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Complete systems of invariants of m-tuples for fundamental groups of a two-dimensional bilinear-metric space over the field of rational numbers

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Abstract. Let Q be the field of rational numbers and Q^2 be the 2-dimensional linear space over Q. A classification of all non-degenerate symmetric bilinear-metric forms over Q^2 have obtained. Let φ be a non-degenerate symmetric bilinear form on Q^2. Denote by $O(2,\varphi,Q)$ the group of all φ-orthogonal (that is the form φ preserving) transformations of Q^2. Put $MO(2,\varphi,Q) = \{F : Q^2 \to Q^2 \mid Fx = gx + b, g \in O(2,\varphi,Q), b \in Q^2\}$, $SO(2,\varphi,Q) = \{g \in O(2,\varphi,Q) | detg = 1\}$ and $MSO(2,\varphi,Q) = \{F \in M(2,\varphi,Q) | detg = 1\}$.The present paper is devoted to solutions of problems of G-equivalence of m-tuples in Q^2 for groups $G = O(2,\varphi,Q), SO(2,\varphi,Q)$, $MO(2,\varphi,Q)$, $MSO(2,\varphi,Q)$. Complete systems of G-invariants of m-tuples in Q^2 for these groups are obtained.

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1. Introduction

Let N be the set of all natural numbers and $m \in N, m \geq 1$. Denote by $(Q^2)^m$ the set of all m-tuples $(u_1, u_2, \ldots, u_m)$ in Q^2, where $u_i \in Q^2, \forall i = 1, 2, \ldots, m$.

Let V be a finite dimensional vector space over a field B and ϕ be a bilinear form on V. Denote by $O(\phi, V)$ the group of all ϕ-orthogonal (that is the form ϕ preserving) transformations of V. Let $MO(\phi, V)$ be the group generated by the group $O(\phi, V)$ and all translations of V. In the paper [5], for the orthogonal group $O(\phi, V)$ in the Euclidean, spherical, hyperbolic and de-Sitter geometries, the orbit of m vectors is characterized by their Gram matrix and an additional subspace. In the book [1, Proposition 9.7.1], for the group $MO(\phi, V)$ in the Euclidean geometry, the orbit of m vectors is characterized by distances between m-vectors. A complete system of relations between elements of this complete system is also given in [1, Theorem 9.7.3.4]. In the paper [7], a complete system of invariants of m-tuples in the two-dimensional pseudo-Euclidean geometry of index 1 and a complete system relations between the obtained complete system of invariants are given. In the paper [8], a complete system of invariants of m-tuples in the one-dimensional projective space and a complete system relations between the obtained complete system of invariants are given. Invariants of m-points appear also in the theory of invariants of Bezier curves ([3], [19]. Complete systems of invariants for various geometric and topological settings have been developed in a series of works. In [9], the authors construct complete systems of invariants for m-tuples associated with the fundamental groups of the two-dimensional Euclidean space. The study in [10] presents complete systems of Galilean invariants describing the motion of parametric figures in three-dimensional Euclidean space. In [11], the authors investigate global invariants of topological figures in the two-dimensional Euclidean space, focusing on properties preserved under continuous deformations. Similarly, in [12], global invariants of objects are analyzed in the context of the two-dimensional Minkowski space, taking into account the Lorentzian structure. The papers [13] and [14] extend the study of invariants to immersions into n-dimensional affine manifolds and to mappings from arbitrary sets into the two-dimensional Euclidean space, respectively. Invariants of m-vectors in Lorentzian geometry are considered in [20], where algebraic invariants under Lorentz transformations are analyzed. Moreover, the concept of m-vector invariants appears prominently in applied disciplines such as computer vision ([16], [21]), where they are used for recognizing and comparing geometric configurations under affine or projective transformations, and in computational geometry ([18]), where such invariants aid in the analysis of shape and spatial relationships. General theory of m-point invariants considered in the invariant theory (see [2], [5], [6], [17], [23], [24]).This paper is a continuation of the paper [15]. The present paper is devoted to solutions of problems of G-equivalence of m-tuples in Q^2 for groups $G = O(2,\varphi,Q), SO(2,\varphi,Q)$, $MO(2,\varphi,Q)$, $MSO(2,\varphi,Q)$. Complete systems of G-invariants of m-tuples in Q^2 for these groups are obtained.

1.1. **A classification bilinear-metric spaces over the field of rational numbers.**

Let Q be the field of rational numbers, Q^2 be the 2-dimensional linear space over Q and $\varphi(x,y)$ be a symmetric bilinear form on Q^2.

If we replace the argument $y \in Q^2$ in the symmetric bilinear form $\varphi(x,y)$ by x, where $x=(x_1,x_2)\in Q^2$, we obtain the quadratic form $\varphi(x,x)$.

Theorem 1.1. *(see [4], p.196) For every quadratic form $\varphi(x,x)$ on Q^2, there exists a basis in Q^2 such that it has following form*

$$\varphi(x,x)=\lambda_1 x_1^2+\lambda_2 x_2^2$$

for some $\lambda_1,\lambda_2\in Q$, where x_1,x_2 are the coordinates of the vector x in this basis.

In this case, there exist only following two cases: 1) $rank(\varphi(x,x))=1$ and 2) $rank(\varphi(x,x))=2$. In the case 1) $rank(\varphi(x,x))=1$, there exists a basis in Q^2 such that $\varphi(x,x)$ has following form: $\varphi(x,x)=\lambda_1 x_1$, where $\lambda_1\in Q$ and $\lambda_1\neq 0$.

Consider the case $rank(\varphi(x,x))=2$. In this case, there exists a basis e_1,e_2 in Q^2 such that $\varphi(x,x)$ has following form $\varphi(x,x)=\lambda_1x_1^2+\lambda_2x_2^2$, where $\lambda_1\in Q$, $\lambda_1\neq 0$ and $\lambda_2\in Q$, $\lambda_2\neq 0$. The equality $\varphi(x,x)=\lambda_1x_1^2+\lambda_2x_2^2$ implies following equality: $\varphi(x,x)=\lambda_1(x_1^2+\frac{\lambda_2}{\lambda_1}x_2^2)$. Since $\frac{\lambda_2}{\lambda_1}$ is a rational number, there are a,b integer numbers such that $\frac{\lambda_2}{\lambda_1}=\frac{a}{b}$. Then we have: $\varphi(x,x)=\lambda_1(x_1^2+\frac{a}{b}x_2^2)$.

We may then introduce a new basis e_1',e_2' by setting $e_1'=e_1, e_2'=be_2$, where b is the above integer number. This implies that the quadratic form $\varphi(x,x)$ can be written in this basis in the form $\varphi(x,x)=\lambda_1(x_1^2+abx_2^2)$. We now consider the case of a positive rational number $a\cdot b$. If the prime factors of the product ab have a square of an integer, then we create $\varphi(x,x)=\lambda_1(x_1^2+px_2^2)$ by introducing a new basis, where $p=1$ or $p=p_1\cdot p_2\cdot\ldots\cdot p_n$ such that $p_j, j=1,\ldots,n,-$ prime numbers and $p_k\neq p_l$ for all $l\neq k, k=1,\ldots,n, l=1,\ldots,n$. As a result, there are infinitely non-congruent symmetric bilinear forms over the field of rational numbers and bilinear-metric spaces relatively.

1.2. **A linear representation of the field $Q(\sqrt{-p})$ in two-dimensional linear space Q^2.**

Let Q be the field of rational numbers and $p=1$ or $p=p_1\cdot p_2\cdot\ldots\cdot p_n$, where p_j- prime numbers and $p_k\neq p_l$ for all $k\neq l$. Denote by $Q(\sqrt{-p})$ the set $\{a+b\sqrt{-p}\mid a,b\in Q\}$. Let $a=a_1+\sqrt{-p}a_2\in Q(\sqrt{-p})$ and $b=b_1+\sqrt{-p}b_2\in Q(\sqrt{-p})$. We define addition and multiplication operations on $Q(\sqrt{-p})$ as follows: put $a+b=(a_1+\sqrt{-p}a_2)+(b_1+\sqrt{-p}b_2)=(a_1+b_1)+\sqrt{-p}(a_2+b_2)$. A multiplication in $Q(\sqrt{-p})$ define as follows: $a\circ b=(a_1+\sqrt{-p}a_2)\circ(b_1+\sqrt{-p}b_2)=(a_1b_1-pa_2b_2)+\sqrt{-p}(a_1b_2+a_2b_1)$.

We will present the Propositions 1.2 - 1.9 mentioned in paper [15], as these propositions will be necessary for us

Proposition 1.2. *The set $Q(\sqrt{-p})$ is a field with respect to the defined above addition $a+b$ and multiplication $a\circ b$ operations.*

Let $a=a_1+\sqrt{-p}a_2$. We denote by M_a the matrix of the form $\begin{pmatrix} a_1 & -pa_2 \\ a_2 & a_1\end{pmatrix}$. Let $M(Q,p)$ denote the set of all matrices M_a, where $a\in Q(\sqrt{-p})$. We consider on the set $M(Q,p)$ standard matrix operations: the component-wise addition and the multiplication operations of matrices. Then $M(Q,p)$ is a field with the unit element, where the unit element is the unit matrix. The following proposition is obvious.

Proposition 1.3. *The mapping $M: Q(\sqrt{-p})\to M(Q,p)$, where $M: a\to M_a, \forall a\in Q(\sqrt{-p})$, is an isomorphism of fields $Q(\sqrt{-p})$ and $M(Q,p)$.*

For $a=a_1+\sqrt{-p}a_2, b=b_1+\sqrt{-p}b_2\in Q(\sqrt{-p})$, we put $\langle a,b\rangle_p=a_1b_1+pa_2b_2$. Then $\langle a,b\rangle_p$ is a bilinear form on $Q(\sqrt{-p})$ and $\langle a,a\rangle_p=a_1^2+pa_2^2$ is a quadratic form on $Q(\sqrt{-p})$. For convenience, we denote by $\Psi(a)$ the quadratic form $\langle a,a\rangle_p$.

Proposition 1.4. *Let $M: Q(\sqrt{-p})\to M(Q,p)$ be the isomorphism $M: x\to M_x$ of fields $Q(\sqrt{-p})$ and $M(Q,p)$. Then $\Psi(x)=det(M_x)$ and $\Psi(x\circ y)=\Psi(x)\Psi(y)$for all $x,y\in Q(\sqrt{-p})$.*

For an arbitrary element $a=a_1+\sqrt{-p}a_2\in Q(\sqrt{-p})$, we set $W(a)=\overline{a}=a_1-\sqrt{-p}a_2$.

Proposition 1.5. *For an arbitrary element $a=a_1+\sqrt{-p}a_2\in Q(\sqrt{-p})$ following equalities hold: $a+\overline{a}=2a_1, \langle a,a\rangle_p=a\circ\overline{a}=a_1^2+pa_2^2\in Q$.*

Proposition 1.6. *The function $\Psi(x)$ has the following properties:*

(1) $\Psi(\lambda x) = \lambda^2 \Psi(x)$, $\forall \lambda \in Q, \forall x \in Q(\sqrt{-p})$;

(2) $\Psi(e) = 1$ *for the unit element* $e \in Q(\sqrt{-p})$;

(3) $\Psi(x) = x \circ \overline{x} = \overline{x} \circ x$ *hold for all* $x \in Q(\sqrt{-p})$;

(4) $\Psi(x) = \Psi(Wx) = \Psi(\overline{x})$ *hold for all* $x \in Q(\sqrt{-p})$.

Proposition 1.7. *Let $x \in Q(\sqrt{-p})$. Then the element x^{-1} exists if and only if $\Psi(x) \neq 0$. In the case $\Psi(x) \neq 0$, the equalities $x^{-1} = \frac{\overline{x}}{\Psi(x)}$ and $\Psi(x^{-1}) = \frac{1}{\Psi(x)}$ hold.*

Put $Q^*(\sqrt{-p}) = \{x \in Q(\sqrt{-p}) \mid \Psi(x) \neq 0\}$. $Q^*(\sqrt{-p})$ is a group with respect to the multiplication operation $\circ$ in the field $Q(\sqrt{-p})$. Denote by $M(Q^*, p)$ the set of all matrices M_a, where $a \in Q^*(\sqrt{-p})$. Consider elements $a = a_1 + \sqrt{-p}a_2 \in Q^*(\sqrt{-p})$ and $x = x_1 + \sqrt{-p}x_2 \in Q(\sqrt{-p})$ as column vectors $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Let M_a be the matrix $\begin{pmatrix} a_1 & -pa_2 \\ a_2 & a_1 \end{pmatrix}$. Since $a \in Q^*\sqrt{-p}$, we have $\Psi(a) = a_1^2 + pa_2^2 \neq 0$ and $\Psi(a) = det(M_a) \neq 0$.

Then the equality $a \circ x = (a_1 + \sqrt{-p}a_2) \circ (x_1 + \sqrt{-p}x_2) = (a_1x_1 - pa_2x_2) + \sqrt{-p}(a_1x_2 + a_2x_1)$ has the following form

$$a \circ x = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1x_1 - pa_2x_2 \\ a_1x_2 + a_2x_1 \end{pmatrix} = \begin{pmatrix} a_1 & -pa_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M_a x, \tag{1.1}$$

where $M_a x$ is the multiplication of matrices M_a and x. Hence $M_a \in M(Q^*, p)$ and the mapping $M : Q^*(\sqrt{-p}) \to M(Q^*, p)$, where $M(a) = M_a$, is a linear representation of the group $Q^*(\sqrt{-p})$ in Q^2.

Proposition 1.8. *$M(Q^*, p)$ is a group with respect to the multiplication operation in the field $M(Q)$.*

Put $S(Q^*, \sqrt{-p}) = \{x \in Q(\sqrt{-p}) \mid \Psi(x) = 1\}$. It is a subgroup of the group $Q^*(\sqrt{-p})$.

Proposition 1.9. *Let $M : Q(\sqrt{-p}) \to M(Q, p)$ be the isomorphism $M : x \to M_x$ of fields $Q(\sqrt{-p})$ and $M(Q, p)$. Then $M(S(Q^*, \sqrt{-p}))$ is a subgroup of the group $M(Q^*, p)$ and the mapping M: $S(Q^*, \sqrt{-p}) \to M(Q^*, p)$, where $M(a) = M_a$ is a linear representation of the group $S(Q^*, \sqrt{-p})$ in Q^2.*

Let $p = 1$ or $p = p_1 \cdot p_2 \cdot \ldots \cdot p_n$, where $p_j -$ prime numbers and $p_k \neq p_l$ for all $k \neq l$. The symmetric bilinear form $x_1y_1 + px_2y_2$ denote by $\langle x, y \rangle_p$. Denote by Q_p^2 the 2-dimensional linear space Q^2 over Q with the bilinear form $\langle x, y \rangle_p = x_1y_1 + px_2y_2$, where $x = (x_1, x_2), y = (y_1, y_2) \in Q^2$.

2. Fundamental groups of transformations of the 2-dimensional bilinear-metric space Q^2

Definition 2.1. A mapping $F : Q(\sqrt{-p}) \to Q(\sqrt{-p})$ is called p-orthogonal if $\langle Fx, Fy \rangle_p = \langle x, y \rangle_p$ for all $x, y \in Q(\sqrt{-p})$.

We denote the set of all p-orthogonal transformations of Q^2 by $O(2, p, Q)$. Let $I : Q^2 \to Q^2$ be the unit transformation $I(x) = x, \forall x \in Q^2$. Then $I \in O(2, p, Q)$. Let $T_1, T_2 \in O(2, p, Q)$ and $T_1 \cdot T_2 : Q^2 \to Q^2$ be such that $(T_1 \cdot T_2)(x) = T_1(T_2(x)), \forall x \in Q^2$. Then it is easy to see that $T_1 \cdot T_2 \in O(2, p, Q)$.

The following propositions are well known.

Proposition 2.2. *$O(2, p, Q)$ is a group with respect to the composition operation $T_1 \cdot T_2$, where $T_1, T_2 \in O(2, p, Q)$.*

Proposition 2.3. *([25], p.221) Every p-orthogonal transformation of Q_p^2 is linear.*

Let $x=(x_1,x_2)\in Q^2, y=(y_1,y_2)\in Q^2$. Denote the matrix of the bilinear form $\langle x,y\rangle_p = x_1y_1+px_2y_2$ by $\Delta_p=\|\delta_{ij}\|_{i,j=1,2}$, where $\delta_{11}=1, \delta_{12}=\delta_{21}=0, \delta_{22}=p$. By Proposition 2.3, we can consider an element of $O(2,p,Q)$ as a 2×2-matrix. Let $H\in O(2,p,Q)$, where $H=\|h_{ij}\|_{i,j=1,2}$. Let H^T be the transpose matrix of H. It is known that the equality $\langle Hx,Hy\rangle_p=\langle x,y\rangle_p$ for all $x,y\in Q^2$ is equivalent to the equality

$$H^T\Delta_pH=\Delta_p. \tag{2.1}$$

The following proposition follows from the equation 2.1.

Proposition 2.4. *Let $H\in O(2,p,Q)$. Then $det(H)=1$ or $det(H)=-1$.*

We denote by $SO(2,p,Q)$ the set $\{H\in O(2,p,Q): det(H)=1\}$. $SO(2,p,Q)$ is a subgroup of $O(2,p,Q)$. $O(2,p,Q)=SO(2,p,Q)\cup\{HW \mid H\in SO(2,p,Q)\}$, where HW is the multiplication of matrices H and W, where $W=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$.

Theorem 2.5. *(see [15]). The equality $SO(2,p,Q)=M(S(Q^*,\sqrt{-p}))$ holds.*

Hence, we conclude from the above theorem that every special orthogonal transformations will be matrices $\begin{pmatrix}a & -pb\\ b & a\end{pmatrix}$ such that $a^2+pb^2=1, a,b\in Q$. In that case, is the solution of the equation $a^2+pb^2=1$ in the rational numbers field? We can answer this question by the following theorem.

Theorem 2.6. *The description of the elements of the group $SO(2,p,Q)$ is as follows.*
(i) There is no element $x=(x_1,x_2)\in Q^2$, such that $x_1=0$ and $M_x\in SO(2,p,Q)$, where $p\neq 1$. There are only two elements $(x_1,x_2)\in Q^2$, such that $x_2=0$ and $M_x\in SO(2,p,Q)$. These are $(1,0)$ and $(-1,0)$.
(ii) Assume that $x=(x_1,x_2)\in Q^2$ such that $x_2\neq 0$ and $M_x\in SO(2,p,Q)$. Then there is the number $r\in Q$, where $r\neq 0$, such that the equalities are satisfied:

$$x_1=\frac{p-r^2}{p+r^2},\qquad x_2=\frac{2r}{p+r^2}\quad (I).$$

(iii) Conversely, assume that r is an arbitrary nonzero element in Q and for $x=(x_1,x_2)\in Q^2$ the equalities are satisfied (I). Then $M_x\in SO(2,p,Q)$.

Proof. (i) This is obvious.

(ii) Assume that $x=(x_1,x_2)\in Q$ such that $x_2\neq 0$ and $x_1^2+px_2^2=1$.

First, we prove that in this case $x_1^2\neq 1$. Suppose $x_1^2=1$. Then from the equation $x_1^2+px_2^2=1$, we obtain that $x_2^2=0$. It follows that $x_2=0$. This contradicts to $x_2\neq 0$. So we proved $x_1^2\neq 1$, $x_1\neq 1$ and $x_1\neq -1$.

From the equation $x_1^2+px_2^2=1$ and from the inequalities $x_1\neq 1$, $x_1\neq -1$ we obtain the following equalities: $1-x_1^2=px_2^2 \Rightarrow px_2^2=(1-x_1)(1+x_1) \Rightarrow \frac{px_2}{1+x_1}=\frac{1-x_1}{x_2}$.
Put $r=\frac{px_2}{1+x_1}$. Then we have $r=\frac{1-x_1}{x_2}$. From these two equalities we obtain the following equalities $\frac{1}{x_2}+\frac{x_1}{x_2}=\frac{p}{r}$, $\frac{1}{x_2}-\frac{x_1}{x_2}=r$. From last equalities we obtain $\frac{2}{x_2}=\frac{p}{r}+r$, $\frac{2x_1}{x_2}=\frac{p}{r}-r$. We find x_1,x_2 from these two equalities and we obtain the following equalities $x_1=\frac{p-r^2}{p+r^2}$, $x_2=\frac{2r}{p+r^2}$. The (ii) is proved.

(iii) Conversely, let $r\in Q$ be an arbitrary nonzero rational number. Put $x_1=\frac{p-r^2}{p+r^2}$, $x_2=\frac{2r}{p+r^2}$. We have $x_1^2+px_2^2=(\frac{p-r^2}{p+r^2})^2+p(\frac{2r}{p+r^2})^2=\frac{p^2-2pr^2+r^4+4pr^2}{(p+r^2)^2}=\frac{p^2+2pr^2+r^4}{(p+r^2)^2}=1$. Therefore, $M_x\in SO(2,p,Q)$.

Hence, all special orthogonal matrices given as follows:

$$SO(2,p,Q)=\left\{\begin{pmatrix}\frac{p-r^2}{p+r^2} & \frac{-2pr}{p+r^2}\\ \frac{2r}{p+r^2} & \frac{p-r^2}{p+r^2}\end{pmatrix} \mid \ \forall r\in Q, r\neq 0\right\}$$

and all orthogonal matrices are given as follows:

$$O(2,p,Q)=\left\{\begin{pmatrix} \frac{p-r^2}{p+r^2} & \frac{-2pr}{p+r^2} \\ \frac{2r}{p+r^2} & \frac{p-r^2}{p+r^2} \end{pmatrix} \mid \forall r\in Q, r\neq 0\right\}\cup\left\{\begin{pmatrix} \frac{p-r^2}{p+r^2} & \frac{2pr}{p+r^2} \\ -\frac{2pr}{p+r^2} & -\frac{p-r^2}{p+r^2} \end{pmatrix} \mid \forall r\in Q, r\neq 0\right\}.$$

□

2.1. **Complete systems of invariants of an m-tuple in Q_p^2 for groups $SO(2,p,Q)$ and $MSO(2,p,Q)$.**

Let N be the set of all natural numbers and $m\in N, m\geq 1$. Put $N_m=\{j\in N|1\leq j\leq m\}$.

Definition 2.7. A mapping $u:N_m\to Q^2$ will be called an m-tuple in Q^2. Denote it in the following form: $u=(u_1,u_2,\dots u_m)$.

Denote by $(Q^2)^m$ the set of all m-tuples in Q^2. Let G be a subgroup of the group $MO(2,p,R)$.

Definition 2.8. Two m-tuples $u=(u_1,u_2,\dots u_m)$ and $v=(v_1,v_2,\dots v_m)$ in Q^2 is called G-equivalent if there exists $g\in G$ such that $v_j=gu_j,\forall j\in N_m$. In this case, we write $v=g(u)$ or $u\overset{G}{\sim}v$.

Definition 2.9. A subset $C\subseteq(Q^2)^m$ is called G-invariant if $g(u)\in C,\forall u\in C,\forall g\in G$.

Definition 2.10. Let Ω be a set and it has at least two elements and C be a G-invariant subset of $(Q^2)^m$. A mapping $f:C\to\Omega$ is called G-invariant on C if $u\in C, v\in C$ and $u\overset{G}{\sim}v$, implies $f(u)=f(v)$.

Let C be a G-invariant subset of $(Q^2)^m$ and Ω be a set such that it has at least two elements. Denote the set of all G-invariant functions $f:C\to\Omega$ on C by $Map(C,\Omega)^G$.

Example 2.11. Definitions of the groups $H=O(2,p,Q), SO(2,p,Q)$ imply that the quadratic form $\Psi(x)=\langle x,x\rangle_p$ and the bilinear form $\langle x,y\rangle_p$ are H-invariant functions on the set Q^2.

Example 2.12. Let $[x\,y]$ be the determinant $\begin{vmatrix} x_1 & y_1\\ x_2 & y_2\end{vmatrix}$ of $x=\begin{pmatrix} x_1\\ x_2\end{pmatrix}, y=\begin{pmatrix} y_1\\ y_2\end{pmatrix}\in Q^2$. Since $det(g)=1$ for all $g\in SO(2,p,Q)$, we have $[(gx)\,(gy)]=det(g)[x\,y]=[x\,y]$ for all $g\in SO(2,p,Q)$. Hence $[x\,y]$ is an $SO(2,p,Q)$-invariant function on the set $(Q^2)^2$.

Example 2.13. Definitions of the groups $H=MO(2,p,Q), MSO(2,p,Q)$ imply that the function $f(x,y)=\langle x-y,x-y\rangle_p$ is an H-invariant function on the set $(Q^2)^2$.

Definition 2.14. (see [22, 1.1]). Let C be a G-invariant subset of $(Q^2)^m$. A system $\{f_j|j\in J\}$, where $f_j\in Map(C,Q)^G,\forall j\in J$, will be called *a complete system* of G-invariant functions on C if $u\in C, v\in C$ and equalities $f_j(u)=f_j(v),\forall j\in J$, imply $u\overset{G}{\sim}v$.

Definition 2.15. (see [22, 1.1]) Let C be a G-invariant subset of $(Q^2)^m$ and $L=\{f_j|j\in J\}$ be a complete system of G-invariant functions on C. L is called *a minimal complete system* of G-invariant functions on C if $L\setminus\{f_j\}$ is not a complete system of G-invariant functions on C for any $j\in J$.

Put $\theta=(0,0)$, where $(0,0)\in Q^2$. Denote by θ_m the m-tuple $u=(u_1,u_2,\dots,u_m)\in(Q^2)^m$ such that $u_j=\theta,\forall j\in N_m$. Define the function $B:(Q^2)^m\to N_m\cup\{0\}$ as follows: put $B(0_m)=0$. Let $u=(u_1,u_2,\dots,u_m)\in(Q^2)^m$ be such that $u\neq\theta_m$. In this case, we put $B(u)=k$, where $k\in N_m$ such that $u_j=\theta,\forall j=1,\dots k-1$ and $u_k\neq\theta$.

Proposition 2.16. *Let G be a subgroup of $O(2,p,Q)$. The function $B(u)$ is a G-invariant function on $(Q^2)^m$.*

Proof. It is obvious. □

Denote by $U(m;0)$ the set $\{\theta_m\}$. Let $k\in N_m$. Denote by $U(m;k)$ the set $\{u\in(Q^2)^m|B(u)=k\}$.

Proposition 2.17. *Let G be a subgroup of $O(2,p,Q)$. Then:*

(1) *The set $U(m;k)$ is a G-invariant subset of $(Q^2)^m$ for $k=0$ and all $k\in N_m$.*

(2) $U(m;0) \cap U(m;l) = \emptyset, \forall l \in N_m$ *and* $U(m;k) \cap U(m;l) = \emptyset, \forall k, l \in N_m, k \neq l$.

(3) $U(m;0) \cup (\cup_{k \in N_m} U(m;k)) = (Q^2)^m$.

Proof. It is obvious. $\square$

Proposition 2.18. *Let* $x, y \in Q(\sqrt{-p})$ *such that* $x \neq 0$. *Then*

(1) *The element* yx^{-1} *exists, the equality* $yx^{-1} = \frac{\langle x,y \rangle}{\Psi(x)} + \sqrt{-p}\frac{[x\, y]}{\Psi(x)}$ *and the following equality hold*

$$M_{yx^{-1}} = \begin{pmatrix} \frac{\langle x,y \rangle_p}{\Psi(x)} & -\frac{p[x\, y]}{\Psi(x)} \\ \frac{[x\, y]}{\Psi(x)} & \frac{\langle x,y \rangle_p}{\Psi(x)} \end{pmatrix}. \tag{2.2}$$

(2) $\det(M_{yx^{-1}}) = (\frac{\langle x,y \rangle_p}{\Psi(x)})^2 + p(\frac{[x\, y]}{\Psi(x)})^2 \neq 0$ *if and only if* $\Psi(y) \neq 0$.

Proof. (1) Let $x = x_1 + \sqrt{-p}x_2, y = y_1 + \sqrt{-p}y_2 \in Q(\sqrt{-p})$ such that $x \neq 0$. Then x^{-1} exists. Hence yx^{-1} exists. By Proposition 1.7, $x^{-1} = \frac{W(x)}{\Psi(x)}$. Using $W(x) = x_1 - \sqrt{-p}x_2$ and the multiplication in the field $Q(\sqrt{-p})$, we obtain the equalities $yx^{-1} = \frac{\langle x,y \rangle_p}{\Psi(x)} + \sqrt{-p}\frac{[x\, y]}{\Psi(x)}$ and $Eq.(2.2)$.

(2) Let $\Psi(x) \neq 0$. Using Proposition 1.4 and $Eq.(2.2)$, we obtain $(\frac{\langle x,y \rangle_p}{\Psi(x)})^2 + p(\frac{[x\, y]}{\Psi(x)})^2 = \det(M_{yx^{-1}}) = \Psi(yx^{-1}) = \Psi(y)\Psi(x^{-1}) = \frac{\Psi(y)}{\Psi(x)}$. Hence $(\frac{\langle x,y \rangle_p}{\Psi(x)})^2 + p(\frac{[x\, y]}{\Psi(x)})^2 = \frac{\Psi(y)}{\Psi(x)}$. This equality implies that $\det(M_{yx^{-1}}) = \frac{\Psi(y)}{\Psi(x)} \neq 0$ if and only if $\Psi(y) \neq 0$. $\square$

Now we consider the G-equivalence problem of m-tuples for the group $SO(2, p, Q)$.

Proposition 2.19. *Let* G *be a subgroup of* $O(2, p, Q)$. *Assume that* $u = (u_1, \dots, u_m), v = (v_1, \dots, v_m) \in (Q^2)^m$ *be* m*-tuples such that* $u \overset{G}{\sim} v$. *Then* $B(u) = B(v)$.

Proof. Assume that $u \overset{G}{\sim} v$. By Proposition 2.16, the function $B(u)$ is G-invariant. The G-equivalence of u, v and the G-invariance of $B(u)$ imply the equality $B(u) = B(v)$. $\square$

Let $u = (u_1, \dots, u_m), v = (v_1, \dots, v_m) \in (Q^2)^m$ be m-tuples such that $B(u) = B(v) = 0$. Then $u = v = \theta_m$. Hence $u \overset{G}{\sim} v$. Now we consider the case $B(u) = B(v) \neq 0$.

Theorem 2.20. *Let* $u = (u_1, u_2, \dots, u_m), v = (v_1, v_2, \dots, v_m) \in (Q^2)^m$ *be two* m*-tuples in* Q^2 *such that* $B(u) = B(v) = k$, *where* $k \in N_m$.

(*i*) *Assume that* $u \overset{SO(2,p,Q)}{\sim} v$. *Then*

(*i.1*) *In the case* $k = m$, *the equality* $\Psi(u_m) = \Psi(v_m)$ *holds.*

(*i.2*) *In the case* $k < m$, *the following equalities hold*

$$\begin{cases} \Psi(u_k) = \Psi(v_k), \\ \langle u_k, u_j \rangle_p = \langle v_k, v_j \rangle_p, \forall j \in N_m, k < j, \\ [u_k u_j] = [v_k\, v_j], \forall j \in N_m, k < j. \end{cases} \tag{2.3}$$

(*ii*) *Conversely, assume that the equality* $\Psi(u_m) = \Psi(v_m)$ *holds in the case* $k = m$ *and equalities* $Eq.(2.3)$ *hold in the case* $k < m$. *Then, in the every of these cases, there exists the unique matrix* $F \in SO(2, p, Q)$ *such that* $v_j = Fu_j, \forall j \in N_m$. *In these cases,* F *has the following form*

$$F = \begin{pmatrix} \frac{\langle u_k,v_k \rangle_p}{\Psi(u_k)} & -\frac{p[u_k\, v_k]}{\Psi(u_k)} \\ \frac{[u_k\, v_k]}{\Psi(u_k)} & \frac{\langle u_k,v_k \rangle_p}{\Psi(u_k)} \end{pmatrix}, \tag{2.4}$$

where $\det(F) = (\frac{\langle u_k,v_k \rangle_p}{\Psi(u_k)})^2 + p(\frac{[u_k\, v_k]}{\Psi(u_k)})^2 = 1$.

Proof. (i) Assume that $u \overset{SO(2,p,Q)}{\sim} v$. In the case $(i.1)$, the function $\Psi(u_m)$ is $SO(2,p,Q)$-invariant. Hence the equality $\Psi(u_m) = \Psi(v_m)$ holds.

In the case $(i.2)$, functions $\Psi(u_k)$, $\langle u_k, u_j\rangle_p$ and $[u_k\, u_j]$ are also $SO(2,p,Q)$-invariant for all $j \in N_m, k < j$. Hence equalities $Eq.(2.3)$ hold.

(ii) Conversely, assume that the equality $\Psi(u_m) = \Psi(v_m)$ holds in the case $k = m$ and equalities $Eq.(2.3)$ hold in the case $k < m$.

Let $k = m$. Consider the element $g = v_k u_k^{-1} \in Q^*(\sqrt{-p})$. Since $v_k = v_k(u_k^{-1}u_k) = (v_k u_k^{-1})u_k$, we have $v_k = gu_k$. Then by $Eq.(2.2)$, we obtain that $v_k = M_g u_k$, where $M_g \in M(Q^*(\sqrt{-p}))$. Using the equality $\Psi(u_k) = \Psi(v_k)$ and Proposition 1.4, we obtain $\det(M_g) = \Psi(g) = \Psi(v_k u_k^{-1}) = \Psi(v_k)\Psi(u_k^{-1}) = \Psi(v_k)\Psi(u_k)^{-1} = 1$. Hence $g \in S(Q^*(\sqrt{-p}))$. By Theorem 2.5, $M_g \in SO(2,p,Q)$. This implies that $v_k = M_g u_k$. Since $B(u) = B(v) = k$, we have $u_j = v_j = \theta, \forall j \in N_m, j < k$. These equalities, the equality $v_k = M_g u_k$ and the equality $k = m$ imply equalities $v_j = M_g u_j, \forall j \in N_m$. Hence $u \overset{SO(2,p,Q)}{\sim} v$ in the case $k = m$. By $g = v_k u_k^{-1}$ and Proposition 2.18, M_g has the form (2.4). By $det(M_g) = 1$ and Proposition 2.18, we obtain the equality $det(M_g) = (\frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)})^2 + p(\frac{[u_k\, v_k]}{\Psi(u_k)})^2 = 1$.

Let $k < m$. Using equalities $Eq.(2.3)$ and equalities $u_k^{-1}u_j = \frac{\langle u_k, u_j\rangle_p}{\Psi(u_k)} + \sqrt{-p}\frac{[u_k\, u_j]}{\Psi(u_k)}, \forall j \in N_m, k < j$, equalities $v_k^{-1}v_j = \frac{\langle v_k, v_j\rangle_p}{\Psi(v_k)} + \sqrt{-p}\frac{[v_k\, v_j]}{\Psi(v_k)}, \forall j \in N_m, k < j$, in Proposition 2.18, we obtain following equalities

$$u_k^{-1}u_j = v_k^{-1}v_j, \forall j \in N_m, k < j. \tag{2.5}$$

Consider the element $g = v_k u_k^{-1} \in Q^*(\sqrt{-p})$. Since $v_k = v_k(u_k^{-1}u_k) = (v_k u_k^{-1})u_k$, we have $v_k = gu_k$. Using equalities $Eq.(2.5)$, we obtain $v_k(u_k^{-1}u_j) = v_k(v_k^{-1}v_j), \forall j \in N_m, k < j$. These equalities and the above equality $g = v_k u_k^{-1}$ imply $v_j = (v_k v_k^{-1})v_j = v_k(v_k^{-1}v_j) = v_k(u_k^{-1}u_j) = (v_k u_k^{-1})u_j = gu_j$ for all $j \in N_m, k < j$. Thus we have $v_j = gu_j, \forall j \in N_m, k < j$, where $g = v_k u_k^{-1} \in Q^*(\sqrt{-p})$. The equality $g = v_k u_k^{-1}$ implies $v_k = gu_k$. This equality and the equalities $v_j = gu_j, \forall j \in N_m, k < j$ imply equalities $v_j = gu_j, \forall j \in N_m, k \le j$. Then by $Eq.(1.1)$, we obtain that $v_j = M_g u_j, \forall j \in N_m, k \le j$, where $M_g \in M(Q^*(\sqrt{-p}))$. These equalities and the equality $B(u) = B(v) = k$ imply that $v_j = M_g u_j$ for all $j \in N_m$. So we obtain that $\det(M_g) = 1$. Since $\det(M_g) = 1$, by Theorem 2.5, $M_g \in SO(2, p.Q)$. Hence we obtain $u \overset{SO(2,p,Q)}{\sim} v$.

Prove the uniqueness of $U \in SO(2,p,Q)$ satisfying the conditions $v_j = Uu_j, \forall j \in N_m$. Assume that $U \in SO(2,p,Q)$ such that $v_j = Uu_j, \forall j \in N_m$. Then by $Eq.(1.1)$ and Theorem 2.5, there exists the unique $b \in S(Q^*(\sqrt{-p}))$ such that $U = M_b$. Hence we have $v_j = M_b u_j, \forall j \in N_m$. By $Eq.(1.1)$, we obtain $v_j = bu_j, \forall j \in N_m$. Since $\Psi(u_k) \neq 0$, the equality $v_k = bu_k$ implies that $b = v_k u_k^{-1} = g \in S(Q^*(\sqrt{-p}))$. The uniqueness of U is proved.

Let us obtain the evident form of M_g. By Proposition 2.18, the element $g = v_k u_k^{-1}$ is equal to $\frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)} + \sqrt{-p}\frac{[u_k\, v_k]}{\Psi(u_k)}$. Hence the matrix M_g has the form $Eq.(2.4)$. Since $g \in S(Q^*(\sqrt{-p}))$, by Theorem 2.5, $\det(M_g) = 1$. □

Remark 2.21. Let $k, m \in N, m > 1, 1 \le k \le m$. By Theorem 2.20, the function $\Psi(u_k)$ is a complete system of $SO(2,p,Q)$-invariant functions on the set $U(k;k)$ in the case $k = m$. By Theorem 2.20, the system

$$\{\Psi(u_k), \langle u_k, u_j\rangle_p, [u_k\, u_j], j \in N_m, k < j\}. \tag{2.6}$$

is a complete system of $SO(2,p,Q)$-invariant functions on the set $U(m;k)$ in the case $k < m$.

Let $G = O(2,p,Q)$ or $G = SO(2,p,Q)$. Denote by $G \vee Tr(2,p,Q)$ the group of all transformations of Q^2 generated by elements of G and all translations of Q^2. In particularly, $MO(2,p,Q) = O(2,p,Q) \vee Tr(2,p,Q)$ and $MSO(2,p,Q) = SO(2,p,Q) \vee Tr(2,p,Q)$. Now we consider H-equivalence problem of m-tuples for the group $H = G \vee Tr(2,p,Q)$. Let u and v be m-tuples, where $m = 1$. Then it is obvious that they are $G \vee Tr(2,p,Q)$-equivalent.

Let $z \in Q^2$. Denote by $z \cdot 1_m$ the m-tuple $(y_1, y_2, \ldots, y_m)$ such that $y_j = z, \forall j \in N_m$. Let $u = (u_1, u_2, \ldots, u_m)$ be an m-tuple. Denote by $u - u_m \cdot 1_m$ the m-tuple $(u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m, 0)$.

Proposition 2.22. *Let $G = O(2,p,Q)$ or $G = SO(2,p,Q)$. Assume that $m > 1$ and $u = (u_1, u_2, \dots, u_m), v = (v_1, v_2, \dots, v_m) \in (Q^2)^m$. Then $u \overset{G\vee Tr(2,p,Q)}{\sim} v$ if and only if m-tuples $u - u_m \cdot 1_m$ and $v - v_m \cdot 1_m$ are G-equivalent.*

Proof. $\Rightarrow$ Assume that $u \overset{G\vee Tr(2,p,Q)}{\sim} v$. Then there exists $F \in G$ and $a \in Q^2$ such that $v_j = Fu_j + a, \forall j \in N_m$. In particularly, for $j = m$, we have $v_m = Fu_m + a$. This equality implies $a = v_m - Fu_m$. This equality and equalities $v_j = Fu_j + a, \forall j \in N_m$, imply equalities $v_j = Fu_j + v_m - Fu_m, \forall j \in N_m$. These equalities imply equalities $v_j - v_m = F(u_j - u_m), \forall j \in N_m$. That is $(u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m, 0) \overset{G}{\sim} (v_1 - v_m, v_2 - v_m, \dots, v_{m-1} - v_m, 0)$.

$\Leftarrow$ Assume that $(u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m, 0) \overset{G}{\sim} (v_1 - v_m, v_2 - v_m, \dots, v_{m-1} - v_m, 0)$. Then there exists $F \in G$ such that $v_j - v_m = F(u_j - u_m), \forall j \in N_m$. Put $a = v_m - Fu_m$. This equality implies $v_m = Fu_m + a$. The equality $a = v_m - Fu_m$ and equalities $v_j - v_m = F(u_j - u_m), \forall j \in N_m$, $v_m = Fu_m + a$ imply equalities $v_j = Fu_j + a, \forall j \in N_m$. Hence $u \overset{G\vee Tr(2,p,Q)}{\sim} v$. □

Corollary 2.23. *Let $G = O(2,p,Q)$ or $G = SO(2,p,Q)$. Assume that $m > 1$ and $u = (u_1, u_2, \dots, u_m)$, $v = (v_1, v_2, \dots, v_m) \in (Q^2)^m$. Then $u \overset{G\vee Tr(2,p,Q)}{\sim} v$ if and only if $(m-1)$-tuples $(u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m)$ and $(v_1 - v_m, v_2 - v_m, \dots, v_{m-1} - v_m)$ are G-equivalent.*

Proof. It follows from Proposition 2.22. □

Proposition 2.24. *Let $G = SO(2,p,Q)$ or $G = O(2,p,Q)$. Assume that $u \overset{G\vee Tr(2,p,Q)}{\sim} v$. Then $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m)$.*

Proof. This statement follows from Propositions 2.19 and 2.22. □

Let $u = (u_1, u_2, \dots, u_m) \in (Q^2)^m$ and G denote either the special orthogonal group $SO(2,p,Q)$ or the orthogonal group $O(2,p,Q)$. By Proposition 2.24, the function $B(u - u_m \cdot 1_m)$ is a $G \vee Tr(2,p,Q)$-invariant function of $u \in (Q^2)^m$.

It is obvious that $B(u - u_m \cdot 1_m) \leq m - 1, \forall u \in (Q^2)^m$. We note that $B(u - u_m \cdot 1_m) = 0$ if and only if $u - u_m \cdot 1_m = 0_m$ that is $u = u_m \cdot 1_m = (u_1, u_2, \dots, u_m)$, where $u_j = u_m, \forall j \in N_m$.

Proposition 2.25. *Let $G = SO(2,p,Q)$ or $G = O(2,p,Q)$. Assume that $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = 0$. Then $u \overset{G\vee Tr(2,p,Q)}{\sim} v$.*

Proof. In the case $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = 0$, the m-tuple u has the form $u = (u_m, u_m, \dots, u_m)$ and the m-tuple v has the form $v = (v_m, v_m, \dots, v_m)$. Then we have $v_j = F(u_j), \forall j \in N_m$, where $F \in Tr(2,p,Q)$ has the following form: $v_j = v_m = u_m + a = u_j + a, \forall j \in N_m$, where $a = v_m - u_m$. Hence u and v are $G \vee Tr(2,p,Q)$-equivalent. □

Denote by $\Omega(m;0)$ the set of all $u \in (Q^2)^m$ such that $B(u - u_m \cdot 1_m) = 0$. Let $k = 0$ or $k \in N_m$ such that $k \leq m - 1$. Put $\Omega(m;k) = \{u \in (Q^2)^m | B(u - u_m \cdot 1_m) = k\}$.

Proposition 2.26. (1) *Let $G = SO(2,p,Q)$ or $G = O(2,p,Q)$. Then every set $\Omega(m;k)$ is an $G \vee Tr(2,p,Q)$-invariant subset of $(Q^2)^m$ for $k = 0$ and all $k \in N_m, k \leq m-1$.*

(2) $\Omega(m;0) \cap \Omega(m;l) = \emptyset, \forall l \in N_m, l \leq m - 1$.

(3) $\Omega(m;k) \cap \Omega(m;l) = \emptyset, \forall k, l \in N_m$, *where* $k \neq l$, $k \leq m-1, l \leq m-1$.

(4) $\cup_{k=0}^{m-1} \Omega(m;k) = (Q^2)^m$.

Proof. It follows from Proposition 2.17 □

Let $u, v \in (Q^2)^m$ such that $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = 0$. Then, by Proposition 2.24, $u \overset{SO(2,p,Q)\vee Tr(2,p,Q)}{\sim} v$. Now we consider the case $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = k$, where $k \in N_m, k \leq m - 1$.

Theorem 2.27. *Let* $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in (Q^2)^m$ *be two m-tuples such that* $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = k$, *where* $k \in N_m, k \leq m-1$.

(*i*) *Assume that* $u \overset{MSO(2,p,Q)}{\sim} v$. *Then*

(*i*.1) *In the case* $m = k+1$, *the equality* $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ *holds.*

(*i*.2) *In the case* $k+1 < m$, *the following equalities hold*

$$\begin{cases} \Psi(u_k - u_m) = \Psi(v_k - v_m); \\ \langle u_k - u_m, u_j - u_m \rangle_p = \langle v_k - v_m, v_j - v_m \rangle_p, \forall j \in N_m, k < j \leq m-1; \\ \left[(u_k - u_m)\,(u_j - u_m) \right] = \left[(v_k - v_m)\,(v_j - v_m) \right], \forall j \in N_m, k < j \leq m-1. \end{cases} \tag{2.7}$$

(*ii*) *Conversely, assume that the equality* $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ *holds in the case* $k+1 = m$ *and equalities Eq.*(2.7) *hold in the case* $k+1 < m$. *Then there exists the unique matrix* $F \in SO(2,p,Q)$ *and the unique element* $b \in Q^2$ *such that* $v_j = Fu_j + b, \forall j \in N_m$. *In this case, F has the following form*

$$F = \begin{pmatrix} \frac{\langle u_k - u_m, v_k - v_m \rangle_p}{\Psi(u_k - u_m)} & -p\frac{[(u_k - u_m)\,(v_k - v_m)]}{\Psi(u_k - u_m)} \\ \frac{[(u_k - u_m)\,(v_k - v_m)]}{\Psi(u_k - u_m)} & \frac{\langle u_k - u_m, v_k - v_m \rangle_p}{\Psi(u_k - u_m)} \end{pmatrix}, \tag{2.8}$$

where $\det(F) = (\frac{\langle u_k - u_m, v_k - v_m \rangle_p}{\Psi(u_k - u_m)})^2 + p(\frac{[(u_k - u_m)\,(v_k - v_m)]}{\Psi(u_k - u_m)})^2 = 1$ *and element* $b \in Q^2$ *is equal to* $v_m - Fu_m$.

Proof. (*i*) Assume that $u \overset{MSO(2,p,Q)}{\sim} v$. Then, by Proposition 2.22, the m-tuples $(u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m, 0)$ and $(v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m, 0)$ are $SO(2,p,Q)$-equivalent. This equivalence and Theorem 2.20 imply the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ in the case $m = k+1$ and the equalities *Eq.*(2.7) in the case $k+1 < m$.

(*ii*) Conversely, assume that the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ holds in the case $m = k+1$ and the equalities *Eq.*(2.7) hold in the case $k+1 < m$. Then, by Theorem 2.20, in every these cases there exists the unique matrix $F \in SO(2,p,Q)$ such that $v_j - v_m = F(u_j - u_m), \forall j \in N_m$. By Theorem 2.20, F has the form (2.8), where $\det(F) = (\frac{\langle u_k - u_m, v_k - v_m \rangle_p}{\Psi(u_k - u_m)})^2 + p(\frac{[(u_k - u_m)\,(v_k - v_m)]}{\Psi(u_k - u_m)})^2 = 1$. Put $b = v_m - Fu_m$. Then this equality and equalities $v_j - v_m = F(u_j - u_m), \forall j \in N_m$, imply equalities $v_j = F(u_j) + b, \forall j \in N_m$. The uniqueness of F such that $v_j - v_m = F(u_j - u_m), \forall j \in N_m$ implies the uniqueness of b such that $v_j = F(u_j) + b, \forall j \in N_m$. □

Remark 2.28. Let $k, m \in N, m > 1, 1 \leq k \leq m-1$. By Theorem 2.27, the function $\Psi(u_k - u_m)$ is a complete system of $MSO(2,p,Q)$-invariant functions on the set $\Omega(m;k)$ in the case $m = k+1$. By Theorem 2.27, the system

$$\{\Psi(u_k - u_m), \langle u_k - u_m, u_j - u_m \rangle_p, \left[(u_k - u_m)(u_j - u_m) \right], k+1 \leq j \leq m-1\} \tag{2.9}$$

is a complete system of $MSO(2,p,Q)$-invariant functions on the set $\Omega(m;k)$ in the case $k+1 < m$.

3. Complete systems of invariants of an m-tuple in Q^2 for groups $O(2,p,Q)$ and $MO(2,p,Q)$

First we consider the case $m = 1$.

Theorem 3.1. *Let* $u, v \in Q^2$.

(*i*) *Assume that* $u \overset{O(2,p,Q)}{\sim} v$. *Then the equality* $\Psi(u) = \Psi(v)$ *holds.*

(*ii*) *Conversely, assume that the equality* $\Psi(u) = \Psi(v)$ *holds. In this case,* $\Psi(u) = 0$ *or* $\Psi(u) \neq 0$

(*ii*.1) *Let* $\Psi(u) = 0$. *Then* $u = \theta, v = \theta$, *where* $\theta = (0,0)$, *and* $u \overset{O(2,p,Q)}{\sim} v$.

(*ii*.2) *Let* $\Psi(u) \neq 0$. *Then* $u \neq \theta, v \neq \theta$ *and* $u \overset{O(2,p,Q)}{\sim} v$. *In this case, only two matrices* $F_1 \in$

$O(2,p,Q)$ and $F_2 \in O(2,p,Q)$ exist such that $v = F_1u$, and $v = F_2u$. Here $F_1 \in SO(2,p,Q)$ and it has the following form

$$F_1 = \begin{pmatrix} \frac{\langle u,v\rangle_p}{\Psi(u)} & -\frac{p[u\,v]}{\Psi(u)} \\ \frac{[u\,v]}{\Psi(u)} & \frac{\langle u,v\rangle_p}{\Psi(u)} \end{pmatrix}, \tag{3.1}$$

where $det(F_1) = (\frac{\langle u,v\rangle_p}{\Psi(u)})^2 + p(\frac{[u\,v]}{\Psi(u)})^2 = 1$.

Here $F_2 \in O(2,p,Q)$ and it has the following form $F_2 = HW$, where $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, $H \in SO(2,p,Q)$ and H has the following form

$$H = \begin{pmatrix} \frac{\langle (Wu),v\rangle_p}{\Psi(Wu)} & -\frac{p[(Wu)\,v]}{\Psi(Wu)} \\ \frac{[(Wu)\,v]}{\Psi(Wu)} & \frac{\langle (Wu),v\rangle_p}{\Psi(Wu)} \end{pmatrix}, \tag{3.2}$$

where $det(H) = (\frac{\langle (Wu),v\rangle_p}{\Psi(Wu)})^2 + p(\frac{[(Wu)\,v]}{\Psi(Wu)})^2 = 1$ and $det(F_2) = -1$.

Proof. (i) Assume that $u \overset{O(2,p,Q)}{\sim} v$. By Example 2.11, the function $\Psi(x)$ is $O(2,p,Q)$-invariant. Hence the equality $\Psi(u) = \Psi(v)$ holds.

(ii) Assume that the equality $\Psi(u) = \Psi(v)$ holds.

$(ii.1)$ Let $\Psi(u) = 0$. Then $u = v = \theta$. Then it is obvious that $u \overset{O(2,p,Q)}{\sim} v$.

$(ii.2)$ Let $\Psi(u) \neq 0$. This inequality and the equality $\Psi(u) = \Psi(v)$ imply inequalities $u \neq \theta$ and $v \neq \theta$. By Theorem 2.20, there exists the unique $F_1 \in SO(2,p,Q)$ such that $v = F_1u$. Since $F_1 \in SO(2,p,Q) \subset O(2,p,Q)$, we obtain that $u \overset{O(2,p,Q)}{\sim} v$. Put $g = vu^{-1}$. By this equality and Proposition 2.18, $F_1 = M_g$. Hence we have $v = M_gu$. By Theorem 2.20, we obtain that M_g has the form (3.1) and the properties $det(M_g) = (\frac{\langle u,v\rangle_p}{\Psi(u)})^2 + p(\frac{[u\,v]}{\Psi(u)})^2 = 1$, $M_g \in SO(2,p,Q)$, $v = M_gu$ hold.

Now we investigate an existence of $F_2 \in O(2,p,Q)$ of the form $F_2 = HW$ such that $v = F_2u$, where $H \in SO(2,p,Q)$. For given above u, v, the equality $\Psi(u) = \Psi(v)$ holds. Using Proposition 1.6(4), we obtain the equality $\Psi(v) = \Psi(u) = \Psi(Wu)$. By the equality $\Psi(v) = \Psi(Wu)$ and Theorem 2.20, there exists the unique $H \in SO(2,p,Q)$ such that $v = H(Wu)$. Put $F_2 = HW$. Then $F_2 \in \{HW | H \in SO(2,p,Q)\}$. Hence there exists $F_2 \in O(2,p,Q)$ of the form $F_2 = HW$, where $H \in SO(2,p,Q)$, such that $v = F_2u$.

Prove the uniqueness of $F_2 \in \{HW | H \in SO(2,p,Q)\}$ such that $v = F_2u$. Assume that $F_2 = H_2W \in \{HW | H \in SO(2,p,Q)\}$ and $F_3 = H_3W \in \{HW | H \in SO(2,p,Q)\}$ such that $v = H_2Wu$ and $v = H_3Wu$, where $H_2, H_3 \in SO(2,p,Q)$. Then we have $v = H_2(Wu) = H_3(Wu)$. Using the uniqueness in Theorem 2.20, we obtain $H_2 = H_3$. This means that the unique $F_2 = H_2W \in \{HW | H \in SO(2,p,Q)\}$ exists such that $v = F_2(u)$. By Theorem 2.20, we obtain that H_2 has the form (3.2) and the properties $det(H_2) = (\frac{\langle Wu,v\rangle_p}{\Psi(Wu)})^2 + p(\frac{[(Wu)\,v]}{\Psi(Wu)})^2 = 1$, $H_2W \in O(2,p,Q)$, $det(F_2) = -1$, $v = H_2W(u)$ hold. □

Remark 3.2. Theorem 3.1 means that the function $\Psi(u)$ is a complete system of $O(2,p,Q)$-invariant functions on the set $U(1;1)$.

Let $u = (u_1, u_2, \ldots, u_m) \in (Q^2)^m$. Denote by $r(u)$ the rank of the system $\{u_1, u_2, \ldots, u_m\}$ in the space Q^2. For $u = \theta_m$, we put $r(\theta_m) = 0$. Assume that $u \neq \theta_m$. Then $r(u) = 1$ or $r(u) = 2$. It is obvious that the rank $r(u)$ is $O(2,p,Q)$-invariant of u. Put $U_0(m) = U(m,0)$. For $k \in N_m$, $l = 1,2$, denote by $U_l(m,k)$ the set of elements $u \in U(m,k)$ such that $r(u) = l$. It is obvious that $U_0(m) \cap U_l(m,k) = \emptyset$ for $l = 1,2$ and $U_l(m,k) \cap U_q(m,k) = \emptyset$ for $l, q = 1,2$ such that $l \neq q$. The following equalities $U_1(m,k) \cup U_2(m,k) = U(m,k), \forall k \in N_m$ and $U_0(m) \cup (\cup_{k=1}^m U_1(m,k)) \cup (\cup_{k=1}^m U_2(m,k)) = (Q^2)^m$ hold.

Let $u \in U_0(m), v \in U_0(m)$. Then $u = v = \theta_m$. Hence $u \overset{O(2,p,Q)}{\sim} v$.

Theorem 3.3. *Let $m > 1$ and $u = (u_1, u_2, \ldots, u_m) \in U_1(m,k), v = (v_1, v_2, \ldots, v_m) \in U_1(m,k)$, where $k \in N_m$.*

(*i*) *Assume that* $k = m$ *and* $u \overset{O(2,p,Q)}{\sim} v$. *Then the equality* $\Psi(u_m) = \Psi(v_m)$ *holds. Conversely, assume that* $k = m$ *and the equality* $\Psi(u_m) = \Psi(v_m)$ *holds. In this case, only two matrices* $F_1 \in O(2,p,Q)$ *and* $F_2 \in O(2,p,Q)$ *exist such that* $v_j = F_1(u_j), \forall j \in N_m$, *and* $v_j = F_2(u_j), \forall j \in N_m$. *Here* $F_1 \in SO(2,p,Q)$ *and it has the following form*

$$F_1 = \begin{pmatrix} \frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)} & -p\frac{[u_k\, v_k]}{\Psi(u_k)} \\ \frac{[u_k\, v_k]}{\Psi(u_k)} & \frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)} \end{pmatrix}, \tag{3.3}$$

where $det(F_1) = (\frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)})^2 + p(\frac{[u_k\, v_k]}{\Psi(u_k)})^2 = 1$.

Here $F_2 \in O(2,p,Q)$ *and it has the following form* $F_2 = HW$, *where* $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, $H \in SO(2,p,Q)$ *and* H *has the following form*

$$H = \begin{pmatrix} \frac{\langle (Wu_k), v_k\rangle_p}{\Psi(Wu_k)} & -\frac{p[(Wu_k)\, v_k]}{\Psi(Wu_k)} \\ \frac{[(Wu_k)\, v_k]}{\Psi(Wu_k)} & \frac{\langle (Wu_k), v\rangle_p}{\Psi(Wu_k)} \end{pmatrix}, \tag{3.4}$$

where $det(H) = (\frac{\langle (Wu_k), v_k\rangle_p}{\Psi(Wu_k)})^2 + p(\frac{[(Wu_k)\, v_k]}{\Psi(Wu_k)})^2 = 1$ *and* $det(F_2) = -1$.

(*ii*) *Assume that* $k < m$ *and* $u \overset{O(2,p,Q)}{\sim} v$. *Then the following equalities hold*

$$\begin{cases} \Psi(u_k) = \Psi(v_k), \\ \langle u_k, u_j\rangle_p = \langle v_k, v_j\rangle_p, \forall j \in N_m, k < j. \end{cases} \tag{3.5}$$

Conversely, assume that the equalities Eq.(3.5) *hold. In this case, only two matrices* $F_1 \in O(2,p,Q)$ *and* $F_2 \in O(2,p,Q)$ *exist such that* $v = F_1 u$ *and* $v = F_2 u$. *Here* $F_1 \in SO(2,Q)$ *and it is has the form Eq.*(3.3). *Here* $F_2 \in O(2,p,Q)$ *and it has the following form* $F_2 = HW$, *where* $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, $H \in SO(2,p,Q)$ *and* H *has the form Eq.*(3.4).

Proof. (*i*) In this case, m-tuples u and v have following forms: $u = (u_1, u_2, \ldots, u_m)$, where $u_j = \theta, \forall j \in N_m, j < m$, $u_m \neq \theta$, $v = (v_1, v_2, \ldots, v_m)$, where $v_j = \theta, \forall j \in N_m, j < m$, $v_m \neq \theta$. These forms imply that the statement (*i*) follows from Theorem 3.1.

(*ii*) Assume that the equalities *Eq.*(3.5) hold. Since $r(u) = r(v) = 1$, m-tuples u and v have following forms: $u = (u_1, u_2, \ldots, u_m)$, where $u_j = \theta, \forall j \in N_m, j < k$, $u_k \neq \theta$, $u_j = a_j u_k, \forall j \in N_m, k < j$, for some $a_j \in Q, \forall j \in N_m, k < j$, and $v = (v_1, v_2, \ldots, v_m)$, where $v_j = \theta, \forall j \in N_m, j < k$, $v_k \neq \theta$, $v_j = b_j v_k, \forall j \in N_m, k < j$ for some $b_j \in R, \forall j \in N_m, k < j$. It is easy to see that equalities *Eq.*(3.5) and the inequality $\Psi(u_k) \neq 0$ imply equalities $a_j = b_j, \forall j \in N_m, k < j$. Hence m-tuples u, v have following forms $u = (u_1, u_2, \ldots, u_m)$, where $u_j = \theta, \forall j \in N_m, j < k$, $u_k \neq \theta$, $u_j = a_j u_k, \forall j \in N_m, k < j$, and $v = (v_1, v_2, \ldots, v_m)$, where $v_j = \theta, \forall j \in N_m, j < k$, $v_k \neq \theta$, $v_j = a_j v_k, \forall j \in N_m, k < j$.

By using Eq.(3.5), we obtain equality $\Psi(u_k) = \Psi(v_k)$. Since $u_k \neq \theta$, we obtain that $\Psi(u_k) = \Psi(v_k) \neq 0$. Then, by Theorem 3.1(*ii*.2), only two matrices $F_1 \in O(2,p,Q)$ and $F_2 \in O(2,p,Q)$ exist such that $v_k = F_1(u_k)$ and $v_k = F_2(u_k)$. Here $F_1 \in SO(2,p,Q)$ and it has the form Eq.(3.3) and $F_2 \in O(2,p,Q)$ has the form $F_2 = HW$, where $H \in SO(2,p,Q)$ and H has the form Eq.(3.4). Equalities $v_k = F_1(u_k)$, $v_k = F_2(u_k)$ and equalities $u_j = \theta, \forall j \in N_m, j < k$, $u_k \neq \theta$, $u_j = a_j u_k, \forall j \in N_m, k < j$, $v_j = \theta, \forall j \in N_m, j < k$, $v_k \neq \theta$, $v_j = a_j v_k, \forall j \in N_m, k < j$ imply equalities $v_j = F_1(u_j), \forall j \in N_m$ and $v_j = F_2(u_j), \forall j \in N_m$.

Now we prove that if a matrix $A \in O(2,p,Q)$ such that $v_j = A(u_j), \forall j \in N_m$, then $A = F_1$ or $A = F_2$. Equalities $v_j = A(u_j), \forall j \in N_m$, implies the equality $v_k = A(u_k)$, where $k \in N_m$. Then, by Theorem 3.1(*ii*.2), $A = F_1$ or $A = F_2$. □

Theorem 3.4. *Let* $m > 1, k \in N_m$ *and* $u = (u_1, u_2, \ldots, u_m) \in (Q^2)^m$. *Let* $\left\{\Psi(u_k), \langle u_k, u_j\rangle_p \,|j \in N_m, k < j\right\}$ *be the complete system of* $O(2,p,Q)$*-invariants on the set* $U_1(m;k)$ *given in Theorem 3.3. Then* $\Psi(u_k) > 0$ *for all* $u \in U_1(m;k)$.

Proof. A proof is similar to the proof of Theorem 2.20 and it is omitted. □

Let $m > 1$ and $u = (u_1, u_2, \ldots, u_m) \in (Q^2)^m$. Assume that $r(u) = 2$ and $B(u) = k$, where $k \in N_m$. Put $B_1(u) = B(u)$. Denote by $\{\lambda u_k | \lambda \in Q\}$ the linear subspace of Q^2 generated by u_k. Denote by $B_2(u)$ the smallest of $s, s \in N_m$, such that $u_s \notin \{\lambda u_k | \lambda \in Q\}$. Then $B_1(u) < B_2(u) \le m$ for all $u \in U_2(m, k)$. The number $B_2(u)$ is an $O(2, p, Q)$-invariant of an m-tuple u. The pair $(B_1(u), B_2(u))$ will be called the type of an m-tuple $u \in U_2(m, k)$ of $r(u) = 2$. The type $(B_1(u), B_2(u))$ is an $O(2, p, Q)$-invariant of an m-tuple $u \in U_2(m, k)$. Let $k, l \in N_m$ such that $k < l$. Then there exists an m-tuple $u = (u_1, u_2, \ldots, u_m) \in (Q^2)^m$ such that $B_1(u) = k$ and $B_2(u) = l$. In this case, vectors $u_k = (u_{k1}, u_{k2})$ and $u_l = (u_{l1}, u_{l2})$ are linearly independent.

Denote by $E(u; k, l)$ the following 2×2-matrix

$$\begin{pmatrix} u_{k1} & u_{k2} \\ u_{l1} & u_{l2} \end{pmatrix}.$$

Since vectors $u_k = (u_{k1}, u_{k2})$ and $u_l = (u_{l1}, u_{l2})$ are linearly independent, $det(E(u; k, l)) \neq 0$.

Denote by $\Phi(u; k, l)$ the following 2×2-matrix

$$\begin{pmatrix} u_{k1} & pu_{k2} \\ u_{l1} & pu_{l2} \end{pmatrix}. \tag{3.6}$$

We have $det(\Phi(u; k, l)) = pu_{k1}u_{i2} - pu_{k2}u_{l1} = pdet(E(u; k, l))$. Since $det(E(u; k, l)) \neq 0$, we obtain that $det(\Phi(u; k, l)) \neq 0$. Hence the inverse matrix $\Phi^{-1}(u; k, l)$ exits.

Denote by $U_2(m, k, l)$ the set of all m-tuples u such that $B_1(u) = k$ and $B_2(u) = l$.

Theorem 3.5. *Let $m > 1$ and $u = (u_1, u_2, \ldots, u_m) \in U_2(m, k, l), v = (v_1, v_2, \ldots, v_m) \in U_2(m, k, l)$, where $k, l \in N_m, k < l$, $u_k = (u_{k1}, u_{k2})$, $u_l = (u_{l1}, u_{l2})$, $v_k = (v_{k1}, v_{k2})$, $v_l = (v_{l1}, v_{l2})$.*

(*i*) *Assume that $u \overset{O(2,p,Q)}{\sim} v$. Then the following equalities hold:*

$$\begin{cases} \langle u_k, u_j \rangle_p = \langle v_k, v_j \rangle_p, \forall j \in N_m, k \le j; \\ \langle u_l, u_j \rangle_p = \langle v_l, v_j \rangle_p, \forall j \in N_m, k < j. \end{cases} \tag{3.7}$$

(*ii*) *Conversely, assume that the equalities Eq.(3.7) hold. Then $u \overset{O(2,p,Q)}{\sim} v$. In this case, the unique $F \in O(2, p, Q)$ exists such that $v_j = F(u_j), \forall j \in N_m$, and F has the following form $F = \Phi^{-1}(v; k, l)\Phi(u; k, l)$.*

Proof. (*i*) Assume that $u \overset{O(2,p,Q)}{\sim} v$. Since the functions $\langle u_k, u_j \rangle_p$ and $\langle u_l, u_j \rangle_p$ are $O(2, p, Q)$-invariant, equalities Eq.(3.7) hold.

(*ii*) Conversely, assume that the equalities Eq.(3.7) hold. For $j \in N_m$, consider the vectors $u_j = (u_{j1}, u_{j2})$ and $v_j = (v_{j1}, v_{j2})$. Transposes of vectors u_j and v_j denote by $u_j^\top$ and $v_j^\top$.

$$u_j^\top = \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix}, v_j^\top = \begin{pmatrix} v_{j1} \\ v_{j2} \end{pmatrix}.$$

Consider the following vectors: $u_k = (u_{k1}, pu_{k2})$ and $u_l = (u_{l1}, pu_{l2})$. The multiplication of matrices u_k and $u_j^\top$ is equal to $\langle u_k, u_j \rangle_p$:

$$u_k \cdot u_j^\top = (u_{k1}, pu_{k2}) \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix} = \langle u_k, u_j \rangle_p, \forall j \in N_m. \tag{3.8}$$

Similarly we obtain

$$u_l \cdot u_j^\top = (u_{l1}, pu_{l2}) \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix} = \langle u_l, u_j \rangle_p, \forall j \in N_m. \tag{3.9}$$

Using equalities Eq.(3.8) and Eq.(3.9) to matrices $\Phi(u;k,l)$, $u_k^\top$ and vectors u_j, we obtain the following equalities

$$\Phi(u;k,l)u_j = \begin{pmatrix} u_{k1} & pu_{k2} \\ u_{l1} & pu_{l2} \end{pmatrix} \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix} = \begin{pmatrix} \langle u_k, u_j\rangle_p \\ \langle u_l, u_j\rangle_p \end{pmatrix}, \forall j \in N_m. \tag{3.10}$$

Similarly we obtain the following equalities

$$\Phi(v;k,l)v_j = \begin{pmatrix} v_{k1} & pv_{k2} \\ v_{l1} & pv_{l2} \end{pmatrix} \begin{pmatrix} v_{j1} \\ v_{j2} \end{pmatrix} = \begin{pmatrix} \langle v_k, v_j\rangle_p \\ \langle v_l, v_j\rangle_p \end{pmatrix}, \forall j \in N_m. \tag{3.11}$$

Since $B_1(u) = B_1(v) = k$, we have $u_j = v_j = 0, \forall j \in N_m, j < k$. These equalities imply the following equalities

$$\begin{cases} \langle u_k, u_j\rangle_p = \langle v_k, v_j\rangle_p = 0, \forall j \in N_m, j < k, \\ \langle u_l, u_j\rangle_p = \langle v_l, v_j\rangle_p = 0, \forall j \in N_m, j < l. \end{cases} \tag{3.12}$$

These equalities and the equalities Eq.(3.7) imply following equalities

$$\begin{cases} \langle u_k, u_j\rangle_p = \langle v_k, v_j\rangle_p, \forall j \in N_m, \\ \langle u_l, u_j\rangle_p = \langle v_l, v_j\rangle_p, \forall j \in N_m. \end{cases} \tag{3.13}$$

These equalities and equalities Eq.(3.10), Eq.(3.11) imply following equalities

$$\Phi(u;k,l)u_j = \begin{pmatrix} \langle u_k, u_j\rangle_p \\ \langle u_l, u_j\rangle_p \end{pmatrix} = \begin{pmatrix} \langle v_k, v_j\rangle_p \\ \langle v_l, v_j\rangle_p \end{pmatrix} = \Phi(v;k,l)v_j, \forall j \in N_m. \tag{3.14}$$

Since vectors $u_k = (u_{k1}, pu_{k2})$ and $u_l = (u_{l1}, pu_{l2})$ are linearly independent, the inverse of the matrix $\Phi^{-1}(u;k,l)$ exists. The equalities $\Phi(u;k,l)u_j = \Phi(v;k,l)v_j, \forall j \in N_m$ in Eq.(3.14) implies the following equalities

$$v_j = \Phi^{-1}(v;k,l))\Phi(u;k,l)u_j, \forall j \in N_m. \tag{3.15}$$

We prove that the matrix $\Phi^{-1}(v;k,l)\Phi(u;k,l)$ is orthogonal. Using the equalities Eq. (3.15) and Eq. (3.7), we obtain the following equality

$$\langle \Phi^{-1}(v;k,l)\Phi(u;k,l)u_j, \Phi^{-1}(v;k,l)\Phi(u;k,l)u_s\rangle = \langle v_j, v_s\rangle = \langle u_j, u_s\rangle \tag{3.16}$$

for $j = k,l$ and $s = k,l$. Since the system of two vectors u_k and u_l are a basis in Q^2, equalities Eq.(3.16) imply the following equalities

$$\langle \Phi^{-1}(v;k,l))\Phi(u;k,l)x, \Phi^{-1}(v;k,l))\Phi(u;k,l)y\rangle_p = \langle x, y\rangle_p, \forall x, y \in Q^2. \tag{3.17}$$

This means that the matrix $\Phi^{-1}(v;k,l))\Phi(u;k,l)$ is orthogonal.

Now we prove the uniqueness of a 2×2-orthogonal matrix F such that $v_j = Fu_j, \forall j \in N_m$. Assume that a 2×2-orthogonal matrix F such that $v_j = Fu_j, \forall j \in N_m$. In particularly, we have $v_j = Fu_j$ for $j = k,l$. These equalities and equalities Eq.(3.15) imply equalities

$$Fu_j = \Phi^{-1}(v;k,l)\Phi(u;k,l)u_j, \forall j = k,l. \tag{3.18}$$

Since the system of two vectors u_k and u_l are a basis in Q^2, equalities Eq.(3.18) imply following equalities:

$$Fx = \Phi^{-1}(v;k,l)\Phi(u;k,l)x, \forall x \in Q^2. \tag{3.19}$$

This means that $F = \Phi^{-1}(v;k,l)\Phi(u;k,l)$. The uniqueness of a 2×2-orthogonal matrix F such that $v_j = Fu_j, \forall j \in N_m$, is proved. □

Remark 3.6. By theorem 3.5, the system of $O(2,p,Q)$-invariants obtained in Theorem 3.5 is a complete system of $O(2,p,Q)$-invariants of m-tuples.

Now we investigate complete systems of invariants of the group $MO(2,p,Q) = O(2,p,Q) \vee Tr(2,p,Q)$ on the set $(Q^2)^m$.

Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in (Q^2)^m$. By Proposition 2.22, $u \overset{MO(2,p,Q)}{\sim} v$ if and only if $(u - u_m \cdot 1_m) \overset{O(2,p,Q)}{\sim} (v - v_m \cdot 1_m)$. By Proposition 2.24, the function $B(u - u_m \cdot 1_m)$ is an $MO(2,p,Q)$-invariant.

Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in U_0(m,k)$. Then $u = v = \theta_m$. This implies that $u \overset{MO(2,p,Q)}{\sim} v$.

Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in U_1(m,k)$. Assume that u and v be m-tuples such that $m = 1$. Then it is obvious that they are $MO(2,p,Q)$-equivalent. Assume that $u = (u_1, u_2, \ldots, u_m) \in U_1(m,m)$ and $v = (v_1, v_2, \ldots, v_m) \in U_1(m,m)$. Then it is obvious that they are $MO(2,p,Q)$-equivalent.

Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in (Q^2)^m$. By Corollary 2.23, $u \overset{MO(2,p,Q)}{\sim} v$ if and only if $(m-1)$-tuples $(u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m)$ and $(v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m)$ are $O(2,p,Q)$-equivalent. This Corollary 2.23 and Theorems 3.1, 3.3, 3.5 imply Theorems 3.7, 3.9, 3.10 given below.

Consider the case $m = 2$.

Theorem 3.7. *Let $u = (u_1, u_2) \in (Q^2)^2$ and $v = (v_1, v_2) \in (Q^2)^2$.*

(i) *Assume that $u \overset{MO(2,p,Q)}{\sim} v$. Then the equality $\Psi(u_1 - u_2) = \Psi(v_1 - v_2)$ holds.*

(ii) *Conversely, assume that the equality $\Psi(u_1 - u_2) = \Psi(v_1 - v_2)$ holds. In this case, $\Psi(u_1 - u_2) = 0$ or $\Psi(u_1 - u_2) \neq 0$*

(ii.1) Let $\Psi(u_1 - u_2) = 0$. Then $u_1 - u_2 = v_1 - v_2 = 0$ and $u \overset{MO(2,p,Q)}{\sim} v$. In this case the unique $a \in Q^2$ exists such that $v_j = u_j + a, \forall j = 1,2$. It is equal to $v_2 - u_2$.

(ii.2) Let $\Psi(u_1 - u_2) \neq 0$. Then $u \overset{MO(2,p,Q)}{\sim} v$. In this case, only two elements $F_1 \in MO(2,p,Q)$ and $F_2 \in MO(2,p,Q)$ exist such that $v_j = F_1 u_j, \forall j = 1,2$, and $v_j = F_2 u_j, \forall j = 1,2$. Here $F_1(u_j) = H_1(u_j) + a_1, j = 1,2$, where $H_1 \in SO(2,p,Q)$, $a_1 \in Q^2$, and H_1 has the following form

$$H_1 = \begin{pmatrix} \frac{\langle u_1 - u_2, v_1 - v_2 \rangle_p}{\Psi(u_1 - u_2)} & -p\frac{[(u_1 - u_2)\ (v_1 - v_2)]}{\Psi(u_1 - u_2)} \\ \frac{[(u_1 - u_2)\ (v_1 - v_2)]}{\Psi(u_1 - u_2)} & \frac{\langle u_1 - u_2, v_1 - v_2 \rangle_p}{\Psi(u_1 - u_2)} \end{pmatrix},$$

$$det(H_1) = \left(\frac{\langle u_1 - u_2, v_1 - v_2 \rangle_p}{Q(u_1 - u_2)}\right)^2 + p\left(\frac{[(u_1 - u_2)\ (v_1 - v_2)]}{Q(u_1 - u_2)}\right)^2 = 1,\ a_1 = v_1 - H_1 u_1.$$

Here $F_2(u_j) = H_2 W(u_j) + a_2, j = 1,2$, where $H_2 \in SO(2,p,Q)$, $a_2 \in Q^2$, $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, and H_2 has the following form

$$H_2 = \begin{pmatrix} \frac{\langle W(u_1 - u_2), v_1 - v_2 \rangle_p}{\Psi(W(u_1 - u_2))} & -p\frac{[(W(u_1 - u_2))\ (v_1 - v_2)]}{\Psi(W(u_1 - u_2))} \\ \frac{[(W(u_1 - u_2))\ (v_1 - v_2)]}{\Psi(W(u_1 - u_2))} & \frac{\langle W(u_1 - u_2), v_1 - v_2 \rangle_p}{\Psi(W(u_1 - u_2))} \end{pmatrix},$$

$$det(H_2) = \left(\frac{\langle W(u_1 - u_2), v_1 - v_2 \rangle_p}{\Psi(W(u_1 - u_2))}\right)^2 + p\left(\frac{[(W(u_1 - u_2))\ (v_1 - v_2)]}{\Psi(W(u_1 - u_2))}\right)^2 = 1,\ a_2 = v_1 - H_2 W u_1.$$

Proof. By Corollary 2.23, two 2-tuples $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are $MO(2,p,Q)$-equivalent if and only if vectors $u_1 - u_2$ and $v_1 - v_2$ are $O(2,p,Q)$-equivalent. Hence Theorem 3.1 implies this theorem. □

Let $m > 2$ and $u = (u_1, u_2, \ldots, u_m), v \in (Q^2)^m$ be two m-tuples such that $u = (u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m) \in U_0(m-1,k), v = (v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m) \in U_0(m-1,k)$, where $1 \leq k \leq m-1$. Then m-tuples u and v have forms $u = (u_m, u_m, \ldots, u_m)$ and $v = (v_m, v_m, \ldots, v_m)$. It is obvious that they are $MO(2,p,Q)$-equivalent.

Remark 3.8. Theorem 3.7 means that the function $\Psi(u_1 - u_2)$ is a complete system of $MO(2,p,Q)$-invariant functions on the set $(Q^2)^2$.

Theorem 3.9. *Let $m > 2$ and $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in (Q^2)^m$ be two m-tuples such that $u = (u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m), v = (v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m) \in U_1(m-1, k)$, where $1 \le k \le m-1$.*

(*i*) *Assume that $k = m-1$ and $u \overset{MO(2,p,Q)}{\sim} v$. Then the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ holds. Conversely, assume that $k = m-1$ and the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ holds. In this case, $\Psi(u_k - u_m) = 0$ or $\Psi(u_k - u_m) \neq 0$*

(i.1) Let $\Psi(u_k - u_m) = 0$. Then $u_k - u_m = v_k - v_m = 0$ and $u \overset{MO(2,p,Q)}{\sim} v$. In this case the unique $a \in Q^2$ exists such that $v_j = u_j + a, \forall j \in N_m$. It is equal to $v_m - u_m$.

(i.2) Let $\Psi(u_k - u_m) \neq 0$. Then $u \overset{MO(2,p,Q)}{\sim} v$. In this case, only two elements $F_1 \in MO(2,p,Q)$ and $F_2 \in MO(2,p,Q)$ exist such that $v_j = F_1 u_j$ and $v_j = F_2 u_j, \forall j \in N_m$. Here $F_1(u_j) = H_1(u_j) + a_1, \forall j \in N_m$, where $H_1 \in SO(2,p,Q)$, $a_1 \in Q^2$, and H_1 has the following form

$$H_1 = \begin{pmatrix} \frac{\langle u_k - u_m, v_1 - v_m\rangle_p}{\Psi(u_k - u_m)} & -p\frac{[(u_k - u_m)\ (v_k - v_m)]}{\Psi(u_k - u_m)} \\ \frac{[(u_k - u_m)\ (v_k - v_m)]}{\Psi(u_k - u_m)} & \frac{\langle u_k - u_m, v_k - v_m\rangle_p}{\Psi(u_k - u_m)} \end{pmatrix}, \tag{3.20}$$

$det(H_1) = (\frac{\langle u_k - u_m, v_k - v_m\rangle_p}{\Psi(u_k - u_m)})^2 + p(\frac{[(u_k - u_m)\ (v_k - v_m)]}{\Psi(u_k - u_m)})^2 = 1$, $a_1 = v_k - H_1 u_k$.

Here $F_2(u_j) = H_2 W(u_j) + a_2, \forall j \in N_m$, where $H_2 \in SO(2,p,Q)$, $a_2 \in Q^2$, $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, and H_2 has the following form

$$H_2 = \begin{pmatrix} \frac{\langle W(u_k - u_m), v_k - v_m\rangle_p}{\Psi(W(u_k - u_m))} & -p\frac{[(W(u_k - u_m))\ (v_k - v_m)]}{\Psi(W(u_k - u_m))} \\ \frac{[(W(u_k - u_m))\ (v_k - v_m)]}{\Psi(W(u_k - u_m))} & \frac{\langle W(u_k - u_m), v_k - v_m\rangle_p}{\Psi(W(u_k - u_m))} \end{pmatrix}, \tag{3.21}$$

$det(H_2) = (\frac{\langle W(u_k - u_m), v_k - v_m\rangle_p}{\Psi(W(u_k - u_m))})^2 + p(\frac{[(W(u_k - u_m))\ (v_k - v_m)]}{\Psi(W(u_k - u_m))})^2 = 1$, $a_2 = v_k - H_2 W u_k$.

(*ii*) *Assume that $k + 1 < m$ and $u \overset{MO(2,p,Q)}{\sim} v$. Then the following equalities hold*

$$\begin{cases} \Psi(u_k - u_m) = \Psi(v_k - v_m), \\ \langle u_k - u_m, u_j - u_m\rangle_p = \langle v_k - v_m, v_j - v_m\rangle_p, \forall j \in N_m, k < j < m. \end{cases} \tag{3.22}$$

Conversely, assume that the equalities Eq.(3.22) hold. In this case, only two elements $F_1 \in MO(2,p,Q)$ and $F_2 \in MO(2,p,Q)$ exist such that $v_j = F_1 u_j, \forall j \in N_m$, and $v_j = F_2 u_j, \forall j \in N_m$. Here $F_1(u_j) = H_1(u_j) + a_1, \forall j \in N_m$, where $H_1 \in SO(2,p,Q)$, $a_1 \in Q^2$, and H_1 has the following form Eq.(3.20). Here $F_2(u_j) = H_2 W(u_j) + a_1, \forall j \in N_m$, where $H_2 \in SO(2,p,Q)$, $a_2 \in Q^2$, $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, H_2 has the form Eq.(3.21) and $a_2 = v_k - H_2 W u_k$.

Proof. A proof follows from Corollary 2.23, Theorem 3.1 and Theorem 3.3. □

Let $m > 2$ and $u = (u_1, u_2, \ldots, u_m) \in (Q^2)^m$ be an m-tuple in Q^2 such that $(u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m) \in U_2(m-1, k, l)$. In this case, vectors $u_k - u_m = (u_{k1} - u_{m1}, u_{k2} - u_{m2})$ and $u_l - u_m = (u_{l1} - u_{m1}, u_{l2} - u_{m2})$ are linearly independent. Denote by $E(u - u_m 1_m; k, l)$ the following 2×2-matrix

$$\begin{pmatrix} u_{k1} - u_{m1} & u_{k2} - u_{m2} \\ u_{l1} - u_{m1} & u_{l2} - u_{m2} \end{pmatrix}.$$

Since the vectors $u_k - u_m$ and $u_l - u_m$ are linearly independent, $det(E(u - u_m 1_m; k, l)) \neq 0$. Denote by $\Phi(u - u_m 1_m; k, l)$ the following 2×2-matrix

$$\begin{pmatrix} u_{k1} - u_{m1} & p(u_{k2} - u_{m2}) \\ u_{l1} - u_{m1} & p(u_{l2} - u_{m2}) \end{pmatrix}.$$

Since $det(\Phi(u - u_m 1_m; k, l) = p \cdot det(E(u - u_m 1_m; k, l))$ and $det(E(u - u_m 1_m; k, l)) \neq 0$, we obtain that $det(\Phi(u - u_m 1_m; k, l) \neq 0$. This implies that the inverse matrix $\Phi^{-1}(u - u_m 1_m; k, l)$ exists.

Theorem 3.10. *Let $m > 2$. Assume that $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ are $(m-1)$-tuples such that $(u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m) \in U_2(m-1, k, l), (v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m) \in U_2(m-1, k, l)$, where $1 \leq k < l \leq m-1$.*

(i) *Assume that $u \overset{MO(2,p,Q)}{\sim} v$. Then the following equalities hold:*

$$\begin{cases} \langle u_k - u_m, u_j - u_m \rangle_p = \langle v_k - v_m, v_j - v_m \rangle_p, \forall j \in N_m, k \leq j \leq m-1; \\ \langle u_l - u_m, u_j - u_m \rangle_p = \langle v_l - v_m, v_j - v_m \rangle_p, \forall j \in N_m, l \leq j \leq m-1. \end{cases} \tag{3.23}$$

(ii) *Conversely, assume that the equalities Eq.(3.23) hold. Then $u \overset{MO(2,p,Q)}{\sim} v$. In this case, the unique $F \in MO(2,p,Q))$ exists such that $v_j = F(u_j), \forall j \in N_m$, and F has the following form $F(u_j) = (\Phi^{-1}(v - v_m 1_m; k, l)\Phi(u - u_m 1_m; k, l))(u_j) + a, \forall j \in N_m$, where $a = v_k - (\Phi^{-1}(v - v_m 1_m; k, l)\Phi(u - u_m 1_m; k, l))(u_k)$.*

Proof. A proof follows from Corollary 2.23 and Theorem 3.5. □

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