Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.22-34
DOI: 10.29229/uzmj.2025-4-3

Classification of Frobenius algebra structures on two-dimensional vector space over any base field

Asrorov D., Bekbaev U., Rakhimov I.

Abstract. In this paper, we first classify all associative algebra structures on a two-dimensional vector space over an arbitrary base field equipped with a non-degenerate bilinear form. We then determine which of these are Frobenius algebras. We provide lists of canonical representatives of the isomorphism classes of these algebras over an arbitrary base field.

Keywords: Frobenius algebra, Non-degenerate bilinear form, Classification, Automorphism.
MSC (2020): 16W20; 16S34; 20C05.

1. Introduction

Frobenius algebras are a fundamental concept in mathematics, appearing in areas such as algebra, topology, and theoretical physics, particularly in the study of categories, representation theory, and quantum field theory. They are named after the German mathematician Ferdinand Georg Frobenius.

Initially it was introduced as a finite-dimensional associative algebra equipped with a linear functional whose "kernel" contains no nontrivial ideals. Later, a few equivalent definitions of these algebras appear depending their applications in various areas of science. Frobenius algebras were first studied by Frobenius [13] around 1900. Further properties and relations go back to Nakayama [16] in the 1930s. The characterisation of Frobenius algebras in terms of comultiplication goes back to Lawvere [15] (1967), and it was rediscovered by Quinn [22] and Abrams [1] in the 1990s (we refer the reader to [12] as a most fundamental work on Frobenius algebras and their connections).

Frobenius algebras began to be studied in the 1930s by R. Brauer and C. Nesbitt [6]. T. Nakayama discovered the beginnings of a rich duality theory [16, 17]. J. Dieudonné used this to characterize Frobenius algebras [10]. Frobenius algebras were generalized to quasi-Frobenius rings, those Noetherian rings whose right regular representation is injective. There are works on generalization of the concept Frobenius algebra to some specific classes of algebras (see [3, 8, 14, 18]). In recent times, interest has been renewed in Frobenius algebras due to connections to Topological Quantum Field Theory. TQFTs are functors from the category of cobordisms to the category of vector spaces. It has been found that they play an important role in the algebraic treatment and axiomatic foundation of Topological Quantum Field Theory [1, 4, 21]. Frobenius algebras underlie the algebraic structure of 2D Topological Quantum Field Theory's (TQFT's). They provide a bridge between physics and algebraic topology by encoding information about 2-dimensional surfaces and their invariants. Let us mention a few results, illustrating the importance of the concept. In [19] the author introduces foundational concepts related to Frobenius algebras in the context of Hopf algebra theory. A. Atiyah in [2] discussed the role of Frobenius algebras in the development of TQFT's and first described their axiomatic foundation. The authors of [7] present a unified approach to the study of separable and Frobenius algebras.

In the paper, we first classify all associative algebra structures on two-dimensional vector spaces over any base field equipped with a non-degenerate bilinear form (Section 3). Then we identify which of those are Frobenius algebras (Section 4). Section 5 contains a comparison of two lists of two-dimensional associative algebras over any base fields obtained in [11] and [20].

1.1. Preliminaries.

Let $\mathbb{A}$ be a PI-algebra, with a given set of polynomial identities

$$\{P_j[u_1,u_2,...,u_n]=0: \ j\in J\} \text{ over a field } \mathbb{F}$$

and

$$P_j[u_1,u_2,...,u_n]=\sum_{i=1}^{k_j} Q_j^i[u_1,u_2,...,u_n]R_j^i[u_1,u_2,...,u_n], \text{ where } j\in J.$$

In some applications this kind of algebras appear with a non-degenerate bilinear form $\sigma : \mathbb{A}\times\mathbb{A}\to\mathbb{F}$ such that $\sum_{i=1}^{k_j}\sigma(Q_j^i[a_1,a_2,...,a_n], R_j^i[a_1,a_2,...,a_n]) = 0$ at all $a_1,a_2,...,a_n\in\mathbb{A}$. A pair $(\mathbb{A},\sigma)$ is said to be a Frobenius PI-algebra. The classification of a given class of Frobenius PI-algebras is of great interest. In this paper we consider as a class PI-algebras the class of associative algebras and provide a complete classification of such algebras on two-dimensional vector space over any base field. Therefore, further an algebra $\mathbb{A}$ always is assumed to be associative. The next theorem establishes the equivalence of several important and useful characterizations of Frobenius algebras (see [9]).

Theorem 1.1. *The following statements about a finite-dimensional unital algebra $\mathbb{A}$ are equivalent:*

- *$\mathbb{A}$ is a Frobenius algebra, i.e., there exists a linear functional $\varepsilon : \mathbb{A}\longrightarrow\mathbb{F}$ whose "kernel" contains no nontrivial ideals.*
- *There exists a non-degenerate bilinear form, $\sigma : \mathbb{A}\times\mathbb{A}\longrightarrow\mathbb{F}$ such that $\sigma(\mathrm{xy},\mathrm{z}) = \sigma(\mathrm{x},\mathrm{yz})$ for all $\mathrm{x},\mathrm{y},\mathrm{z}\in\mathbb{A}$.*
- *For all left ideals L and right ideals R in $\mathbb{A}$ we have*

$$l(r(L)) = L, \text{ and } (r(L):\mathbb{F}) + (L:\mathbb{F}) = (\mathbb{A}:\mathbb{F});$$

$$r(l(R)) = R, \text{ and } (l(R):\mathbb{F}) + (R:\mathbb{F}) = (\mathbb{A}:\mathbb{F}),$$

where $r(P) = \{\mathrm{x}\in\mathbb{A} : P\mathrm{x} = 0\}$ and $l(P) = \{\mathrm{x}\in\mathbb{A} : \mathrm{x}P = 0\}$ are right and left annihilators of a subset $P\subset\mathbb{A}$, respectively and $(\_ : \mathbb{F})$ is the dimension over $\mathbb{F}$.

Definition 1.2. Let $(\mathbb{A},\sigma)$ and $(\mathbb{B},\tau)$ be Frobenius algebras on a vector space $\mathbb{V}$ over a field $\mathbb{F}$ with non-degenerate bilinear forms $\sigma : \mathbb{A}\times\mathbb{A}\to\mathbb{F}$ and $\tau : \mathbb{B}\times\mathbb{B}\to\mathbb{F}$, respectively. Pairs $(\mathbb{A},\sigma)$ and $(\mathbb{B},\tau)$ are said to be isomorphic if there exists isomorphism of algebras $f : \mathbb{A}\longrightarrow\mathbb{B}$ such that $\sigma(\mathrm{xy}) = \tau(f(\mathrm{x})f(\mathrm{y}))$ for all $\mathrm{x},\mathrm{y}\in\mathbb{A}$ (we denote it by $\mathbb{A}\cong\mathbb{B}$).

Let $\mathbb{A}$ be an n-dimensional algebra and $\mathbf{e} = (\mathrm{e}_1,\mathrm{e}_2,...,\mathrm{e}_n)$ its basis. Then x, y and z can be presented by their coordinate vectors $x = (x_1,x_2,...,x_n)^T$, $y = (y_1,y_2,...,y_n)^T$ and $z = (z_1,z_2,...,z_n)^T$ as

$$\mathrm{x} = \mathbf{e}x, \mathrm{y} = \mathbf{e}y \text{ and } \mathrm{z} = \mathbf{e}z, \text{ respectevely.}$$

Here and onward we use the notions x and x for a vector and its coordinate vector on the basis $\mathbf{e}$. Therefore, $\mathrm{xy} = \mathbf{e}A(x\otimes y)$, where the entries a_{ij}^k of

$$A = \begin{pmatrix} a_{11}^1 & a_{12}^1 & ... & a_{1n}^1 & a_{21}^1 & a_{22}^1 & ... & a_{2n}^1 & ... & a_{n1}^1 & a_{n2}^1 & ... & a_{nn}^1 \\ a_{11}^2 & a_{12}^2 & ... & a_{1n}^2 & a_{21}^2 & a_{22}^2 & ... & a_{2n}^2 & ... & a_{n1}^2 & a_{n2}^2 & ... & a_{nn}^2 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{11}^n & a_{12}^n & ... & a_{1n}^n & a_{21}^n & a_{22}^n & ... & a_{2n}^n & ... & a_{n1}^n & a_{n2}^n & ... & a_{nn}^n \end{pmatrix}$$

are defined by:

$$\mathrm{e}_i\mathrm{e}_j = \sum_{k=1}^n a_{ij}^k\mathrm{e}_k, \text{ where } i,j = 1,2,...,n.$$

The matrix A is said to be the matrix of structure constants (MSC) of $\mathbb{A}$ on the basis $\mathbf{e}$.

If $\mathbb{A}$ is a Frobenius algebra then the Frobenius map σ also is presented by its matrix S: $\sigma(\mathrm{x},\mathrm{y}) = x^TSy$. Then

$$\sigma(\mathrm{xy},\mathrm{z}) = (x^T\otimes y^T)A^TSz \text{ and } \sigma(\mathrm{x},\mathrm{yz}) = x^TSA(y\otimes z).$$

Therefore, one has

Lemma 1.3. *An algebra $\mathbb{A}$ is Frobenius if and only if*

$$(x^T\otimes y^T)A^TSz = x^TSA(y\otimes z). \tag{1.1}$$

Recently, in [5] a result on classification of two-dimensional algebras over any base field $\mathbb{F}$ appeared. Using this classification in [20] the author gave the representatives of isomorphism classes of all associative algebra structures on two-dimensional vector space over any base field and their automorphism groups. Below we give results from [20] which we make use in the paper.

Theorem 1.4. *Any non-trivial 2-dimensional associative algebra over a field $\mathbb{F}$ ($Char(\mathbb{F}) \neq 2$) is isomorphic to only one of the following listed, by their matrices of structure constants, algebras:*

(1) $As_{13}^1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$,

(2) $As_3^2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

(3) $As_3^3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$,

(4) $As_3^4 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

(5) $As_3^5(\alpha_4) := \begin{pmatrix} 1 & 0 & 0 & \alpha_4 \\ 0 & 1 & 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1 & 1 & 0 \end{pmatrix}$, *where* $\alpha_4 \in \mathbb{F}$, $a \in \mathbb{F}$ *and* $a \neq 0$.

Theorem 1.5. *Any non-trivial 2-dimensional associative algebra over a field $\mathbb{F}$, ($Char(\mathbb{F}) = 2$) is isomorphic to only one of the following listed by their matrices of structure constants, such algebras:*

(1) $As_{12,2}^1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$,

(2) $As_{11,2}^2(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, *where* $a, b \in \mathbb{F}$ *and* $b \neq 0$,

(3) $As_{6,2}^3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

(4) $As_{4,2}^4(\beta_1) := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 0 & 0 & 1 \end{pmatrix}$, *where* $a, \beta_1 \in \mathbb{F}$,

(5) $As_{3,2}^5 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

(6) $As_{3,2}^6 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

We also need the automorphism groups of the algebras given in the theorems above.

Theorem 1.6. *The automorphism groups of the algebras listed in Theorem 1.4 are given as follows*

(1) $Aut(As_{13}^1) = \left\{ \begin{pmatrix} p & 0 \\ s & p^2 \end{pmatrix} \mid p, s \in \mathbb{F},\ p \neq 0 \right\}$,

(2) $Aut(As_3^2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{F},\ t \neq 0 \right\}$,

(3) $Aut(As_3^3) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid s, t \in \mathbb{F},\ t \neq 0 \right\}$,

(4) $Aut(As_3^4) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid s, t \in \mathbb{F},\ t \neq 0 \right\}$,

(5) $Aut(As_3^5(0)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{F}, t \neq 0 \right\}$,

(6) $Aut(As_3^5(\alpha_4)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$, $\alpha_4 \neq 0$.

Theorem 1.7. *The automorphism groups of the algebras listed in Theorem 1.5 are given as follows*

(1) $Aut(As^1_{12,2}) = \left\{ \begin{pmatrix} p & 0 \\ s & p^2 \end{pmatrix} \mid p \neq 0, s \in \mathbb{F} \right\}$,

(2) $Aut(As^2_{11,2}) = \left\{ \begin{pmatrix} p & 0 \\ \beta_1(p-1) & 1 \end{pmatrix} \mid p \neq 0 \in \mathbb{F} \right\}$,

(3) $Aut(As^3_{6,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid t \neq 0, s \in \mathbb{F} \right\}$,

(4) $Aut(As^4_{4,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$,

(5) $Aut(As^5_{3,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \neq 0 \in \mathbb{F} \right\}$,

(6) $Aut(As^6_{3,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid s, t \in \mathbb{F} \text{ and } t \neq 0 \right\}$.

Let $\mathbb{A}$ be a two-dimensional algebra over a field $\mathbb{F}$, $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2)$ its basis, $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$ MSC of $\mathbb{A}$ and $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{F})$ the matrix of σ. Then

$$\begin{array}{rcl} (x^T \otimes y^T)A^T S z & = & (\alpha_1 a + \beta_1 c)x_1y_1z_1 + (\alpha_2 a + \beta_2 c)x_1y_2z_1 + (\alpha_3 a + \beta_3 c)x_2y_1z_1 \\ & & +(\alpha_4 a + \beta_4 c)x_2y_2z_1 + (\alpha_1 b + \beta_1 d)x_1y_1z_2 + (\alpha_2 b + \beta_2 d)x_1y_2z_2 \\ & & +(\alpha_3 b + \beta_3 d)x_2y_1z_2 + (\alpha_4 b + \beta_4 d)x_2y_2z_2 \\ x^T S A(y \otimes z) & = & (\alpha_1 a + \beta_1 b)x_1y_1z_1 + (\alpha_3 a + \beta_3 b)x_1y_2z_1 + (\alpha_1 c + \beta_1 d)x_2y_1z_1 \\ & & +(\alpha_3 c + \beta_3 d)x_2y_2z_1 + (\alpha_2 a + \beta_2 b)x_1y_1z_2 \\ & & +(\alpha_4 a + \beta_4 b)x_1y_2z_2 + (\alpha_2 c + \beta_2 d)x_2y_1z_2 + (\alpha_4 c + \beta_4 d)x_2y_2z_2 \end{array} \tag{1.2}$$

and (1.3) can be written as follows

$$\left\{ \begin{array}{rcl} \alpha_1 a + \beta_1 c - \alpha_1 a - \beta_1 b & = & 0 \\ \alpha_2 a + \beta_2 c - \alpha_3 a - \beta_3 b & = & 0 \\ \alpha_3 a + \beta_3 c - \alpha_1 c - \beta_1 d & = & 0 \\ \alpha_4 a + \beta_4 c - \alpha_3 c - \beta_3 d & = & 0 \\ \alpha_1 b + \beta_1 d - \alpha_2 a - \beta_2 b & = & 0 \\ \alpha_2 b + \beta_2 d - \alpha_4 a - \beta_4 b & = & 0 \\ \alpha_3 b + \beta_3 d - \alpha_2 c - \beta_2 d & = & 0 \\ \alpha_4 b + \beta_4 d - \alpha_4 c - \beta_4 d & = & 0 \end{array} \right. \tag{1.3}$$

as far as the system of monomial functions $\{x_1y_1z_1, x_1y_2z_1, x_2y_1z_1, x_1y_1z_2, x_2y_2z_1, x_1y_2z_2, x_2y_1z_2, x_2y_2z_2\}$ is linearly independent over $\mathbb{F}$.

2. Classification of two-dimensional associative algebras equipped by a non-degenerate form σ

In this section first we classify non-degenerate bilinear forms given as $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the transformations $g^T S g$, $g \in G$, where G is a fixed nontrivial automorphism group from Theorems 1.6 and 1.7 (the transformation $S' = g^T S g$ is denoted by $S' \simeq S$). For the further usage the list of all nontrivial autmorphism groups in Theorems 1.6 and 1.7 we enumerate as follows

- $G_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} : \ 0 \neq t \in \mathbb{F} \right\}$,
- $G_2 = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} : \ s, t \in \mathbb{F}, \ t \neq 0 \right\}$,
- $G_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} (Char(\mathbb{F}) = 2)$,
- $G_4 = \left\{ \begin{pmatrix} p & 0 \\ s & p^2 \end{pmatrix} : \ p, s \in \mathbb{F}, p \neq 0 \right\}$,
- $G_{5,\beta_1} = \left\{ \begin{pmatrix} p & 0 \\ \beta_1(p-1) & 1 \end{pmatrix} : \ p \in \mathbb{F}, p \neq 0 \right\}$ $(Char(\mathbb{F}) = 2)$,
- $G_6 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} (Char(\mathbb{F}) \neq 2)$.

Now we treat the action $g \cdot S = g^T S g$ for each $G_i, i = 1, 2, ..., 6$ $(g \in G_i)$ one by one and find the canonical representatives of the equivalent classes with respect to this action.

Let $g = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \in G_1$. Then $g^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} a & tb \\ tc & t^2 d \end{pmatrix}$ and the following canonical forms occur:

- $\begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$, where $ad - c \neq 0$,
- $\begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \simeq \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}$, where $0 \neq t \in \mathbb{F}, \ ad \neq 0$.

If $g = \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \in G_2$ then $g^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} a + sc + sb + s^2 d & tb + std \\ tc + std & t^2 d \end{pmatrix}$ and one comes to the following canonical forms:

- $\begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \simeq \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}$, where $0 \neq t \in \mathbb{F}, \ ad \neq 0$,
- $\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$, where $c \neq 0, \ c + 1 \neq 0$,
- $\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$, where $a \in \mathbb{F}$.

Let $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in G_3$. Then $g^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} a + c + b + d & b + d \\ c + d & d \end{pmatrix}$ and we get the following canonical forms:

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \simeq \begin{pmatrix} a + c + b + d & b + d \\ c + d & d \end{pmatrix}$, where $ad - bc \neq 0$.

If $g = \begin{pmatrix} p & 0 \\ s & p^2 \end{pmatrix} \in G_4$, where $p, s \in \mathbb{F}$, $p \neq 0$, then $g^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} p^2 a + psc + psb + s^2 d & p^3 b + p^2 sd \\ p^3 c + p^2 sd & p^4 d \end{pmatrix}$ and the following canonical forms occur:

- $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \simeq \begin{pmatrix} p^2 a & 0 \\ p^3 c & p^4 d \end{pmatrix}$, where $p \neq 0$,
- $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & p^3 b \\ p^3 c & 0 \end{pmatrix}$, where $p \neq 0$, $bc \neq 0$,

- $\begin{pmatrix} a & b \\ -b & 0 \end{pmatrix} \simeq \begin{pmatrix} p^2a & p^3b \\ -p^3b & 0 \end{pmatrix}$, where $p \neq 0$.

If $g = \left\{ \begin{pmatrix} p & 0 \\ \beta_1(p-1) & 1 \end{pmatrix} : \ p \neq 0 \in \mathbb{F} \right\} \in G_{5,\beta_1}$ then

$$g^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} p^2a + \beta_1(p+1)(pc + pb + \beta_1 d(p+1)) & pb + \beta_1 d(p+1) \\ pc + \beta_1 d(p+1) & d \end{pmatrix}$$

and one gets the following canonical forms:

- $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & \beta_1 d \\ 0 & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & \beta_1 d \\ \beta_1 d & d \end{pmatrix} \simeq \begin{pmatrix} p^2(a - \beta_1^2 d) + \beta_1^2 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}$, where $ad - \beta_1^2 d^2 \neq 0$ and the polynomial $u^2(a - \beta_1^2 d) + \beta_1^2 d$ has no root in $\mathbb{F}$,
- $\begin{pmatrix} 0 & \beta_1 d \\ \beta_1 d & d \end{pmatrix}$, where $\beta_1 d \neq 0$,
- $\begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \simeq \begin{pmatrix} p^2a & 0 \\ 0 & d \end{pmatrix}$, where $ad \neq 0$.

Finally, taking $g = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \in G_6$ we get $g^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} a & \pm b \\ \pm c & d \end{pmatrix}$ and only the canonical form appears:

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \simeq \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, where $ad - bc \neq 0$.

Now we turn to pairs $(\mathbb{A}, \sigma)$, where $\mathbb{A}$ is a two-dimensional associative algebra, $\sigma : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{F}$ is a non-degenerate bilinear form, up to isomorphism. Taking into account the canonical forms of σ along with Theorems 1.4 and 1.7 we state the following results.

Lemma 2.1. *The representatives of isomorphism classes of pairs $(\mathbb{A}, \sigma)$, where $\mathbb{A}$ is a two dimensional associative algebra over a field $\mathbb{F}$ ($Char(\mathbb{F}) \neq 2$), $\sigma : \mathbb{A} \times \mathbb{A} \to \mathbb{F}$ is a non-degenerate form, are given as follows:*

(1) $\left(As_3^2, \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \right)$, *where* $a, c, d \in \mathbb{F}$, $ad - c \neq 0$,

(2) $\left(As_3^2, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix} \right)$, *where* $a, d \in \mathbb{F}$, $ad \neq 0$,

(3) $\left(As_3^2, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) \cong \left(As_3^2, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix} \right)$, *where* $a, d, t \in \mathbb{F}$, $ad \neq 0$, $t \neq 0$,

(4) $\left(As_3^3, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix} \right)$, *where* $a, d \in \mathbb{F}$, $ad \neq 0$,

(5) $\left(As_3^3, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As_3^3, \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(6) $\left(As_3^3, \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right)$, *where* $c \in \mathbb{F},\ c \neq 0,\ c+1 \neq 0$,

(7) $\left(As_3^3, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$,

(8) $\left(As_3^4, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(9) $\left(As_3^4, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As_3^4, \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(10) $\left(As_3^4, \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right)$, *where* $c \in \mathbb{F},\ c \neq 0,\ c+1 \neq 0$,

(11) $\left(As_3^4, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$,

(12) $\left(As_3^5(0), \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}\right)$, *where* $a, c, d \in \mathbb{F},\ ad - c \neq 0$,

(13) $\left(As_3^5(0), \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(14) $\left(As_3^5(0), \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As_3^5(0), \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(15) $\left(As_3^5(\alpha_4), \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cong \left(As_3^5(\alpha_4), \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\right)$, *where* $\alpha_4, a, b, c, d \in \mathbb{F},\ \alpha_4 \neq 0,\ ad - bc \neq 0$,

(16) $\left(As_{13}^1, \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right) \cong \left(As_{13}^1, \begin{pmatrix} p^2a & 0 \\ p^3c & p^4d \end{pmatrix}\right)$, *where* $a, c, d, p \in \mathbb{F},\ ad \neq 0,\ p \neq 0$,

(17) $\left(As_{13}^1, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right) \cong \left(As_{13}^1, \begin{pmatrix} 0 & p^3b \\ p^3c & 0 \end{pmatrix}\right)$, *where* $b, c, p \in \mathbb{F},\ bc \neq 0,\ p \neq 0$,

(18) $\left(As_{13}^1, \begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}\right) \cong \left(As_{13}^1, \begin{pmatrix} p^2a & p^3b \\ -p^3b & 0 \end{pmatrix}\right)$, *where* $a, b, p \in \mathbb{F},\ a \neq 0,\ b \neq 0,\ p \neq 0$.

Lemma 2.2. *The representatives of isomorphism classes of pairs* $(\mathbb{A}, \sigma)$, *where* $\mathbb{A}$ *is a two dimensional associative algebra over a field* $\mathbb{F}$ $(Char(\mathbb{F}) = 2)$, $\sigma : \mathbb{A} \times \mathbb{A} \to \mathbb{F}$ *is a non-degenerate form, are given as follows:*

(1) $\left(As_{12,2}^1, \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right) \cong \left(As_{12,2}^1, \begin{pmatrix} p^2a & 0 \\ p^3c & p^4d \end{pmatrix}\right)$, *where* $a, c, d, p \in \mathbb{F},\ ad \neq 0,\ p \neq 0$,

(2) $\left(As_{12,2}^1, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right) \cong \left(As_{12,2}^1, \begin{pmatrix} 0 & p^3b \\ p^3c & 0 \end{pmatrix}\right)$, *where* $b, c, p \in \mathbb{F},\ bc \neq 0,\ p \neq 0$,

(3) $\left(As_{12,2}^1, \begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}\right) \cong \left(As_{12,2}^1, \begin{pmatrix} p^2a & p^3b \\ -p^3b & 0 \end{pmatrix}\right)$, *where* $a, b, p \in \mathbb{F},\ b^2 \neq 0,\ p \neq 0$,

(4) $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right)$, *where* $\beta_1, a, c, d \in \mathbb{F},\ \beta_1 ad \neq 0$,

(5) $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} a & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right) \cong \left(As^2_{11,2}(\beta_1), \begin{pmatrix} p^2(a-\beta_1^2 d)+\beta_1^2 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right)$, *where* $\beta_1(ad-\beta_1^2 d^2) \neq 0$ *and the polynomial* $u^2(a-\beta_1^2 d)+\beta_1^2 d$ *has no root in* $\mathbb{F}$, $\beta_1, a, d, p \in \mathbb{F}$,

(6) $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} a & \beta_1 d \\ 0 & d \end{pmatrix}\right)$, *where* $\beta_1, a, d \in \mathbb{F},\ \beta_1 ad \neq 0$,

(7) $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} 0 & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right)$, *where* $\beta_1, d \in \mathbb{F},\ \beta_1 d \neq 0$,

(8) $\left(As^2_{11,2}(0), \begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(9) $\left(As^2_{11,2}(0), \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(10) $\left(As^2_{11,2}(0), \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^2_{11,2}(0), \begin{pmatrix} p^2 a & 0 \\ 0 & d \end{pmatrix}\right)$, *where* $p, a, d \in \mathbb{F},\ ad \neq 0$,

(11) $\left(As^3_{6,2}, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(12) $\left(As^3_{6,2}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^3_{6,2}, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(13) $\left(As^3_{6,2}, \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right)$, *where* $c \in \mathbb{F},\ c \neq 0,\ c+1 \neq 0$,

(14) $\left(As^3_{6,2}, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$,

(15) $\left(As^4_{4,2}(\beta_1), \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cong \left(As^4_{4,2}(\beta_1), \begin{pmatrix} a+c+b+d & b+d \\ c+d & d \end{pmatrix}\right)$, *where* $\beta_1, a, b, c, d \in \mathbb{F}$, $ad-bc \neq 0$,

(16) $\left(As^5_{3,2}, \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}\right)$, *where* $a, c, d \in \mathbb{F},\ ad-c \neq 0$,

(17) $\left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(18) $\left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(19) $\left(As^6_{3,2}, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *wherea*$, d \in \mathbb{F},\ ad \neq 0$,

(20) $\left(As^6_{3,2}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^6_{3,2}, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(21) $\left(As^6_{3,2}, \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right)$, *where* $c \in \mathbb{F},\ c \neq 0,\ c+1 \neq 0,$

(22) $\left(As^6_{3,2}, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$.

3. Classification of two-dimensional Frobenius algebras

Now we determine those pairs from the lemmas above which are Frobenious algebras.

Theorem 3.1.

- *If* $Char(\mathbb{F}) \neq 2$ *then any two-dimensional Frobenius algebra over* $\mathbb{F}$ *is isomorphic to only one of the following such algebras:*
  - $\left(As^2_3, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^2_3, \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ t \neq 0,\ ad \neq 0,$
  - $\left(As^5_3(0), \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$,
  - $\left(As^5_3(\alpha_4), \begin{pmatrix} a & b \\ b & 2\alpha_4 a \end{pmatrix}\right) \cong \left(As^5_3(\alpha_4), \begin{pmatrix} a & -b \\ -b & 2\alpha_4 a \end{pmatrix}\right)$, *where* $\alpha_4, a, b \in \mathbb{F}$, $2\alpha_4 a^2 - b^2 \neq 0,\ \alpha_4 \neq 0,$
  - $\left(As^1_{13}, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\right) \cong \left(As^1_{13}, \begin{pmatrix} 0 & p^3b \\ p^3b & 0 \end{pmatrix}\right)$, *where* $b, p \in \mathbb{F},\ p \neq 0,\ b \neq 0.$
- *If* $Char(\mathbb{F}) = 2$ *then any two-dimensional Frobenius algebra over* $\mathbb{F}$ *is isomorphic to only one of the following such algebras:*
  - $\left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ t \neq 0,\ ad \neq 0,$
  - $\left(As^4_{4,2}(\beta_1), \begin{pmatrix} b+\beta_1 d & b \\ b & d \end{pmatrix}\right) \cong \left(As^4_{4,2}(\beta_1), \begin{pmatrix} b+d+\beta_1 d & b+d \\ b+d & d \end{pmatrix}\right)$, *where* $\beta_1, b, d \in \mathbb{F}$, $\beta_1 d^2 + bd - b^2 \neq 0,$
  - $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} \beta_1 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right) \cong \left(As^2_{11,2}(\beta_1), \begin{pmatrix} p^2(\beta_1 d - \beta_1^2 d) + \beta_1^2 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right)$, *where the polynomial* $u^2(1-\beta_1) + \beta_1$ *has no root in* $\mathbb{F}$, $\beta_1, d, p \in \mathbb{F}$, $\beta_1 \neq 0, 1,\ d \neq 0,$
  - $\left(As^1_{12,2}, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\right) \cong \left(As^1_{12,2}, \begin{pmatrix} 0 & p^3b \\ p^3b & 0 \end{pmatrix}\right)$, *where* $b, p \in \mathbb{F},\ p \neq 0,\ b \neq 0.$

Proof. Let now check the condition $\sigma(\mathrm{xy}, \mathrm{z}) = \sigma(\mathrm{x}, \mathrm{yz})$ for the pairs appeared in Lemma 2.1 ($Char\mathbb{F} \neq 2$) and Lemma 2.2 ($Char\mathbb{F} = 2$), i.e., find the solutions to the system of equations (1.3).

• Let $Char\mathbb{F} \neq 2$. In this case the system of equations (1.3) is consistent only for pairs (3), (12), (15) and (17) of Lemma 2.1. The solutions to the system are given as follows. For the pair

$$(3)\ \left(As^2_3, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^2_3, \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right),$$

where $a, d, t \in \mathbb{F}, ad \neq 0, t \neq 0$ of Lemma 2.1 the system becomes an identity, therefore, the pair (3) is a Frobenius algebra.

Consider the pair (12) $\left(As^5_3(0), \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}\right)$, where $a, c, d \in \mathbb{F}$, $ad - c \neq 0$ of Lemma 2.1. As a solution to the system (1.3) we get $c = 1$ and $d = 0$ and the corresponding Frobenius algebras are

$$\left(As^5_3(0), \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\right), \text{ where } a \in \mathbb{F}.$$

Among the pairs

$$(15)\left(As_3^5(\alpha_4),\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)\cong\left(As_3^5(\alpha_4),\begin{pmatrix}a & -b\\ -c & d\end{pmatrix}\right),$$

where $a,b,c,d,\alpha_4\in\mathbb{F},\ ad-bc\neq 0,\ \alpha_4\neq 0$ of Lemma 2.1 those satisfying $c=b$, and $d=2\alpha_4 a$ are Frobenius. Thus, we get

$$\left(As_3^5(\alpha_4),\begin{pmatrix}a & b\\ b & 2\alpha_4 a\end{pmatrix}\right)\cong\left(As_3^5(\alpha_4),\begin{pmatrix}a & -b\\ -b & 2\alpha_4 a\end{pmatrix}\right),$$

where $a,b,\alpha_4\in\mathbb{F},\ 2\alpha_4 a^2-b^2\neq 0,\alpha_4\neq 0$.

Considering the system of equations (1.3) for the pair

$$(17)\left(As_{13}^1,\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}\right)\cong\left(As_{13}^1,\begin{pmatrix}0 & p^3b\\ p^3c & 0\end{pmatrix}\right),$$

where $p\neq 0$, of Lemma 2.1 we obtain $c=b$ and hence,

$$\left(As_{13}^1,\begin{pmatrix}0 & b\\ b & 0\end{pmatrix}\right)\cong\left(As_{13}^1,\begin{pmatrix}0 & p^3b\\ p^3b & 0\end{pmatrix}\right)\ \text{where } b,p\in\mathbb{F}\ p\neq 0,\ \text{and } b\neq 0$$

is a Frobenius algebra.

• Let $Char\mathbb{F}=2$. Considering the pairs

$$(2)\left(As_{12,2}^1,\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}\right)\cong\left(As_{12,2}^1,\begin{pmatrix}0 & p^3b\\ p^3c & 0\end{pmatrix}\right),$$

where $b,c,p\in\mathbb{F},\ bc\neq 0,p\neq 0$ of Lemma 2.2 we generate the following Frobenius algebras

$$\left(As_{12,2}^1,\begin{pmatrix}0 & b\\ b & 0\end{pmatrix}\right)\cong\left(As_{12,2}^1,\begin{pmatrix}0 & p^3b\\ p^3b & 0\end{pmatrix}\right),$$

where $b,p\in\mathbb{F},\ p\neq 0,b\neq 0$ as far as the solution to the system of equations (1.3) in this case is $c=b=0$.

From the pair

$$(5)\left(As_{11,2}^2(\beta_1),\begin{pmatrix}a & \beta_1 d\\ \beta_1 d & d\end{pmatrix}\right)\cong\left(As_{11,2}^2(\beta_1),\begin{pmatrix}p^2(a-\beta_1^2d)+\beta_1^2d & \beta_1 d\\ \beta_1 d & d\end{pmatrix}\right),$$

where $\beta_1(ad-\beta_1^2d^2)\neq 0$ and the polynomial $u^2(a-\beta_1^2d)+\beta_1^2d$ has no root in $\mathbb{F}$, $\beta_1,a,d,p\in\mathbb{F}$, of Lemma 2.2 subjecting to the system (1.3) we get $a=\beta_1 d$ and obtain the following Frobenius algebra

$$\left(As_{11,2}^2(\beta_1),\begin{pmatrix}\beta_1 d & \beta_1 d\\ \beta_1 d & d\end{pmatrix}\right)\cong\left(As_{11,2}^2(\beta_1),\begin{pmatrix}p^2(\beta_1 d-\beta_1^2d)+\beta_1^2d & \beta_1 d\\ \beta_1 d & d\end{pmatrix}\right),$$

where the polynomial $u^2(1-\beta_1)+\beta_1$ has no root in $\mathbb{F}$, $\beta_1,d,p\in\mathbb{F}$ and $\beta_1 d^2-\beta_1^2d^2\neq 0$.

Let us consider the pair (15) $\left(As_{4,2}^4(\beta_1),\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)\cong\left(As_{4,2}^4(\beta_1),\begin{pmatrix}a+c+b+d & b+d\\ c+d & d\end{pmatrix}\right)$, where $\beta_1,a,b,c,d\in\mathbb{F},\ ad-bc\neq 0$ of Lemma 2.2. Then the system of equations (1.3) is equivalent to $b=c$ and $a=b+\beta_1 d$. Therefore,

$$\left(As_{4,2}^4(\beta_1),\begin{pmatrix}b+\beta_1 d & b\\ b & d\end{pmatrix}\right)\cong\left(As_{4,2}^4(\beta_1),\begin{pmatrix}b+d+\beta_1 d & b+d\\ b+d & d\end{pmatrix}\right),$$

where $\beta_1,b,d\in\mathbb{F},\ \beta_1 d^2+bd-b^2\neq 0$ are Frobenius algebras.

Finally, considering the pair

$$(18)\left(As_{3,2}^5,\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\right)\cong\left(As_{3,2}^5,\begin{pmatrix}a & 0\\ 0 & t^2d\end{pmatrix}\right),$$

where $a,d,t\in\mathbb{F},t\neq 0,ad\neq 0$ of Lemma 2.2 we find the system of equations (1.3) to be an identity and hence, they all are Frobenius algebras.

Note that for all the other classes from Lemma 2.2 the system of equations (1.3) is inconsistent.

□

4. Remarks

(1) The authors of the paper are informed by the referee on a classification of two-dimensional associative algebras obtained earlier by M. Gerstenhaber and F. Kubo in [11]. On the way we compare the classification of [11] with that obtained in [20]. Here are the comparisons and some corrections.

• $Char(\mathbb{F}) \neq 2$:

Algebra from [11]	Algebra from [20]	Isomorphism
$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$As_3^5(\frac{1}{4})$	$P_1 = g^{-1}As_3^5(\frac{1}{4})g^{\otimes 2}$, where $g = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{pmatrix}$
$P_2(d) = \begin{pmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 1 & 0 \end{pmatrix}, d \neq \frac{1}{4}$ $\cong \begin{pmatrix} 1 & 0 & 0 & r^2d \\ 0 & 1 & 1 & 0 \end{pmatrix}, \ r \neq 0$	$As_3^5(d), d \neq 0, \frac{1}{4}$	$P_2(d) = As_3^5(d)$
$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$	$As_3^5(0)$	$P_3 = g^{-1}As_3^5(0)g^{\otimes 2}$, where $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	As_3^3	$P_4 = As_3^3$
$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	As_3^4	$P_5 = g^{-1}As_3^4g^{\otimes 2}$, where $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	As_3^2	$P_6 = As_3^2$
$P_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	As_{13}^1	$P_7 = As_{13}^1$

• $Char(\mathbb{F}) = 2$:

Algebra from [11]	Algebra from [20]	Isomorphism
$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$	$As_{4,2}^4(1)$	$Q_1 = gAs_{4,2}^4(1)(g^{-1})^{\otimes 2}$, where $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$Q_2(d) = \begin{pmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 1 & 0 \end{pmatrix}, d \neq 0$ $\cong \begin{pmatrix} 1 & 0 & 0 & r^2d \\ 0 & 1 & 1 & 0 \end{pmatrix}, \ r \neq 0$	$As_{11,2}^2(d), d \neq 0$	$Q_2(d) = g^{-1}As_{11,2}^2(d)g^{\otimes 2}$, where $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$	$As_{11,2}^2(0)$	$P_3 = g^{-1}As_{11,2}^2(0)g^{\otimes 2}$, where $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$As_{3,2}^6$	$P_4 = As_{3,2}^6$
$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$As_{6,2}^3$	$P_5 = As_{6,2}^3$
$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$As_{3,2}^5$	$P_6 = As_{3,2}^5$
$P_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$As_{12,2}^1$	$P_7 = As_{12,2}^1$

Conclusion: In the case of $Char(\mathbb{F}) = 2$ in [11] the family of algebras $As^4_{4,2}(\beta_1), \beta_1 \neq 1$ is missing.

(2) There are the following misprintings in the list of automorphism groups in [20]: in the case of $Char(\mathbb{F}) = 3$ the groups

$$Aut(As^1_{13,3}) = \left\{ \begin{pmatrix} p & 0 \\ s & 2p^2 \end{pmatrix} \mid p, s \in \mathbb{F}, p \neq 0 \right\} \text{and } Aut(As^4_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ 1+2t & t \end{pmatrix} \mid t \in \mathbb{F}, t \neq 0 \right\}$$

must be read as follows

$$Aut(As^1_{13,3}) = \left\{ \begin{pmatrix} p & 0 \\ s & p^2 \end{pmatrix} \mid p, s \in \mathbb{F}, p \neq 0 \right\} \text{and } Aut(As^4_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid s, t \in \mathbb{F}, t \neq 0 \right\},$$

respectively.

(3) In the paper we used Maple software for some computations.

5. Acknowledgements

The authors would like to thank anonymous reviewer for his/her useful and valuable comments.

References

[1] Abrams L.; Two-dimensional Topological Quantum Field Theories and Frobenius algebras. Journal of Knot Theory and Its Ramifications., –1996. –5(5). –P.569-587.

[2] Atiyah M.; Topological Quantum Field Theories. Publications mathématiques de l' I.H.É.S., –1988. –68. –P.175-186.

[3] Bai C.; Double constructions of Frobenius algebras. Connes cocycles and their duality. J. Noncommut. Geom., –2020. –4. –P.475-530. DOI 10.4171/JNCG/64.

[4] Bar-Natan D.; Khovanov's homology for tangles and cobordisms. Geom. Topol., –2005. –9(3). –P1443-1499. arXiv:math/0410495, doi:10.2140/gt.2005.9.1443.

[5] Bekbaev U.; Classification of two-dimensional algebras over any base field, 2023. AIP Conference Proceedings., –2880. –030001. https://doi.org/10.1063/5.0165726.

[6] Brauer R., Nesbitt C.; On the regular representations of algebras. Proc. Natl. Acad. Sci. USA., –1937. –23(4). –P.236–240. doi:10.1073/pnas.23.4.236.

[7] Caenepeel S., Ion B., Militaru G.; The structure of Frobenius algebras and separable algebras. Journal of K-Theory., –2000. –19(4). –P.365–402. DOI: 10.1023/A:1007849203555

[8] Csikós B., Verhóczki L.; Classification of Frobenius Lie algebras of dimension ≤ 6. Publ. Math. Debrecen., –70/3-4. –2007. –P.427–451.

[9] Curtis C.W., Reiner I.; Representation theory of finite groups and associative algebras. John Wiley & Sons, New York · London · Sydney., –1962. –689 pp.

[10] Dieudonne J.; Remarks on quasi-Frobenius rings. Illinois Journal of Mathematics., –1958. –2(3). –P.346-354. https://doi.org/10.1215/ijm/1255454538

[11] Gerstenhaber M., Kubo F.; On the deformation of the two dimensional associative algebras. JP Journal of Algebra, Number Theory and Applications., –2022. –54. –P.1-17.

[12] Kock J.; Frobenius algebras and 2D topological quantum field theories. London Mathematical Society. Cambridge University Press., –2003.

[13] Frobenius G.; Theorie der hyperkomplexen Grossen. Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften., –1903. –24. –P.504-537; 634-645.

[14] Kadison L., Nikshych D.; Frobenius extensions and weak Hopf algebras. Journal of Algebra., –2001. –244. –P.312-342. https://doi.org/10.1006/jabr.2001.8911

[15] Lawvere F.W.; Ordinal sums and equational doctrines. In: Eckmann B. (ed.) Seminar on Triples and Categorical Homology Theory, ETH 1966/67. Lecture Notes in Mathematics, no.80. Springer-Verlag, New York., –1967. –P.141-155.

[16] Nakayama T.; On Frobeniusean algebras I. Annals of Mathematics., –1939. –40. –P.611-633.

[17] Nakayama T.; On Frobeniusean algebras II. Annals of Mathematics., –1941. –42. –P.1-21.

[18] Ooms A.; On Frobenius Lie algebras. Communications in Algebra., –1980. –8(1). –P.13-52. https://doi.org/10.1080/00927878008822445

[19] Radford D.E.; The structure of Hopf algebras with a projection. Journal of Algebra., –1985. –92(2). –P.322-347.

[20] Rakhimov I.S.; Algebraic structures on two-dimensional vector space over any basic field. Malaysian Journal of Mathematical Sciences., –2024. –18(2). –P.227-257. https://doi.org/10.47836/mjms.18.2.02

[21] Turner P.; Five lectures on Khovanov homology. arXiv:math/0606464., –2006.

[22] Quinn F.; Lectures on axiomatic topological quantum field theory. In: Geometry and Quantum Field Theory (Park City, UT, 1991). Amer. Math. Soc., Providence, RI., –1995. –P.323-453.

Asrorov D.U.,
Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan.
email: 96asrorovdiorjon@mail.ru

Bekbaev U.D.,
Turin Polytechnic University in Tashkent, Tashkent, Uzbekistan
email: uralbekbaev@gmail.com

Rakhimov I.S.,
Universiti Teknologi MARA (UiTM), Shah Alam, Malaysia and V.I. Romanovsky Institute of Mathematics, Tashkent, Uzbekistan
email: isamiddin@uitm.edu.my