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On e^*-Semisimple Modules and the Associated Socle

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Abstract. Let R be a ring, and let M be a right R-module. Recently, Baannon and Khalid introduced and studied the concept of e^*-essential submodules, extending the notion of essential submodules. In this context, we consider e^*-semisimple submodules, and we define the e^*-socle of M as the largest e^*-semisimple submodule of M. It turns out that several properties of the socle can be extended to the e^*-socle. It is shown that the e^*-socle of M is exactly the intersection of all its e^*-essential submodules.

Keywords: e^*-essential submodules, e^*-semisimple modules, socle, e^*-socle.
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1. Introduction

Throughout the paper, R will be an associative ring with identity and all modules are unital right R-modules.

Let M be a right R-module. A submodule N of M is said to be essential in M if, for every nonzero submodule K of M, we have $K \cap N \neq 0$. The socle of M, denoted by $\mathrm{Soc}(M)$, is the sum of all minimal (i.e., simple) submodules of M and, consequently, is the largest semisimple submodule of M. It is also equal to the intersection of all essential submodules of M [1, Proposition 9.7]. It is also worth recalling that

$$\mathrm{Soc}(M) = \mathrm{Tr_M}(\mathcal{S}) = \mathrm{Tr_M}\left(\bigoplus_{\mathrm{S}\in\mathcal{S}} \mathrm{S}\right) = \sum_{\mathrm{S}\in\mathcal{S}} \mathrm{Tr_M}(\mathrm{S}),$$

where $\mathcal{S}$ is a set of representatives of the simple right R-modules, and $\mathrm{Tr_M}(\mathcal{S})$ denotes the trace of $\mathcal{S}$ in M [1, Proposition 9.11].

A submodule N of M is called a small submodule if, whenever $N + L = M$ for some submodule L of M, it follows that $M = L$. An R-module M is said to be small if it is a small submodule of some R-module. It was shown that M is small if and only if it is small in its injective hull [4].

In 2022, Baannon and Khalid introduced and studied the notion of e^*-essential submodules; that is, submodules N of a module M such that $N \cap K \neq 0$ for every nonzero cosingular submodule K [2]. Recall that a right R-module M is called cosingular if mR is a small R-module for every element $m \in M$ [5, Definition 2.5].

In this article, we investigate, as an e^*-version of semisimple modules and the socle, the concepts of e^*-semisimple modules and e^*-socle. The motivation behind this study is that the study of the socle of a modules has led to some interesting properties and has shed more light on their structure (see, for instance, [1, Theorem 10.4]).

The paper is organized as follows: In Section 2, we provide a definition of e^*-semisimple submodules of a module M, and establish some fundamental properties. In Section 3, we define the e^*-socle of a module M, denoted by $\mathrm{e^*\text{-}Soc(M)}$, as the largest e^*-semisimple submodule within M. We show that $\mathrm{e^*\text{-}Soc(M)}$ is exactly the intersection of all e^*-essential submodules of M (see Proposition 3.2). Once again, it turns out that

$$\mathrm{e^*\text{-}Soc(M)} = \mathrm{Tr_M}(\mathcal{S}_\mathrm{M}) = \mathrm{Tr_M}(\bigoplus_{\mathrm{S}\in\mathcal{S}_\mathrm{M}} \mathrm{S}) = \sum_{\mathrm{S}\in\mathcal{S}_\mathrm{M}} \mathrm{Tr_M}(\mathrm{S}),$$

where $\mathcal{S}_M$ is a set of representatives of the e^*-simple submodule of M, and $\mathrm{Tr_M}(\mathcal{S}_\mathrm{M})$ denote the trace of $\mathcal{S}_M$ in M.

From now on, for a module M, $\mathrm{Soc(M)}$ will denote the socle of M. We use $N \leqslant M$, $N \leqslant_e M$ and $N \leqslant_{e^*} M$ to mean that N is a nonzero submodule, an essential submodule and an e^*-essential submodule of M, respectively. General background material can be found in [1, 3].

2. e^*-Semisimple modules

In this section we define e^*-semisimple (sub)modules and we provide some useful results concerning this new class of modules. The main result of this section is Theorem 2.10, which provides a characterization of e^*-semisimple submodules in a module M. Recall from [6] that a module M is said to be simple if $M \neq 0$ and has no proper nonzero submodules. Moreover, M is said to be semisimple if it is a direct sum of (possibly infinitely many) simple modules.

Definition 2.1. Let N be a submodule of a module M.

(1) We say that N is e^*-simple in M if N is simple and contained in every e^*-essential submodule of M.

(2) We say that N is e^*-semisimple in M if it is a direct sum of e^*-simple submodules. In particular, if $M = N$, we say that M is e^*-semisimple.

Let $(S_i)_{i\in I}$ be an indexed set of e^*-simple submodules of M. If $N = \bigoplus_{i\in I} S_i$, then $\bigoplus_{i\in I} S_i$ is called an e^*-semisimple decomposition of N. Clearly every e^*-simple submodule in M (resp., e^*-semisimple submodule in M) is simple (resp., semisimple). However, the converse is not necessarily the case, as shown in Example 2.2 (1).

Example 2.2. (1) Let R be a ring and M be an artinian R-module such that M has a non-essential e^*-essential submodule N [2, Examples 1]. Then M has a simple submodule which is not e^*-simple in M. Indeed, by [1, Corollary 10.11], Soc(M), the intersection of all essential submodules of M, is an essential submodule of M. If Soc(M) $\subseteq$ N, then N must be essential. Hence, Soc(M) $\nsubseteq$ N. Since, by [1, Proposition 9.7], Soc(M) is the sum of all simple submodules of M, M has a simple submodule S such that $S \nsubseteq N$. Hence, S is not e^*-simple in M and Soc(M) is a semisimple module which is not e^*-semisimple in M.

(2) Since every essential submodule is e^*-essential,

$$H := \bigcap_{L \leq_{e^*} M} L \subseteq \mathrm{Soc(M)} = \bigcap_{\mathrm{L} \leq_e \mathrm{M}} \mathrm{L}.$$

Thus, if $H \neq 0$, every nonzero submodule of H is an e^*-semisimple in M.

Remark 2.3. If M is a simple R-module, then M is e^*-simple in itself in the sense of Definition 2.1. Indeed, this follows from the fact that M has no proper e^*-essential submodule. This is why we do not define the notion of an e^*-simple module. Moreover, one should be careful to avoid confusion between the notions of "e^*-semisimple" and "e^*-semisimple in".

Lemma 2.4. *Let $N \leq M$ be a submodule. It is easy to verify the following assertions.*

(1) *N is e^*-simple in M if and and only if N is a simple submodule of* $\bigcap_{L \leqslant_{e^*} M} L$.

(2) *N is e^*-semisimple in M if and and only if N a semisimple submodule of* $\bigcap_{L \leq_{e^*} M} L$.

We will use the following lemma to support the conclusions which we will reach in the rest of this paper

Lemma 2.5. (1) *Let K and L be nonzero submodules of a module M. Assume that K is isomorphic to L and denote this isomorphism by $f : K \to L$. Then, K is e^*-simple in M if and only if $f(K) = L$ is e^*-simple in M.*

(2) *Let $g : M \to N$ be an isomorphism of modules and let K be a submodule of M. Then, K is e^*-simple in M if and only if $g(K)$ is e^*-simple in N.*

Consequently, the class of e^*-semisimple modules is closed under isomorphisms.

Proof. 1. Let K and L be submodules of a module M as in (1). It suffices to show that if K is e^*-simple in M, then so is L. Two cases are then possible.

Case 1. M has no nonzero cosingular submodule (see for instance [2, Example 1]). Then, every submodule of M is e^*-essential. Let P be a nonzero submodule of M. Since K is e^*-simple, $K \subseteq P$ and $K \subseteq L$. Hence, $P \cap L \neq 0$; so $P \cap L = L$ because L is simple. Then $L \subset P$. Hence, L is e^*-simple.

Case 2. M has a nonzero cosingular submodule C. Assume that K is e^*-simple. If P be an e^*-essential submodule of M, then $K \subseteq P$. We need to show that $L \subseteq P$, but, since L is simple, it suffice to prove that $L \cap P \neq 0$. Consider the following diagram:

$$\begin{array}{ccccc} K & \xrightarrow{\ i\ } & M & \overset{h}{\dashrightarrow} & \\ \downarrow f & & \| & & \searrow \\ L & \xrightarrow{\ j\ } & M & \xrightarrow{\ \sigma\ } & \mathrm{E}(M), \end{array}$$

where i and j are the inclusions, σ is the injective envelope of M and h is an induced map such that $hi = \sigma jf$. Since, by [3, Proposition 1.11], $\sigma(M) \leq_e \mathrm{E}(M)$, $\sigma(P) \leq_{e^*} \mathrm{E}(M)$ by [2, Proposition 6]. If $h^{-1}(\sigma(P)) = 0$, h must be injective; so, by [5, Lemma 2.6], $h(C)$ is a nonzero cosingular submodule of $\mathrm{E}(M)$, so $h(C) \cap \sigma(P) \neq 0$, a contradiction. Thus, $h^{-1}(\sigma(P)) \neq 0$ and $h^{-1}(\sigma(P)) \leq_{e^*} M$ by [2, Proposition 2]. Then $K \subseteq h^{-1}(\sigma(P))$; so, $h(K) \subseteq \sigma(P)$.

Now let x be a nonzero element in K. Then, $h(x) = \sigma jf(x) = \sigma f(x) \neq 0$ because f and σ are injective. On the other hand, $h(x) \in \sigma(P)$, so, $h(x) = \sigma f(x) = \sigma(p)$ for some nonzero $p \in P$. But σ is injective; this implies that $f(x) = p$. Finally, $L \cap P \neq 0$ as desired.

2. It suffices to prove that if $K \leqslant \bigcap\limits_{L \leq_{e^*} M} L$, then $g(K) \leqslant \bigcap\limits_{L \leq_{e^*} N} L$. Indeed, suppose that $K \leqslant \bigcap\limits_{L \leq_{e^*} M} L$. Let $x \in K$ and L be an e^*-essential submodule of N; we must show that $g(x) \in L$. By [2, Proposition 2], $g^{-1}(L) \leqslant_{e^*} M$ ($g^{-1}(L) \neq 0$ because g is an isomorphism). Then, $x \in g^{-1}(L)$. Hence, $g(x) \in L$. □

Proposition 2.6. *Let M be a module, and let N and K be nonzero submodules of M with $K \subseteq N$. Then the following assertions hold true:*

(1) *If N is e^*-semisimple in M, then so is K.*

(2) *If $N \leq_{e^*} M$ and K is e^*-semismple in M, then K is e^*-semisimple in N.*

Proof. By [1, Proposition 9.4], K is semisimple. Hence, K is e^*-semisimple by Lemma 2.4.

2. By Lemma 2.4, it suffices to show that $K \subseteq \bigcap\limits_{L \leqslant_{e^*} N} L$. If L is an e^*-essential submodule of N, then, by [2, Proposition 1], L is also an e^*-essential submodule of M. Since K is e^*-semisimple in M, $K \subseteq L$ by Lemma 2.4. □

Proposition 2.7. *Let $(S_i)_{i \in I}$ be an indexed set of e^*-simple submodule of M. If S is a simple submodule of M such that*

$$S \cap \sum_{i \in I} S_i \neq 0$$

then there is an $i \in I$ such that $S \cong S_i$.

Proof. This follows from [1, Corollary 9.5] □

Before giving the main result of this section which characterizes e^*-semisimple submodules in a given module M, we need to recall some notions. Let $(M_i)_{i \in I}$ be an indexed set of submodules of a module M. Recall from [1, page 93] that $(M_i)_{i \in I}$ is said to be independent in case for each $j \in I$

$$M_j \cap \left(\sum_{i \neq j} M_i\right) = 0.$$

One of the most important results in the study of semisimple structures is that: if $(S_i)_{i\in I}$ is an indexed set of simple submodules of a module M such that $M = \sum_{i\in I} S_i$, then for each submodule N of M there is a subset $J \subseteq I$ such that $(S_j)_{j\in J}$ is independent and

$$M = N \bigoplus (\bigoplus_{j\in J} S_j)$$

[1, Lemma 9.2]. Since every e^*-simple submodule is simple, the next result follows immediately.

Proposition 2.8. *Let $(S_i)_{i\in I}$ be an indexed set of e^*-simple submodules of a module M. If $N = \sum_{i\in I} S_i$, then for each submodule K of N there is a subset $J \subseteq I$ such that $(S_j)_{j\in J}$ is independent and*

$$N = K \bigoplus (\bigoplus_{j\in J} S_j).$$

Proof. This is a particular case of [1, Lemma 9.2]. □

An interesting consequence of the Proposition 2.8 arises if the submodule $K = 0$.

Corollary 2.9. *Let M a module. Let $N = \sum_{i\in I} S_i$ for an indexed set $(S_i)_{i\in I}$ of e^*-simple submodules of M, then for some $J \subset I$*

$$N = \bigoplus_{j\in J} S_j;$$

that is, N is e^*-semisimple in M.

Let $\mathcal{X}$ be a class of modules. A module M is generated by $\mathcal{X}$ in case there is an indexed set $(X_i)_{i\in I}$ in $\mathcal{X}$ and an epimorphism $\oplus_{i\in I} X_i \to M$ [1, page 105].

Let $\mathcal{S}$ be a set of representatives of the simple modules. For a given module M, we define the subset $\mathcal{S}_M$ of S as follows:

$$\mathcal{S}_M := \{S \in \mathcal{S} \mid \ S \text{ is isomorphic to an } e^*\text{-simple submodule of M}\}.$$

Having finished all the preparatory work, we can now deduce the main result of this section.

Theorem 2.10. *Let M be a module, and let N be a submodule of M. The following statements are equivalent:*

(1) *N is e^*-semisimple in M.*

(2) *N is generated by $\mathcal{S}_M$.*

(3) *N is the sum of some set of e^*-simple submodules of M (e^*-simple means e^*-simple in M).*

(4) *N is the sum of all e^*-simple submodules included in N (again, e^*-simple means e^*-simple in M).*

Proof. (1) $\Rightarrow$ (2) Follows immediately by the definitions.

(2) $\Rightarrow$ (3) There exists an epimorphism $h : \bigoplus_{i\in I} S_i \to N$ where, for each $i \in I$, $S_i \in \mathcal{S}_M$. Then, we have $N = \sum_{i\in I} h(S_i)$. Then, for each $i \in I$, $h(S_i)$ is either 0 or e^*-simple submodule of M by Lemma 2.5 (1).

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) Follow by Corollary 2.9. □

Corollary 2.11. *The following statements are equivalent for a module M:*

(1) *M is e^*-semisimple;*

(2) *M is generated by $\mathcal{S}_M$;*

(3) *M is the sum of some set of e^*-simple submodules;*

(4) *M is the sum of its e^*-simple submodules;*

Proof. In Theorem 2.10, let $N = M$. □

3. The e^*-Socle

Let R be a ring, and let M be a right R-mdoule. The socle of M, denoted by Soc(M), is the sum of all minimal (i.e., simple) submodules of M and, therefore, is the largest semisimple submodule of M. It is equal to the intersection of all essential submodules of M [1, Proposition 9.7]. In this section, we define the e^*-Socle of a module M, denoted e*-Soc(M), as the largest e^*-semisimple submodule of M and demonstrate that it shares many properties as the socle .

Definition 3.1. We define the e^*-socle of a module M as the largest e^*-semisimple submodule of M, and we denote it by e*-Soc(M).

Proposition 3.2. *Let M be an R-module. Then,*

$$\mathrm{e}^*\text{-Soc(M)} = \sum_{\substack{S \leq M \\ S\ e^*\text{-simple}}} \mathrm{S} = \bigcap_{\mathrm{L} \leq_{\mathrm{e}^*} \mathrm{M}} \mathrm{L}.$$

Proof. The first equality is clear from Theorem 2.10. Then, it suffices to prove the second equality. The inclusion

$$\mathrm{e}^*\text{-Soc(M)} \subseteq \bigcap_{\mathrm{L} \leq_{\mathrm{e}^*} \mathrm{M}} \mathrm{L},$$

follows by definitions. For the reverse inclusion, by Lemma 2.4, we only need to prove that $\bigcap_{L \leq_{e^*} M} L$ is semisimple. But, Soc(M) is semisimple; then $\bigcap_{L \leq_{e^*} M} L$, being a submodule of Soc(M), is also semisimple. □

Unlike the socle [1, Proposition 9.8], the e^*-socle doesn't behave well under homomorphisms (see Example 3.7). But we have,

Proposition 3.3. *Let $f : M \to N$ be a homomorphism of R-modules. Assume that $f^{-1}(L) \neq 0$ for every* e*-*essential submodule L of N. Then*

$$f(\mathrm{e}^*\text{-Soc(M)}) \leq \mathrm{e}^*\text{-Soc(N)}.$$

Proof. Let $x \in$ e*-Soc(M). If L is an e^*-essential submodule of N, then, by hypothesis, $f^{-1}(L) \neq 0$. Hence, by [2, Proposition 2], $f^{-1}(L)$ is an e^*-essential submodule of M. Then, by Proposition 3.2, $x \in f^{-1}(L)$; so $f(x) \in L$. Hence, again by Proposition 3.2, $f(x) \in$ e*-Soc(N). □

Corollary 3.4. *Let $f : M \to N$ be a homomorphism of R-modules. Assume that M has a nonzero cosingualr submodule C. Then*

$$f(\mathrm{e}^*\text{-Soc(M)}) \leq \mathrm{e}^*\text{-Soc(N)}.$$

Proof. Let L be an e^*-essential submodule of N. If $f^{-1}(L) = 0$, then f is injective since $\mathrm{ker(f)} \subset \mathrm{f^{-1}(L)} = 0$. Hence, by [5, Lemma 2.6], $f(C)$ is a nonzero cosingular submodule of N. Since $L \leq_{e^*} N$, $L \cap f(C) \neq 0$, which is clearly impossible. Thus, $f^{-1}(L) \neq 0$ for every e^*-essential submodule L of N. Now, the rsult follows from Proposition 3.3. □

Corollary 3.5. *Let M be a module. Suppose that $K \cap L \neq 0$ for every* e*-*essential submodule L of M. If $K \leq_{e^*} M$, then*

$$\mathrm{e}^*\text{-Soc(K)} = \mathrm{K} \cap \mathrm{e}^*\text{-Soc(M)}.$$

In particular, if e*-Soc(M) $\leq_{\mathrm{e}^*}$ M, *then* e*-Soc(e*-Soc(M)) = e*-Soc(M).

Proof. The inclusion e*-Soc(K) $\subseteq$ K $\cap$ e*-Soc(M) follows from Proposition 3.3. By Proposition 2.6 (1), $K \cap$ e*-Soc(M) is e^*-semisimple in M because e*-Soc(M) is e^*-semisimple in M. Then, by Proposition 2.6 (2), $K \cap$ e*-Soc(M) is e^*-semisimple in K . Hence, by Proposition 3.2, $K \cap$ e*-Soc(M) $\subseteq$ e*-Soc(K). □

Remark 3.6. Let $K \leq M$ be a submodule. Suppose that K contains a nonzero cosingular submodule, and $K \leq_{e^*} M$. Then, $\mathrm{e^*\text{-}Soc(K) = K \cap e^*\text{-}Soc(M)}$. Indeed, since K contains a nonzero consingular submodule, the zero submodule can't be e^*-essential. Therefore, for every e^*-essential submodule L, we have $K \cap L \neq 0$. Then Corollary 3.5 applies.

The promised example is as follows:

Example 3.7. Let $M = M_1 \oplus M_2$, where, for each i, M_i is an artinian modules wich has no nonzero cosingular submodule (see for instance [2, Example 1]). By [1, Corollary 10.11] M_1 has a simple module S_1. Notice that $\mathrm{e^*\text{-}Soc(S_1) = S_1}$ and $\mathrm{e^*\text{-}Soc(M)} = \bigcap_{\mathrm{K \leq M}} \mathrm{K}$ because every nonzero submodule of M is e^*-essential (M has no nonzero cosingular submodule because, for each i, M_i has no nonzero cosingular submodule, see [5, Lemma 2.6]). Now, suppose that $\mathrm{e^*\text{-}Soc(S_1) = S_1} \subseteq \bigcap_{\mathrm{K \leq M}} \mathrm{K}$; that is, $S \subseteq K$ for every submodule of M, which says that, $S_1 \subseteq M_2$, a contradiction. Therefore, $\mathrm{e^*\text{-}Soc(S_1)} \nsubseteq \mathrm{Soc(M)}$.

Now, the e^*-socle of a module M is the largest submodule of M that is contained in every e^*-essential submodule of M. In general, though, $\mathrm{e^*\text{-}Soc(M)}$ need not be e^*-essential in M; in fact, nonzero modules can have zero e^*-socle (because $\mathrm{e^*\text{-}Soc(M)}$ is submodule of $\mathrm{Soc(M)}$ which can be zero [1, Exercise 9.2]). However, we do have:

Proposition 3.8. *The following assertions are equivalent for a module M.*

(1) $\mathrm{e^*\text{-}Soc(M) \leq_{e^*} M}$

(2) *Every nonzero cosingular submodule of M contains an e^*-simple submodule.*

Proof. (1) ⇒ (2) If C is a cosingular submodule of M, then $C \cap \mathrm{e^*\text{-}Soc(M)} \neq 0$. Hence, by Proposition 2.6, $C \cap \mathrm{e^*\text{-}Soc(M)} \neq 0$ is a nonzero e^*-semisimple submodule of M, which include in C.
(2) ⇒ (1) Follows by Proposition 3.2. □

Let $\mathcal{X}$ be a class of R-modules. The trace of $\mathcal{X}$ in M is defined by

$$\mathrm{Tr_M}(\mathcal{X}) = \sum_{\substack{h \in \mathrm{Hom(X,M)} \\ X \in \mathcal{X}}} \mathrm{h(X)}.$$

It was shown that $\mathrm{Tr_M}(\mathcal{X})$ is the unique largest submodule of M generated by $\mathcal{X}$ [1, Proposition 8.12].
For a given module M, let $\mathcal{S}_M$ be a representatives of e^*-simple submodules of M as it is defined in Section 2. So, as an e^*-version of [1, Proposition 9.11], we have

Proposition 3.9. *Let M be a right R-modules. Then:*

$$\mathrm{e^*\text{-}Soc(M) = Tr_M(\mathcal{S}_M) = Tr_M(\bigoplus_{S \in \mathcal{S}_M} S) = \sum_{S \in \mathcal{S}_M} Tr_M(S)}$$

Proof. The first equality follows from definitions and Theorem 2.10. The rest follows form well-know properties of the trace [1, Proposition 8.20]. □

Corollary 3.10. *For any ring R, the* $\mathrm{e^*\text{-}Soc(R_R)}$*, the e^*-socle of R as right R-module, is a two sided ideal of R.*

Proof. Follows by Proposition 3.9 and [1, Proposition 8.21]. □

Recall that a ring R is said to be *right cosingular* if it is cosingular as a (right) R-module [5, Definition 2.5]. It follows from [5, Corollary 2.7] that if R is right cosingular, then every right R-module is cosingular. Consequently, over a right cosingular ring, the notion of "e^*-semisimple submodules/modules" coincides with the classical notion of semisimple modules.
In the rest of this section, we provide some results concerning this situation, i.e., when these new relative concepts coincide with their classical counterparts.

Proposition 3.11. *Let M be an R-module. The following assertions are equivalent:*

(1) *Every simple submodule N of M is* e**-simple in M.*

(2) *Every semisimple submodule N of M is* e**-semisimple in M.*

(3) e*-Soc(M) = Soc(M).

Proof. (1) $\Leftrightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) This follows from Proposition 3.2 and [1, Proposition 9.7].

(3) $\Rightarrow$ (1) If N is a simple submodule of M, then $N \subseteq$ Soc(M). By (3), we have $N \subseteq$ e*-Soc(M). Then, by Proposition 3.2, N is contained in every e^*-essential submodule of M. Thus, N is e^*-simple in M. □

Using a similar argument as in the proof of Proposition 3.11, one can prove the following:

Proposition 3.12. *The following assertions are equivalent:*

(1) *Every semisimple R-module is* e**-semisimple.*

(2) *For every R-module M, every semisimple submodule of M is* e**-semisimple in M.*

(3) *For every R-module M, every simple submodule of M is* e**-simple in M.*

(4) *For any R-module M,* e*-Soc(M) = Soc(M).

(5) *Every semisimple right R-module has no proper* e**-essential submodule.*

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