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Symmetric Leibniz algebras whose underlying Lie algebra is almost filiform

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Abstract. In this paper, we classify the symmetric Leibniz algebras whose underlying Lie algebra is almost filiform. We also describe the symmetric Leibniz algebras associated with Heisenberg and triangular Lie algebras. Moreover, we prove that there is no symmetric Leibniz algebra whose underlying Lie algebra is perfect.

Keywords: Nilpotent Lie algebras, symmetric Leibniz algebras, perfect algebras, almost filiform Lie algebras.

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1. Introduction

Leibniz algebras are generalizations of Lie algebras, which are defined with the property that any operator of left (or right) multiplication is a derivation. Leibniz algebras were first introduced in the work of Bloh [7] under the name D-algebras in 1965. Then they were rediscovered by Loday [12], who called them Leibniz algebras. Since the left and right Leibniz algebras have opposite properties, researchers investigated just left (or right) Leibniz algebra as a Leibniz algebra. In the recent years, the theory of Leibniz algebras has been actively examined and many results on Lie algebras have been extended to Leibniz algebras [4, 10, 11, 14].

Symmetric Leibniz algebras are intersections of the left and right Leibniz algebras. The first characterization and theory of symmetric Leibniz algebras are found in the paper of Benayadi and Hidri [6]. They proved that quadratic left (or right) Leibniz algebra which has properties of invariant, non-degenerate and symmetric bilinear forms is a symmetric Leibniz algebras. Recently, the theory of symmetric Leibniz algebras has been intensively studied and many works have been devoted to the investigation of this theory [5, 6, 11, 13]. Symmetric Leibniz algebras are associated to Lie racks [1] and any symmetric Leibniz algebra is flexible, power-associative, and a nilalgebra with nilindex 3. Symmetric Leibniz algebra gives a Poisson algebra under commutator and anticommutator multiplications [2].

It is difficult and fundamental problem that to classify up to isomorphism of any class of algebras. There are several methods for the classification of algebras that were used for the Leibniz algebras. Barreiro and Benayadi in [5] gave a method for the classification of symmetric Leibniz algebras, which is based on the property that a symmetric Leibniz algebra forms a Poisson algebra with respect to the commutator and anticommutator. Using this method, the classification of symmetric Leibniz algebras, whose underlying Lie algebra is filiform was obtained in [8]. The complete classification of complex five-dimensional symmetric Leibniz algebras can be found in [3, 9]. In this work, using this method we give the classification of symmetric Leibniz algebras whose underlying Lie algebra is almost filiform. Moreover, we prove that a non-Lie symmetric Leibniz algebra associated with perfect Lie algebras does not exist.

2. Preliminaries

In this section, we give the basic concepts, definitions and preliminary results which are used in this paper.

Definition 2.1. An algebra $(\mathcal{L}, [-,-])$ over a field $\mathbb{F}$ is called Lie algebra if for any $u, v, w \in \mathcal{L}$ the following identities hold:

$$[u, v] = -[v, u],$$

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0.$$

Definition 2.2. An algebra $(\mathcal{L}, \cdot)$ is said to be a symmetric Leibniz algebra, if for any $u, v, w \in \mathcal{L}$ the following identities hold:

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w + v \cdot (u \cdot w),$$
$$(v \cdot w) \cdot u = (v \cdot u) \cdot w + v \cdot (w \cdot u).$$

Let $(\mathcal{L}, \cdot)$ be an algebra. For all $u, v \in \mathcal{L}$, we define $[-,-]$ and $\circ$ as follows

$$[u, v] = \frac{1}{2}(u \cdot v - v \cdot u), \qquad u \circ v = \frac{1}{2}(u \cdot v + v \cdot u).$$

Proposition 2.3. [5] *Let $(\mathcal{L}, \cdot)$ be an algebra. The following assertions are equivalent:*

1. *$(\mathcal{L}, \cdot)$ is a symmetric Leibniz algebra.*
2. *The following conditions hold:*
   (a) *$(\mathcal{L}, [-,-])$ is a Lie algebra.*
   (b) *For any $x, y \in \mathcal{L}$, $x \circ y$ belongs to the center of $(\mathcal{L}, [-,-])$.*
   (c) *For any $x, y, z \in \mathcal{L}$, $([x, y]) \circ z = 0$ and $(x \circ y) \circ z = 0$.*

According to this Proposition, any symmetric Leibniz algebra is given by a Lie algebra $(\mathcal{L}, [-,-])$ and a symmetric bilinear form $\omega : \mathcal{L} \times \mathcal{L} \to Z(\mathcal{L})$, where $Z(\mathcal{L})$ is the center of the Lie algebra, such that for any $x, y, z \in \mathcal{L}$,

$$\omega([x, y], z) = \omega(\omega(x, y), z) = 0. \tag{2.1}$$

Then the product of the symmetric Leibniz algebra is given by

$$x \cdot_\omega y = [x, y] + \omega(x, y).$$

Proposition 2.4. [1] *Let $(\mathcal{L}, [-,-])$ be a Lie algebra and ω and μ two solutions of (2.1). Then $(\mathcal{L}, \cdot_\omega)$ isomorphic to $(\mathcal{L}, \cdot_\mu)$ if and only if there exists an automorphism A of $(\mathcal{L}, [-,-])$ such that*

$$\mu(x, y) = A^{-1}\big(\omega(A(x), A(y))\big).$$

In the following proposition, we consider symmetric Leibniz algebras whose underlying Lie algebra is perfect.

Proposition 2.5. *Let $\mathcal{L}$ be a complex symmetric Leibniz algebra, whose underlying Lie algebra is perfect, then it is a Lie algebra.*

Proof. Let $(\mathcal{L}, [-,-])$ be a perfect Lie algebra, i.e., $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$. Then, for any $x \in \mathcal{L}$, there exist $z_i, t_i \in \mathcal{L}$, such that $x = \sum_i \alpha_i [z_i, t_i]$. Consider a symmetric bilinear form ω satisfying the condition (2.1). Then, we obtain $\omega(x, y) = \sum_i \alpha_i \omega([z_i, t_i], y) = 0$, for any $y \in \mathcal{L}$. Hence, for any $x, y \in \mathcal{L}$, we have $x \cdot y = [x, y] + \omega(x, y) = [x, y]$. Thus, $(\mathcal{L}, \cdot)$ is a Lie algebra. □

Corollary 2.6. *The symmetric Leibniz algebra structure for the following Lie algebras is trivial:*

- *Schrödinger algebra*

$$\mathcal{S}_n : \begin{array}{llll} [x_i, y_i] = z, & [h, x_i] = x_i, & [s_{j,k}, x_i] = \delta_{k,i} x_j - \delta_{j,i} x_k, & \\ [e, f] = h, & [h, y_i] = -y_i, & [s_{j,k}, y_i] = \delta_{k,i} y_j - \delta_{j,i} y_k, & \\ [h, e] = 2e, & [e, y_i] = x_i, & [s_{j,k}, s_{l,m}] = \delta_{l,k} s_{j,m} + \delta_{j,m} s_{k,l} + \delta_{m,k} s_{l,j} + \delta_{l,j} s_{m,k}, & \\ [h, f] = -2f, & [f, x_i] = y_i, & 1 \le i \le n, \quad 1 \le j \le k \le n, \quad 1 \le l \le m \le n. & \end{array}$$

- *n-th Schrödinger algebra*

$$\mathfrak{sch}_n : \begin{array}{lllll} [x_i, y_i] = z, & [h, x_i] = x_i, & [e, f] = h, & [h, y_i] = -y_i, & \\ [h, e] = 2e, & [e, y_i] = x_i, & [h, f] = -2f, & [f, x_i] = y_i, & 1 \le i \le n. \end{array}$$

- *Virasoro algebra,*

$$Vir: \quad [e_i, e_j] = (i-j)e_{i+j} + \delta_{i,-j}\tfrac{i^3-i}{12}c.$$

- *Not-finitely graded Virasoro algebra with basis* $\{e_{\alpha,i}, c \mid \alpha \in \Gamma, i \in Z_+\}$

$$\widehat{W}(\Gamma): \quad [e_{\alpha,i}, e_{\beta,j}] = (\beta-\alpha)e_{\alpha+\beta,i+j} + (j-i)e_{\alpha+\beta,i+j+1} + \delta_{\alpha+\beta,0}\delta_{i+j,0}\tfrac{\alpha^3-\alpha}{12}c,$$

where $\alpha, \beta \in \Gamma, i, j \in \mathcal{Z}_+$.

- *Virasoro-like algebra with basis* $\{e_{\alpha,i}, c \mid \alpha \in \Gamma, i \in Z\}$

$$\begin{aligned}\tilde{\mathcal{W}(\Gamma)}: \quad [e_{\alpha,i}, e_{\beta,j}] &= (\beta-\alpha)e_{\alpha+\beta,i+j} + (j-i)e_{\alpha+\beta,i+j-1} + \delta_{\alpha+\beta,0}(\delta_{i+j,-1}\alpha^3 + 3i\delta_{i+j,0}\alpha^2 + \\ &+3i(i-1)\delta_{i+j,1}\alpha + i(i-1)(i-2)\delta_{i+j,2})c,\end{aligned}$$

where $\alpha, \beta \in \Gamma, i, j \in \mathcal{Z}$.

- *The twisted Heisenberg-Virasoro algebra with basis* $\{e_i, f_j, c, c_1, c_2 \mid i, j \in Z\}$

$$H_{Vir}: \quad \begin{aligned}&[e_i, e_j] = (j-i)e_{i+j} + \delta_{i+j,0}\tfrac{i^3-i}{12}c,\\ &[f_i, f_j] = i\delta_{i+j,0}c_1,\\ &[e_i, f_j] = jf_{i+j} + \delta_{i+j,0}(i^2+i)c_2.\end{aligned}$$

Where $\delta_{i,j}$ *are Kronecker symbols.*

3. Main result

In this section, we classify symmetric Leibniz algebras associated with the almost filiform algebra $\mathbf{n}_{n,3}$. Furthermore, we also describe the symmetric Leibniz algebras whose underlying Lie algebra is Heisenberg and triangular.

The Lie algebra $\mathbf{n}_{n,3}$ is a nilpotent Lie algebra with the table of multiplications:

$$\mathbf{n}_{n,3} \quad : \quad [e_2, e_n] = e_1, \quad [e_3, e_{n-1}] = e_1, \quad [e_k, e_{n-1}] = e_{k-1}, \quad [e_{n-1}, e_n] = e_2, \quad 4 \le k \le n-2.$$

This algebra is called almost filiform and can be found in the monograph by Snobl and Winternitz [15].

Let $\mathcal{M}$ be a complex symmetric Leibniz algebra whose underlying Lie algebra is $\mathbf{n}_{n,3}$. Since $Z(\mathbf{n}_{n,3}) = \mathrm{span}\{e_1\}$, then by straightforward computations, we get that the corresponding symmetric bilinear form $\omega : \mathbf{n}_{n,3} \times \mathbf{n}_{n,3} \to Z(\mathbf{n}_{n,3})$ satisfying equation (2.1) is

$$\omega(e_{n-2}, e_{n-2}) = A_1e_1, \quad \omega(e_{n-2}, e_{n-1}) = A_2e_1, \quad \omega(e_{n-2}, e_n) = A_3e_1,$$

$$\omega(e_{n-1}, e_n - 1) = A_4e_1, \quad \omega(e_{n-1}, e_n) = A_5e_1, \quad \omega(e_n, e_n) = A_6e_1,$$

where $(A_1, A_2, A_3, A_4, A_5, A_6) \neq (0, 0, 0, 0, 0, 0)$.

Then, considering the multiplication $x \cdot y = [x, y] + \omega(x, y)$, we obtain that any symmetric Leibniz algebra $\mathcal{M}$ associated with the almost filiform algebra has the following product

$$\mathcal{M}(A_1, A_2, A_3, A_4, A_5, A_6): \quad \begin{array}{lll} e_2 \cdot e_n = e_1, & e_{n-1} \cdot e_n = e_2 + A_5e_1, & e_{n-2} \cdot e_{n-2} = A_1e_1,\\ e_n \cdot e_2 = -e_1, & e_n \cdot e_{n-1} = -e_2 + A_5e_1, & e_{n-2} \cdot e_n = A_3e_1,\\ e_3 \cdot e_{n-1} = e_1, & e_{n-2} \cdot e_{n-1} = e_{n-3} + A_2e_1, & e_{n-1} \cdot e_{n-1} = A_4e_1,\\ e_{n-1} \cdot e_3 = -e_1, & e_{n-1} \cdot e_{n-2} = -e_{n-3} + A_2e_1, & e_n \cdot e_n = A_6e_1,\\ e_k \cdot e_{n-1} = e_{k-1}, & e_{n-1} \cdot e_k = -e_{k-1}, & 4 \le k \le n-3. \end{array}$$

It is not difficult to obtain the matrix form of the group of automorphisms of the algebra $\mathbf{n}_{n,3}$ is

$$A=\begin{pmatrix}
a^2b^{2n-8} & a_{2,n-1}ab^{n-5}-a_{2,n}d-a_{3,n}b^2 & a_{n-3,n-2}b^{2n-10} & a_{n-4,n-2}b^{2n-12} & \cdots & a_{3,n-2}b^2 & a_{1,n-2} & a_{1,n-1} & a_{1,n}\\
0 & ab^{n-3} & 0 & 0 & \cdots & 0 & 0 & a_{2,n-1} & a_{2,n}\\
0 & -ab^{n-5}d & a^2b^{2n-10} & a_{n-3,n-2}b^{2n-12} & \cdots & a_{4,n-2}b^2 & a_{3,n-2} & a_{3,n-1} & a_{3,n}\\
0 & 0 & 0 & a^2b^{2n-12} & \cdots & a_{5,n-2}b^2 & a_{4,n-2} & a_{4,n-1} & ab^{n-7}d\\
0 & 0 & 0 & 0 & \cdots & a_{6,n-2}b^2 & a_{5,n-2} & a_{5,n-1} & 0\\
0 & 0 & 0 & 0 & \cdots & a_{7,n-2}b^2 & a_{6,n-2} & a_{6,n-1} & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & \cdots & a^2b^2 & a_{n-3,n-2} & a_{n-3,n-1} & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & a^2 & c & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b^2 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & d & ab^{n-5}
\end{pmatrix},$$

Then, using Proposition (2.4), we obtain the following isomorphism criteria for the algebras of the previous classes.

Proposition 3.1. *Two algebras $\mathcal{M}(A_1,A_2,A_3,A_4,A_5,A_6)$ and $\mathcal{M}(A_1',A_2',A_3',A_4',A_5',A_6')$ are isomorphic, if and only if there exist a,b,c,d such that*

$$\begin{array}{lll}
A_1'=\frac{a^2A_1}{b^{2n-8}}, & A_2'=\frac{cA_1+b^2A_2+dA_3}{b^{2n-8}}, & A_3'=\frac{aA_3}{b^{n-3}},\\
A_4'=\frac{c^2A_1+2b^2cA_2+2cdA_3+b^4A_4+2b^2dA_5+d^2A_6}{a^2b^{2n-8}}, & A_5'=\frac{cA_3+b^2A_5+dA_6}{ab^{n-3}}, & A_6'=\frac{A_6}{b^2}.
\end{array} \tag{3.1}$$

In the following theorem, we give the main result of the work.

Theorem 3.2. *Let $\mathcal{L}$ be an n-dimensional complex symmetric Leibniz algebra, whose underlying Lie algebra is $\boldsymbol{n}_{n,3}$, then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{L}_1^{\alpha,\beta}=\mathcal{M}(1,0,1,\alpha,0,\beta)$ $\mathcal{L}_2^{\alpha,\beta}=\mathcal{M}(1,0,1,0,\alpha,\beta)$ $\mathcal{L}_3^{\alpha}=\mathcal{M}(1,0,0,\alpha,0,1)$ $\mathcal{L}_4=\mathcal{M}(1,0,0,0,1,0)$
$\mathcal{L}_5=\mathcal{M}(1,0,0,1,0,0)$ $\mathcal{L}_6=\mathcal{M}(1,0,0,0,0,0)$ $\mathcal{L}_7^{\alpha}=\mathcal{M}(0,0,1,\alpha,0,1)$ $\mathcal{L}_8=\mathcal{M}(0,0,1,1,0,0)$
$\mathcal{L}_9=\mathcal{M}(0,0,1,0,0,0)$ $\mathcal{L}_{10}^{\alpha}=\mathcal{M}(0,\alpha,0,0,0,1)$ $\mathcal{L}_{11}=\mathcal{M}(0,0,0,1,0,1)$ $\mathcal{L}_{12}=\mathcal{M}(0,0,0,0,0,1)$
$\mathcal{L}_{13}=\mathcal{M}(0,1,0,0,1,0)$ $\mathcal{L}_{14}=\mathcal{M}(0,1,0,0,0,0)$ $\mathcal{L}_{15}=\mathcal{M}(0,0,0,0,1,0)$ $\mathcal{L}_{16}=\mathcal{M}(0,0,0,1,0,0)$.

Proof. We make the following denotations:

$$\Delta_1=A_1,\quad \Delta_2=A_3,\quad \Delta_3=A_1A_6-A_3^2.$$

From this and Proposition 3.1, we get that

$$\Delta_1'=\frac{a^2}{b^{2n-8}}\Delta_1,\quad \Delta_2'=\frac{a}{b^{n-3}}\Delta_2,\quad \Delta_3'=\frac{a^2}{b^{2n-6}}\Delta_3.$$

Therefore, the Δ_i are relative invariants under isomorphisms in the given class of algebras. Now, we consider the following cases:

- If $\Delta_1\neq 0$, $\Delta_2\neq 0$, $\Delta_3\neq 0$, then choosing $a=\frac{\Delta_2^{n-4}}{\sqrt{\Delta_1^{n-3}}}$, $b=\frac{\Delta_2}{\sqrt{\Delta_1}}$, $c=-\frac{\Delta_2(A_2\Delta_2+d\Delta_1)}{\Delta_1^2}$, $d=-\frac{\Delta_2^2(A_5\Delta_1-A_2\Delta_2)}{\Delta_1\Delta_3}$, we obtain the family of algebras $\mathcal{L}_1^{\alpha,\beta}$.
- If $\Delta_1\neq 0$, $\Delta_2\neq 0$, $\Delta_3=0$, then defining $\Delta_4=A_1A_5-A_2A_3$, we get $\Delta_4'=\frac{a}{b^{3n-13}}\Delta_4$. Next, we consider the following subcases:
  - If $\Delta_4\neq 0$, then choosing $a=\frac{\Delta_2^{n-4}}{\sqrt{\Delta_1^{n-3}}}$, $b=\frac{\Delta_2}{\sqrt{\Delta_1}}$, $c=-\frac{\Delta_2(A_2\Delta_2+d\Delta_1)}{\Delta_1^2}$, $d=-\frac{\Delta_2^2(A_4\Delta_1-A_2^2)}{2\Delta_1\Delta_4}$, we obtain the family of algebras $\mathcal{L}_2^{\alpha,\beta}$.
- If $\Delta_1\neq 0$, $\Delta_2=0$, $\Delta_3\neq 0$, then choosing $a=\sqrt{\frac{\Delta_3^{n-4}}{\Delta_1^{n-2}}}$, $b=\sqrt{\frac{\Delta_3}{\Delta_1}}$, $c=-\frac{\Delta_3A_2}{\Delta_1^2}$, $d=-A_5$, we obtain the family of algebras $\mathcal{L}_3^{\alpha}$.
- If $\Delta_1\neq 0$, $\Delta_2=0$, $\Delta_3=0$, $A_5\neq 0$ then choosing $a=\frac{b^{n-4}}{\sqrt{\Delta_1}}$, $b=\sqrt[4n-18]{\Delta_1A_5^2}$, $d=-\frac{\sqrt[2n-9]{\Delta_1A_5^2}(A_1A_4-A_2^2)}{2\Delta_1A_5}$, we obtain the algebra $\mathcal{L}_4$.
- If $\Delta_1\neq 0$, $\Delta_2=0$, $\Delta_3=0$, $A_5=0$, then defining $\Delta_5=A_1A_4-A_2^2$, we get $\Delta_5'=\frac{1}{b^{4n-20}}\Delta_5$. Next, we consider the following subcases:
  - If $\Delta_5\neq 0$, then choosing $a=\frac{b^{n-4}}{\sqrt{\Delta_1}}$, $b=\sqrt[4n-20]{\Delta_5}$, we obtain the algebra $\mathcal{L}_5$.

- If $\Delta_5 \neq 0$, then choosing $a = \frac{b^{n-4}}{\sqrt{\Delta_1}}$, we obtain the algebra $\mathcal{L}_6$.

- If $\Delta_1 = 0$, $\Delta_2 \neq 0$, $A_6 \neq 0$, then choosing $a = \frac{\sqrt{A_6^{n-3}}}{\Delta_2}$, $b = \sqrt{A_6}$, we obtain the family of algebras $\mathcal{L}_7^{\alpha}$.
- If $\Delta_1 = 0$, $\Delta_2 \neq 0$, $A_6 = 0$, then defining $\Delta_6 = A_3A_4 - 2A_2A_5$, we get $\Delta_6' = \frac{1}{ab^{3n-15}}\Delta_6$. Next, we consider the following subcases:
  - If $\Delta_6 \neq 0$, then choosing $a = \frac{b^{n-3}}{\sqrt{\Delta_2}}$, $b = \sqrt[4n-18]{\Delta_2\Delta_6}$, we obtain the algebra $\mathcal{L}_8$.
  - If $\Delta_6 = 0$, then choosing $a = \frac{b^{n-3}}{\sqrt{\Delta_2}}$, we obtain the algebra $\mathcal{L}_9$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 \neq 0$, $A_2 \neq 0$, then choosing $b = \sqrt{A_6}$, $c = -\frac{A_4A_6 - A_5^2}{2A_2}$, $d = -A_5$, we obtain the family of algebras $\mathcal{L}_{10}^{\alpha}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 \neq 0$, $A_2 = 0$, then defining $\Delta_7 = A_4A_6 - A_5^2$, we get $\Delta_7' = \frac{1}{a^2b^{2n-10}}\Delta_7$. Next, we consider the following subcases:
  - If $\Delta_7 \neq 0$, then choosing $a = \sqrt{\frac{\Delta_7}{A_6^{n-5}}}$, $b = \sqrt{A_6}$, we obtain the algebra $\mathcal{L}_{11}$.
  - If $\Delta_7 = 0$, then choosing $b = \sqrt{A_6}$, we obtain the algebra $\mathcal{L}_{12}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 = 0$, $A_2 \neq 0$, $A_5 \neq 0$ then choosing $a = \frac{A_5}{\sqrt{A_2}}$, $b = \sqrt[2n-10]{A_2}$, $c = -\frac{A_4 \sqrt[n-5]{A_2} + 2dA_5}{2A_2}$, we obtain the algebra $\mathcal{L}_{13}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 = 0$, $A_2 \neq 0$, $A_5 = 0$ then choosing $b = \sqrt[2n-10]{A_2}$, $c = -\frac{A_4}{2\sqrt[n-5]{A_2^{n-6}}}$, we obtain the algebra $\mathcal{L}_{14}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 = 0$, $A_2 = 0$, $A_5 \neq 0$ then choosing $a = \frac{A_5}{b^{n-5}}$, we obtain the algebra $\mathcal{L}_{15}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 = 0$, $A_2 = 0$, $A_5 = 0$, then $A_4 \neq 0$ and choosing $a = \frac{A_4}{b^{n-6}}$, we obtain the algebra $\mathcal{L}_{16}$. *Box*

Now, we consider a Lie algebra of strictly upper triangular matrices, which is denoted by $\mathbf{n}(n, \mathbb{F})$. It is known that $\mathbf{n}(n, \mathbb{F})$ has a basis $e_{i,j}$ for $i < j$, which the table of multiplication is

$$[e_{i,j}, e_{j,k}] = e_{i,k}, \quad 1 \leq i, j, k \leq n.$$

Consider a symmetric Leibniz algebra $\mathcal{L}$, whose underlying Lie algebra is $\mathbf{n}(n, \mathbb{F})$. Since $Z(\mathbf{n}(n, \mathbb{F})) = \text{span}\{e_{1,n}\}$, then by straightforward computations, we get that the corresponding symmetric bilinear form ω satisfying the condition (2.1) is

$$\omega(e_{i,i+1}, e_{k,k+1}) = A_{i,k}e_{1,n}, \quad 1 \leq i \leq k \leq n-1.$$

Then we obtain the following result.

Proposition 3.3. *Any symmetric Leibniz algebra whose underlying Lie algebra is the algebra of upper triangular matrices has the following multiplications:*

$$\begin{array}{lll}
e_{i,j} \cdot e_{j,k} = -e_{j,k} \cdot e_{i,j} = e_{i,k}, & 1 \leq i, j, k \leq n, & (j-i, k-j) \neq (1,1), \\
e_{i,i+1} \cdot e_{i+1,i+2} = e_{i,i+2} + A_{i,i+1}e_{1,n}, & 1 \leq i \leq n, & \\
e_{i+1,i+2} \cdot e_{i,i+1} = -e_{i,i+2} + A_{i,i+1}e_{1,n}, & 1 \leq i \leq n, & \\
e_{i,i+1} \cdot e_{k,k+1} = e_{k,k+1} \cdot e_{i,i+1} = A_{i,k}e_{1,n}, & i \leq k, & i+1 \neq k.
\end{array}$$

Now, we consider a symmetric Leibniz algebra whose underlying Lie algebra is a Heisenberg algebra. A Heisenberg algebra $\mathfrak{h}$ is a $(2n+1)$-dimensional Lie algebra with basis $\{x_i, y_i, z\}$ and the following multiplication table:

$$[x_i, y_i] = z, \quad 1 \leq i \leq n.$$

Since $Z(\mathfrak{h}) = \text{span}\{z\}$, then by straightforward computations, we get that the corresponding symmetric bilinear form $\omega : \mathfrak{h} \times \mathfrak{h} \to Z(\mathfrak{h})$ satisfying the equation (2.1) is

$$\omega(x_i, x_j) = \alpha_{ij}z, \quad \omega(y_i, y_j) = \beta_{ij}z, \quad \omega(x_i, y_j) = \gamma_{ij}z.$$

Proposition 3.4. *Any symmetric Leibniz algebra whose underlying Lie algebra is the Heisenberg algebra has the following multiplications:*

$$\begin{array}{lll} x_i \cdot y_i = (1+\gamma_{i,i})z, & y_i \cdot x_i = (-1+\gamma_{i,i})z, & 1 \le i \le n, \\ x_i \cdot x_j = x_j \cdot x_i = \alpha_{i,j}z, & y_i \cdot y_j = y_j \cdot y_i = \beta_{i,j}z, & 1 \le i,j \le n, \\ x_i \cdot y_j = y_j \cdot x_i = \gamma_{i,j}z, & 1 \le i \ne j \le n. & \end{array}$$

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