}(U^* + \widetilde{A}(t)).$$

Then system (2.7) with (2.6) for $t \in (0, +\infty)$ can be rewritten as follows:

$$\begin{cases} \dfrac{\partial \widetilde{U}}{\partial t} - \widetilde{\mathbf{M}}_{\widetilde{A}}(t)\dfrac{\partial \widetilde{U}}{\partial x} = 0, & x \in (0,1), \\ \widetilde{U}(0,x) = \widetilde{\Phi}(x), & x \in (0,1), \\ \widetilde{\mathbf{V}}(t) = R\widetilde{\mathbf{M}}_{\widetilde{A}}(t)\left[\widetilde{U}(t,0) + \Delta(t)\right] + (\mathbf{E} - \mathbf{R})\left\{\mathbf{M}^* - \widetilde{\mathbf{M}}_{\widetilde{A}}(t)\right\}U^*, & \\ \widetilde{\mathbf{M}}_{\widetilde{A}}(t) = \mathbf{M}(U^* + \widetilde{A}(t)), \widetilde{\mathbf{M}}_{\widetilde{A}}(t)\widetilde{U}(t,1) = \widetilde{\mathbf{V}}(t) & \\ \widetilde{\mathcal{A}}(t) = \displaystyle\int_0^1 \widetilde{U}(t,x)dx \quad \text{where} \quad \int_0^1 {}_i\widetilde{u}(t,x)dx = {}_iu^*, & i = \overline{1,n}, \\ {}_i\mu(s) = \dfrac{{}_iP}{{}_iQ + s}, \quad \text{with } {}_iP > 0,\ {}_iQ > 0,\ s \in [0, +\infty), & i = \overline{1,n}. \end{cases} \tag{2.8}$$

Using the expressions given for functions ${}_i\mu,\ i = \overline{1,n}$ of characteristic velocities (2.6) in equation (2.8), we have

$$\left\{\mathbf{M}^* - \widetilde{\mathbf{M}}_{\mathcal{A}}(t)\right\}U^* = \left[\begin{matrix} \operatorname{diag}\left(\dfrac{{}_1P}{{}_1Q + {}_1u^*}, \ldots, \dfrac{{}_nP}{{}_nQ + {}_nu^*}\right) \\ -\operatorname{diag}\left(\dfrac{{}_1P}{{}_1Q + {}_1u^* + {}_1\tilde{a}(t)}, \ldots, \dfrac{{}_nP}{{}_nQ + {}_nu^* + {}_n\tilde{a}(t)}\right) \end{matrix}\right]U^* \tag{2.9}$$

$$= \operatorname{diag}\left(\frac{{}_1P\,{}_1\tilde{a}(t)}{({}_1Q + {}_1u^*)({}_1Q + {}_1u^* + {}_1\tilde{a}(t))}, \ldots, \frac{{}_nP\,{}_n\tilde{a}(t)}{({}_nQ + {}_nu^*)({}_nQ + {}_nu^* + {}_n\tilde{a}(t))}\right)U^*$$

$$U^* = \Omega\,\widetilde{\mathbf{M}}_{\mathcal{A}}(t)\,\tilde{\mathcal{A}}(t),$$

where

$$\Omega \triangleq \operatorname{diag}\left({}_1\varpi, {}_2\varpi, \dots, {}_n\varpi\right), \qquad {}_i\varpi = \frac{{}_iu^*}{{}_iQ + {}_iu^*}, \qquad i = \overline{1,n}$$

Note that matrix inequality $\Omega < E$ is true.

For convenience, we omit the "~" symbol. Then for $t \in (0, +\infty)$ the system in equation (2.8) with equation (2.9) can be rewritten as follows:

$$\begin{cases} \dfrac{\partial U}{\partial t} - \mathbf{M}_{\mathcal{A}}(t)\dfrac{\partial U}{\partial x} = 0, & x \in (0,1), \\ U(0,x) = \Phi(x), & x \in (0,1), \\ \mathbf{V}(t) = \mathbf{R}\mathbf{M}_{\mathcal{A}}(t)\left[U(t,0) + \Delta(t)\right] + (\mathbf{E} - \mathbf{R})\Omega\mathbf{M}_{\mathcal{A}}(t)\mathcal{A}(t), & \\ \mathbf{M}_{\mathcal{A}}(t) = \mathbf{M}(U^* + \mathcal{A}(t)), \quad \mathbf{M}_{\mathcal{A}}(t)U(t,1) = \overline{\mathbf{V}}(t), & \\ \mathcal{A}(t) = \displaystyle\int_0^1 U(t,x)\,dx, \quad \text{where} \quad \int_0^1 {}_iu(t,x)\,dx \geq -{}_iu^*, & i = \overline{1,n}, \\ {}_i\mu(s) = \dfrac{{}_iP}{{}_iQ + s}, \quad \text{with } {}_iP > 0,\ {}_iQ > 0,\ s \in [0, +\infty), & i = \overline{1,n}. \end{cases} \tag{2.10}$$

3. EXPONENTIAL STABILITY OF THE NUMERICAL SOLUTION

In this section we establish the exponential stability of the numerical solution of the initial-boundary difference problem.

To obtain the initial-boundary difference problem, we will use the upwind difference scheme for the numerical calculation of system (2.7).

To do this, we cover the spatial region $[0,1]$ using a uniform grid $\Omega_h = \left\{x_j = ih,\ j = \overline{0,J}\right\}$, h is step by x. We calculate the integral $\mathcal{A}(t)$ for each value of $t^k \triangleq k\tau$ (τ is step by time), $k \in \{0,1,2,\dots\}$ using the quadrature formula

$$A^k \triangleq \left({}_1a^k, {}_2a^k, \dots, {}_na^k\right)^T, \qquad {}_ia^k = h\sum_{j=0}^{J} {}_iu_j^k, \quad k \in \{0,1,2,\dots\}. \tag{3.1}$$

Next, we define the discrete value $\mathbf{M}^k$:

$$\mathbf{M}^k \triangleq \mathbf{M}(A^k) \equiv \operatorname{diag}({}_1\mu^k, {}_2\mu^k, \dots, {}_n\mu^k),$$

$${}_i\mu^k \triangleq \mu({}_ia^k) = \frac{{}_iP}{{}_iQ + {}_ia^k}, \qquad {}_iP > 0, \quad {}_iQ > 0, \quad i = \overline{1,n}, \quad k \in \{0,1,2,\dots\}. \tag{3.2}$$

Let us assume that the Courant-Friedrichs-Levy condition is satisfied

$$0 < \Lambda^k \triangleq \frac{\tau}{h}\mathbf{M}^k \leq E, \quad k \in \{0,1,2,\dots\} \tag{3.3}$$

where

$$\Lambda^k = \operatorname{diag}\left({}_1\lambda^k, {}_2\lambda^k, \dots, {}_n\lambda^k\right), \quad {}_i\lambda^k = \frac{\tau}{h}{}_i\mu^k, \quad i = \overline{1,n}, \quad k \in \{1,2,\dots,K\}.$$

To numerically solve system (2.7), we propose an upwind difference scheme

$$\begin{cases} U_j^{k+1} = \left(1 - \Lambda^k\right)U_j^k + \Lambda^k U_{j+1}^k, & j = 0,\dots,J-1; \quad k \in \{0,1,\dots\}, \\ U_J^{k+1} = RU_0^{k+1} + (E - R)\left(\mathbf{M}^k\right)^{-1}\mathbf{M}^*U^* + R\Delta^{k+1}, & k \in \{0,1,\dots\}, \\ U_j^0 = \Phi(x_j), & j = 0,\dots,J. \end{cases} \tag{3.4}$$

$$U_j^k = \left({}_1u_j^k, {}_2u_j^k, \dots, {}_nu_j^k\right)^T, \quad \Delta^k \triangleq \left({}_1\delta^k, {}_2\delta^k, \dots, {}_n\delta^k\right)^T$$

Let us introduce the following matrices:

$$U^k \triangleq \operatorname{diag}\left({}_1u_0^k, {}_2u_0^k, \dots, {}_nu_0^k, {}_1u_1^k, {}_2u_1^k, \dots, {}_nu_1^k, \dots\dots, {}_1u_{J-1}^k, {}_2u_{J-1}^k \dots, {}_nu_{J-1}^k\right),$$

$$U^0 \triangleq \operatorname{diag}\left({}_1\varphi_0, {}_2\varphi_0, \dots, {}_n\varphi_0, {}_1\varphi_1, {}_2\varphi_1, \dots, {}_n\varphi_1, \dots\dots, {}_1\varphi_{J-1}, {}_2\varphi_{J-1}, \dots, {}_n\varphi_{J-1}\right),$$

$$U^* \triangleq \operatorname{diag}\left(\begin{pmatrix} {}_1u^*, {}_2u^*, \dots, {}_nu^* \\ {}_1u^*, {}_2u^*, \dots, {}_nu^* \\ \vdots \\ {}_1u^*, {}_2u^*, \dots, {}_nu^* \end{pmatrix}_{n\times J}\right), \qquad \Delta^k \triangleq \operatorname{diag}\left(\begin{pmatrix} {}_1\delta^k, 0, 0, \dots, 0 \\ 0, {}_2\delta^k, 0, \dots, 0 \\ \vdots \\ 0, 0, 0 \dots, {}_n\delta^k \end{pmatrix}_{n\times J}\right).$$

Definition 3.1. Let $\Xi > 0$. Equilibrium state U^* of the initial-boundary difference problem (3.4) is stable in the l^2-norm with respect to discrete perturbations that satisfy matrix inequalities $\Delta^k \le \Xi$, $k \in \{1, 2, \dots\}$, if there exist positive real constants $\zeta_1 > 0$, $\zeta_2 > 0$, $\zeta_3 > 0$ such that for any initial condition $\Phi(x_j)$, $j = \overline{0, J}$, solution U_j^k, $k \in \{1, 2, \dots\}$, $j = \overline{0, J}$ of the initial-boundary difference problem (3.4) satisfies the inequality

$$\|U^k - U^*\|_{l^2} \le \zeta_2 e^{-\zeta_1 t^k} \|\Phi - U^*\|_{l^2} + \zeta_3 \max_{0 \le s < k} (|\Delta^s|), \quad k \in \{1, 2, \dots\}, \tag{3.5}$$

where

$$U^k \triangleq \begin{pmatrix} U_0^k \\ U_1^k \\ \vdots \\ U_{J-1}^k \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi(x_0) \\ \Phi(x_1) \\ \vdots \\ \Phi(x_{J-1}) \end{pmatrix}, \quad U^* \triangleq \begin{pmatrix} U_0^* \\ U_1^* \\ \vdots \\ U_{n-1}^* \end{pmatrix}_{n\times J}, \quad |\Delta^s| = \max_{1 \le i \le n} |{}_i\delta^s|.$$

and

$$\|U^k - U^*\|_{l^2}^2 \triangleq h \sum_{j=0}^{J-1} \left(\left[U_j^k - U^*\right], \left[U_j^k - U^*\right] \right), \quad k \in \{0, 1, \dots\}.$$

Definition 3.2. (Discrete Lyapunov function). That function $\mathbf{L} : \mathbb{R}^{n\times J} \to \mathbb{R}_0^+$ is a discrete Lyapunov function for the initial-boundary difference problem (3.4) if:

(1) there are positive constants $\chi_1 > 0$ and $\chi_2 > 0$ such that for all $k \in \{0, 1, \dots\}$

$$\chi_1 \|U^k - U^*\|_{l^2}^2 \le \mathbf{L}(U^k) \le \chi_2 \|U^k - U^*\|_{l^2}^2, \tag{3.6}$$

(2) there are positive constants $\eta > 0$ and $\nu > 0$ such that for all $k \in \{0, 1, \dots\}$

$$\frac{\mathbf{L}(U^{k+1}) - \mathbf{L}(U^k)}{\Delta t} \le -\eta \mathbf{L}(U^k) + \nu \left(\Delta^k, \Delta^k\right).$$

To simplify the notation, in what follows we will define the sequence of discrete values $\mathcal{L}^k$ as

$$\mathcal{L}^k = \mathcal{L}(U^k), \quad k \in \{0, 1, \dots\}$$

and where U^k is a given solution of the initial-boundary difference problem (3.4).

Theorem 3.1. (Discrete stability for the case $U^* \ge 0$). Assume that the CFL condition (3.3) is satisfied. Let $\Xi \ge 0$. For each U^* satisfying the matrix inequality $U^* \ge 0$, each R satisfying the matrix inequality $0 \le R < E$, each $u > 0$, and for any initial vector function Φ satisfying the matrix inequality with $U^0 \ge 0$, and

$$\|\Phi - U^*\|_{l^2} < u \tag{3.7}$$

the solution U^k of the initial-boundary value difference problem (3.4) satisfies the matrix inequalities $U^k \ge 0$, $\quad k \in \{0, 1, \dots\}$, and the stationary state U^* of the initial-boundary value difference problem (3.4) is stable in the norm is l^2 with respect to any discrete perturbation function Δ^k, $k \in \{0, 1, \dots\}$, such that the matrix inequality $\Delta^k \le \Xi$ holds.

To analyze the stability of the initial-boundary difference problem (3.4) using the discrete Lyapunov method, we use the following transformation:

$$\widetilde{U}_j^k = U_j^k - U^*, \qquad \widetilde{A}^k = h \sum_{j=0}^{J-1} \widetilde{U}_j^k, \qquad \widetilde{\mathbf{M}}_{\widetilde{A}^k}^k = \mathbf{M}(U^* + \widetilde{A}^k), \qquad \widetilde{\Lambda}^k = \frac{\tau}{h} \widetilde{\mathbf{M}}_{\widetilde{A}^k}^k, \qquad k \in \{0, 1, \dots\},$$

$$\mathbf{M}(U^* + \widetilde{A}^k) \equiv \operatorname{diag}\left(\frac{{}_1P}{{}_1Q + {}_1u^* + {}_1a^k}, \frac{{}_2P}{{}_2Q + {}_2u^* + {}_2a^k}, \ldots, \frac{{}_nP}{{}_nQ + {}_nu^* + {}_na^k}\right) \tag{3.8}$$

For simplicity, we omit the symbol "$\tilde{}$" in notation (3.8) and discretize system (2.10) as follows

$$\begin{cases} U_j^{k+1} = \left(1 - \Lambda^k\right) U_j^k + \Lambda^k U_{j+1}^k, j = \overline{0, J-1}; \quad k \in \{0, 1, \ldots\}; \\ U_J^{k+1} = RU_0^{k+1} + (E - R)\,\Theta A^{k+1} + R\Delta^{k+1}, k \in \{0, 1, \ldots\}; \\ c\,\Theta = \operatorname{diag}\left(\frac{{}_1u^*}{{}_1Q + {}_1u^*}, \frac{{}_2u^*}{{}_2Q + {}_2u^*}, \ldots, \frac{{}_nu^*}{{}_nQ + {}_nu^*}\right); \\ A^k = h\sum_{j=0}^{J-1} U_j^k, \quad M_{A^k}^k = \mathrm{M}(U^* + A^k), \quad \Lambda^k = \frac{\tau}{h} M_{A^k}^k, k \in \{0, 1, \ldots\}; \\ \mathrm{M}(U^* + A^k) \equiv \operatorname{diag}\left(\frac{{}_1P}{{}_1Q + {}_1u^* + {}_1a^k}, \frac{{}_2P}{{}_2Q + {}_2u^* + {}_2a^k}, \ldots, \frac{{}_nP}{{}_nQ + {}_nu^* + {}_na^k}\right); \\ {}_ia^k = h\sum_{j=0}^{J-1} {}_iu_j^k \geq -{}_iu^*, i = \overline{1, n}, \quad k \in \{0, 1, \ldots\}; \\ U_j^0 = \Phi(x_j), \quad j = \overline{0, J}. \end{cases} \tag{3.9}$$

Thus, the assumption in the form of inequality (3.7) in Theorem 1 is now expressed as

$$\|\Phi\|_{\ell^2} < U. \tag{3.10}$$

Note that it can be rewritten in the form

$$\left\|U^k\right\|_{\ell^2} \leq \zeta_2 e^{-\zeta_1 t^k} \|\Phi\|_{\ell^2} + \zeta_3 \max_{0 \leq s < k} \left(|\Delta^s|\right), \quad k \in \{1, 2, \ldots\}.$$

4. COMPUTATIONAL EXPERIMENT

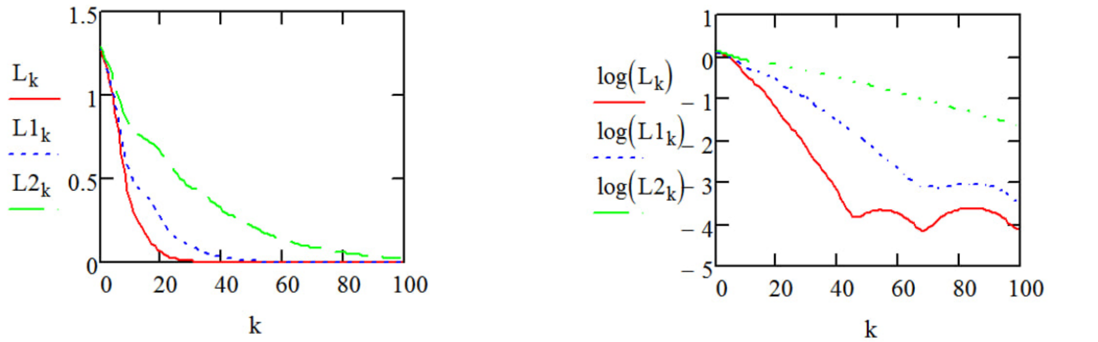
Option. Consider the influence of the parameter r on the numerical solution.

To carry out the computational experiment, the following data were entered: Time $T = 10$ seconds, time step $\tau = 0.1$, spatial domain $[0, 1]$ divided into a grid with step $h = 0.1$. Parameters $P = 1$, $Q = 1$, equilibrium solution $u^* = 0$, limited (known) disturbance $\delta(t) = 2.4 \cdot 10^{-3} \cdot \sin(t)$. Initial condition $u(x) = 1.1 + \sin(2\pi x)$.

Norm of the numerical solution for the case when $r = 0.1$; $r = 0.3$; $r = 0.6$ is shown in Fig. 1. Graph of the numerical solution when $r = 0.1$; $r = 0.3$; $r = 0.6$ is shown in Fig. 2. As can be seen from the graph, the smaller the parameter r, the faster the numerical solution reaches an equilibrium solution.

CONCLUSION

The *Lyapunov* exponential stability of stationary numerical solutions of a mixed problem for a hyperbolic system of linear differential equations with nonlocal matrix coefficients of derivatives with respect to x is studied. In this case, the boundary conditions take into account a limited disturbance of measurement errors. A methodology is proposed that makes it possible to obtain sufficient conditions for the exponential stability of a numerical solution of an initial-boundary difference problem with nonlocal coefficients and limited perturbation of measurement errors in boundary conditions. Sufficient conditions are presented in the form of matrix inequalities, including matrices of boundary conditions. The results are presented in the form of an a priori estimate of the numerical solution in the ℓ^2-norm through the ℓ^2-norm of the functions of the initial data and the norms of perturbation of measurement errors. The proposed method can be used to study the influence of nonlocal coefficients of the original system and perturbation of measurement errors on the exponential stability of the numerical solution of the initial-boundary difference problem for hyperbolic systems. A computational experiment was carried out that confirmed the theoretical results obtained.



(a) *norm of the numerical solution*

(b) *The logarithm of the norm of the numerical solution*

FIGURE 1. Graph of the norm of the numerical solution. Here k is time step, L_k is the norm of the numerical solution at $r = 0.1$; $L1_k$ is the norm of numerical solution at $r = 0.3$; $L2_k$ is the norm of numerical solution at $r = 0.6$.

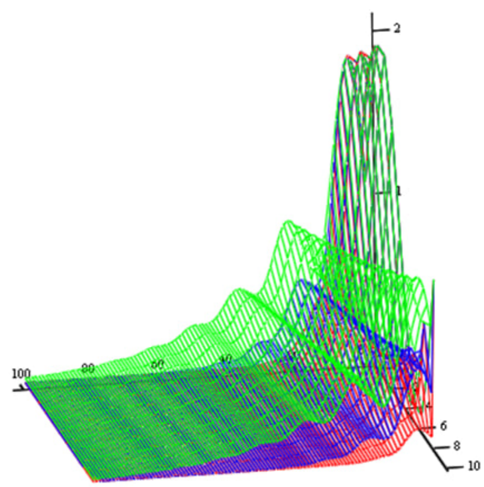


FIGURE 2. Numerical solution graph. Here the graph of the numerical solution at $r = 0.1$ is shown in red; blue shows the graph of the numerical solution at $r = 0.3$; Green shows the graph of the numerical solution at $r = 0.6$

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