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On Finite-Dimensional Approximations of the Time-Optimal Control Problem for the Heat Equation in a Rod

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Abstract. The time-optimal problem is considered for the controllable heat conduction process in a rod. Using the Fourier method, the problem is usually reduced to an infinite system of one-dimensional control equations whose control parameters are connected by complex relationships that make it difficult to solve. This article presents a method for regrouping the terms of the Fourier series of the control function, enabling the reduction of the problem to a finite-dimensional framework. The resulting reduced problem facilitates the more effective construction of a suboptimal control.

Keywords: heat equation, control problem, time optimality, Fourier expansion, regrouping, finite dimensional reduction, suboptimal control.

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1. Statement of the problem

The time-optimal problem for the heat equation is formulated as follows. Consider the equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + v(t,x) \tag{1.1}$$

in the domain $t \geq 0$, $0 \leq x \leq \pi$ with boundary conditions $u(t,0) = u(t,\pi) \equiv 0$ and a given initial state $u(0,x) = \varphi(x)$. Here, the function $v(t,x)$ from the class L_2, playing the role of control, should satisfy the constraint

$$\sup |v(t,x)| \leq v_0. \tag{1.2}$$

If $\varphi(\cdot) \in L_2^0(0,\pi)$ then the equation (1.1) has a unique solution $u(t,x)$ from the Sobolev class $H^{1,2}$ [1, 2, 3]. If at this $u(t,x) \equiv 0$ for some $t = T$, then the function $v(t,x)$ is called an admissible control due to the initial state $\varphi(v)$ and the number T is called a transition time. The time-optimal problem for (1.1) demands to find an admissible control $\hat{v}(t,x)$ such that the transition time is minimal. We call the formulated task a Chernousko problem, as in the work [10] F. Chernousko reduced this problem to an infinite system of one-dimensional control problems

$$\frac{du_n}{dt} = -n^2 u_n + v_n, \quad u_n(0) = \varphi_n \tag{1.3}$$

where u_n, v_n, φ_n are the coefficients of the Fourier expansions of the functions $u(t,x)$, $v(t,x)$, and $\varphi(x)$, respectively, more precisely

$$u_n(t) = \frac{1}{\pi}\int_0^\pi u(t,x)\sin x\,dx,$$
$$v_n(t) = \frac{1}{\pi}\int_0^\pi v(t,x)\sin x\,dx,$$
$$\varphi_n = \frac{1}{\pi}\int_0^\pi \varphi(x)\sin x\,dx.$$

The system (1.3) can be easily solved in the Hilbert space l_2 [11]. However, a control sequence $(v_1, v_2, \ldots, v_k, \ldots) \in l_2$ is subject to the constraint.

$$\sup_{0\leq x\leq\pi} |v_1 \sin x + v_2 \sin 2x + \ldots + v_n \sin nx + \ldots| \leq v_0. \tag{1.4}$$

In other words, the region of values of a control sequence for (1.3), (1.4) is given by the formula

$$V = \left\{ (v_n) \in l_2 \mid \sup_x |v_1 \sin x + v_2 \sin 2x + \ldots + v_n \sin nx + ...| \leq v_0 \right\}. \tag{1.5}$$

Obviously (1.5) is a closed and convex set, but it is unknown whether V is compact. In any case, the time-optimal problem for (1.4), (1.5) remains difficult to solve. That is why it is natural to attempt to find suboptimal controls. It seems logical to truncate the partial sums of the Fourier expansions of $u(t,x)$, $v(t,x)$, and $\varphi(x)$, replacing original the infinite-dimensional problem with a finite-dimensional one. However, it would be difficult to estimate the remainder terms of the Fourier expansions for $v(t,x)$ and $\varphi(x)$.

In the paper [10], the region (1.5) is replaced by the set

$$V_1 = \{(v_n) \in l_2 \mid |v_n| \le U_n,\, n = 1,2,\ldots\} \tag{1.6}$$

where the sequence $\{U_n\}$ should be chosen to satisfy the following condition

$$\sum U_n = v_0 \tag{1.7}$$

due to (1.2). F. Chernousko showed that the sequence U_n can be chosen so that the time-optimal control for the fully separated system of control equations

$$\dot{u}_n = -n^2 u_n + v_n, \quad |v_n| \le U_n, \quad n = 1,2,\ldots \tag{1.8}$$

has a solution possessing the property $u_n(T_1) = 0$ for all n and the same time T_1. The inclusion $V_1 \subset V$, implies that $T_1 \ge T_{\text{opt}}$, which means the solution of the system (1.3), (1.6) can serve as a suboptimal control. Taking V_1 instead of V can be interpreted as replacing the domain V with a Hilbert brick embedded in it. In [12], a stronger result was obtained by replacing V with the Hilbert "octahedron" (cocube):

$$\{v_n \in l_2 \mid |v_1| + |v_2| + \ldots + |v_n| + \ldots \le v_0\} \tag{1.9}$$

Another approach to construct suboptimal controls was suggested in the papers [13], [15], [16], based on the idea of regrouping the terms of the expansion $\sum_k v_k \sin kx$ in such a way that the infinite system of finite-dimensional problems would be reduced to a single finite-dimensional problem. This is based on the following arithmetic assertion, that has an independent interest. A set of positive integers $\{n_1, n_2, \ldots, n_m\}$ such that $n_1 < n_2 < \ldots < n_m$, will be called a decomposition basis if the partition

$$\mathbb{N}^+ = \bigoplus_{k \in K} \{kn_1, kn_2, \ldots, kn_m\} \tag{1.10}$$

holds for some $K \subset \mathbb{N}^+$. (The symbol $\oplus$ denotes a union of non-intersecting subsets.)

Obviously, the sets $\{1\}$ and $\{1, p\}$ (where $p > 1$) are decomposition bases.

Proposition 1.1. *The set* $\{1, 2, 4, \ldots, 2^{m-1}\}$ *forms a decomposition basis.*

Proof. One may assume $m \ge 3$. We will construct the appropriate subset $K = \{k_1, k_2, \ldots, k_n, \ldots\}$ possessing the property (1.10). Every number $n \in \mathbb{N}^+$ can be represented in the form $n = 2^{sm+r}p$, where $s \in \mathbb{N}$ and p is odd, $r = 0, 1, 2, \ldots, m-1$. Thus $n \in K$ if and only if $r = 0$.

The list of respective arrays in the decomposition for $m = 3$ is as follows

$$\mathbf{1}, 2, 4; \quad \mathbf{3}, 6, 12; \quad \mathbf{5}, 10, 20; \quad \mathbf{7}, 14, 28; \quad \mathbf{8}, 16, 32;$$

$$\mathbf{9}, 18, 36; \quad \mathbf{11}, 22, 44; \quad \mathbf{13}, 26, 52; \quad \ldots; \quad \mathbf{23}, 46, 92; \quad \mathbf{24}, 48, 96; \quad \ldots$$

It should be verified that the arrays

$$\{k_i, 2k_i, 4k_i, \ldots, 2^{m-1}k_i\},$$

$$\{k_j, 2k_j, 4k_j, \ldots, 2^{m-1}k_j\}$$

do not intersect for $i < j$. Indeed, the relation $2^\alpha k_i = 2^\beta k_j$ where $i < j$, $\alpha, \beta \in \{0, 1, 2, \ldots, m-1\}$ leads to a contradiction because of the representations $k_i = 2^{s_i m} p_i$ and $k_j = 2^{s_j m} p_j$. □

2. Regrouping method for finite-dimensional approximations

Now using the representation (1.10), terms of the Fourier expansion for a control function $v(t,x)$ will be regrouped:

$$v(t,x) = \sum_{i=1}^{\infty}(v_{k_i}\sin k_i x + v_{2k_i}\sin 2k_i x + \ldots + v_{2^{m-1}k_i}\sin 2^{m-1}k_i x). \tag{2.1}$$

Further, instead of the condition 2.1, we consider the more stronger constraint (naturally loosing optimality)

$$\sum_{i=1}^{\infty}|v_{k_i}\sin k_i x + v_{2k_i}\sin 2k_i x + \ldots + v_{2^{m-1}k_i}\sin 2^{m-1}k_i x| \le v_0 \tag{2.2}$$

for all x. Then, following Chernousko's approach, consider the sequence of constraints

$$|v_{k_i}\sin k_i x + v_{2k_i}\sin 2k_i x + \ldots + v_{2^{m-1}k_i}\sin 2^{m-1}k_i x| \le U_i, \tag{2.3}$$

for $i = 1, 2, \ldots$, with $\sum_{i=1}^{\infty} U_i = v_0$.

In this way, we get the sequence of m-dimensional control problems

$$\begin{cases} \dot u_{k_i} = -k_i^2 u_{k_i} + v_{k_i}, \\ \dot u_{2k_i} = -(2k_i)^2 u_{2k_i} + v_{2k_i}, \\ \ldots \\ \dot u_{2^{m-1}k_i} = -2^{2(m-1)}k_i^2 u_{2^{m-1}k_i} + v_{2^{m-1}k_i}, \end{cases} \tag{2.4}$$

for $i = 1, 2, \ldots$, with the constraints (2.3). Note that the collection (2.4) is equivalent to system (1.3), but conditions (2.3) is stronger than (2.2).

Now let us perform the following transformation.

$$y_1 = \frac{k_i^2}{\mu_{k_i}} u_{k_i}, \quad y_2 = \frac{k_i^2}{\mu_{k_i}} u_{2k_i}, \quad \ldots, \quad y_m = \frac{k_i^2}{\mu_{k_i}} u_{2^{m-1}k_i}, \quad \tau = k_i^2 t.$$

$$v_{k_i} = \mu_{k_i} w_1, \quad v_{2k_i} = \mu_{k_i} w_2, \quad \ldots, \quad v_{2^{m-1}k_i} = \mu_{k_i} w_m.$$

Overall, all systems (2.4) will be reformulated to the single m-dimensional control system:

$$\begin{cases} \dot y_1 = -y_1 + w_1, \\ \dot y_2 = -4y_2 + w_2, \\ \ldots \\ \dot y_m = -4^{m-1} y_m + w_m, \end{cases} \tag{2.5}$$

where

$$w_1 = \frac{1}{\mu_{k_i}} v_{k_i}, \quad w_2 = \frac{1}{\mu_{k_i}} v_{2k_i}, \quad \ldots, \quad w_m = \frac{1}{\mu_{k_i}} v_{2^{m-1}k_i}.$$

Now consider the transformation of the constraints (2.3) for the control parameter of the system (2.5). For that, we replace 2.3 by the even more rigid restriction

$$\max_x \left|w_1 \sin k_i x + w_2 \sin 2k_i x + \ldots + w_m \sin 2^{m-1}k_i x\right| \le 1, \tag{2.6}$$

for $k = 1, 2, \ldots$, that implies 2.3 if $\sum_i \mu_{k_i} = v_0$. Obviously 2.6 can be written in the form

$$\max_{\bar x} \left|w_1 \sin \bar x + w_2 \sin 2\bar x + \ldots + w_m \sin 2^{m-1}\bar x\right| \le 1, \tag{2.7}$$

The time-optimal problem (2.5), (2.7) is a concrete finite-dimensional linear system with the region W of vectors $\mathbf{w} = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$, satisfying 2.7. Obviously, W is a convex compact body (containing the cube $|w_1| + |w_2| + \ldots + |w_m| \le 1$). Therefore, there exists an optimal control $\hat{\mathbf{w}}(t)$ with a bang-bang property, that can be found by the Pontryagin maximum principle [4], [6]. It is clear that the optimal control $\hat{\mathbf{w}}(t)$ of the system (2.5), (2.7) generates optimal controls $(\hat{v}_{k_i}, \hat{v}_{2k_i}, \ldots, \hat{v}_{2^{m-1}k_i})$ for every problem (2.3) - (2.4), $i = 1, 2, \ldots$.

Finally, by selecting the sequence of numbers $U_1, U_2, \ldots$ according to the technique proposed by Chernousko, it is possible to guarantee that the optimal transition time for all systems (2.3) - (2.4) is identical. Furthermore, by combining the control functions

$$(\hat{v}_{k_i},\ \hat{v}_{2k_i},\ \ldots,\ \hat{v}_{2^{m-1}k_i}), \quad i = 1, 2, \ldots,$$

and subsequently applying inverse regrouping, one obtains a suboptimal control for the problem (2.3) – (2.4).

Concluding Remarks

As demonstrated by F. Chernousko, although the Fourier method does not enable the direct construction of an optimal control for the heat conduction equation in a rod, it remains highly effective for developing finite-dimensional approximations. In contrast to the classical approach, when the problem is reduced to a finite-dimensional system by truncating partial sums of the Fourier seriesthe regrouping method proposed in the present work yields an improvement in the quality of suboptimal control.

A separate investigation will be devoted to a detailed comparison of these two approaches to finite-dimensional approximations. It should be noted that, for small values of m, the regrouping method produces systems for which the optimal control can be determined explicitly [13] (see also [14] - [16]).

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