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The decreasing rearrangements of functions for vector-valued measures

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Abstract. Let B be a complete Boolean algebra, let $Q(B)$ be the Stone compact of B, let $C_\infty(Q(B))$ be the commutative unital algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$. We consider Maharam measure m defined on B, which takes on value in the algebra $L^0(\Omega)$ of all real measurable functions on the measurable space (Ω, Σ, μ) with a σ-finite numerical measure μ. The decreasing rearrangements of functions from $C_\infty(Q(B))$, associated with such a measure m and taking values in the algebra $L^0(\Omega)$ are determined. The basic properties of such rearrangements are established, which are similar to the properties of classical decreasing rearrangements of measurable functions. As an application, with the help of the property of equimeasurablity of elements from $C_\infty(Q(B))$, associated with such a measure m, the notion of a symmetric Banach-Kantorovich space $(E, \|\cdot\|_E)$ over $L^0(\Omega)$ is introduced and studied in detail. Here $E \subset C_\infty(Q(B))$, and $\|\cdot\|_E - L^0(\Omega)$-valued norm in E, endowing it with the structure of the Banach-Kantorovich space. Examples of symmetric Banach-Kantorovich spaces are given, which are vector-valued analogues of classical L^p-spaces, $1 \le p \le \infty$, associated with a numerical σ-finite measure.

Keywords: vector integration, vector-valued measure, decreasing rearrangements, equimeasurablity, the Banach-Kantorovich lattice, symmetric space.
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1. Introduction

There are well-known examples of Banach-Kantorovich spaces constructed using integration theory for vector measures with values in order complete vector lattice (K-spaces), in particular, in the algebra $L^0(\Omega)$ of all classes of almost everywhere equal real measurable functions on the measurable space (Ω, Σ, μ) with a σ-finite numerical measure μ. Important examples of the Banach-Kantorovich spaces include the "vector-valued" analogues of the L_p-spaces, $1 \le p < \infty$ [1], [2], and the Orlicz spaces [3], [4], [5], [6].

If Ω is a singleton, then the class of Banach-Kantorovich spaces coincides with the class of real Banach spaces, important examples of which are functional symmetric spaces. The theory of symmetric spaces contains many profound results and has important applications in a wide variety of fields of function theory and functional analysis, in particular, in the theory interpolation of linear operators, ergodic theory and harmonic analysis (see for example, [7], [8], [9].

In the general theory of functional symmetric spaces, the notion non-increasing rearrangements plays an important role as shown in [7], [8]. For an measurable function $f \in L^0(\Omega)$ for which $\mu(\{|f| > \lambda\}) < \infty$ for some $\lambda > 0$, its non-increasing rearrangement $f^*(t)$ is defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu(\{|f| > \lambda\}) \le t\} = \inf\{\ \|f\chi_A\|_\infty : A \in \Sigma,\ f\chi_A \in L^\infty(\Omega),\ \mu\big(\Omega \setminus A\big) \le t\},\ t > 0,$$

where χ_A is a characteristic function of the set A, and $\|\cdot\|_\infty$ is a uniform norm in algebra $L^\infty(\Omega)$ of all essentially bounded functions from $L^0(\Omega)$.

In this paper, we consider a measure m defined on a complete Boolean algebra B, which takes on value in the algebra $L^0(\Omega)$. It is assumed that the measure m has the property Maharam, that is, for any $e \in B$, $f \in L^0(\Omega)$, $0 \le f \le m(e)$, there exists $q \in B$, $q \le e$, such that $m(q) = f$ (such measures are called Maharam measures). For the Maharam measure, there is always a unique injective completely additive Boolean homomorphism φ from the Boolean algebra $B(\Omega) = \{q \in L^0(\Omega) : q = q^2\}$ in Boolean algebra B such that $\nabla(m) = \varphi(B(\Omega))$ is a regular Boolean subalgebra of B, and $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$ [10]. Moreover, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) := C_\infty(Q(\nabla(m)))$ of all continuous functions $x : Q(\nabla(m)) \to [-\infty, +\infty]$, defined on the Stone compact $Q(\nabla(m))$ of a Boolean algebra $\nabla(m)$, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(\nabla(m))$. It is clear that, the algebra $C_\infty(Q(\nabla(m)))$ can be considered as a subalgebra and as a regular vector sublattice of $L^0(B) = C_\infty(Q(B))$ (this means that the exact upper and lower bounds for bounded subsets of $L^0(\nabla(m))$ are the same in $L^0(B)$ and in $L^0(\nabla(m))$).

Let $L^0_{++}(\Omega)$ be the set of all positive elements $f \in L^0(\Omega)_+$ for which the support is $s(f) := \sup\limits_{n\geq 1}\{|f| > n^{-1}\} = \mathbf{1}$. With the help of the $L^0(\Omega)$-valued measure $m : B \to L^0(\Omega)$, non-increasing rearrangements

$$m(x,t) = \inf\{h \in L^0_{++}(\Omega) : m\{|x| > h\} \leq t\cdot \mathbf{1}\},\ t > 0,$$

for elements x from the algebra $L^0(B)$, is determined. Here $\mathbf{1}$ is a unit of Boolean algebra B.

In this paper, we study the properties of $L^0(\Omega)$-valued non-increasing rearrangements of $m(x,t)$, in particular, it is shown that for every fixed $x \in L^0(B)$ and $t > 0$, the equality is true

$$m(x,t) = \inf\{\|xe\|_{\infty,L^0(\Omega)} : e \in B,\ xe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1}-e) \leq t\cdot\mathbf{1}\},\ t > 0,$$

where $L^\infty(B, L^0(\Omega)) = \{x \in L^0(B) : |x| \leq f \text{ for some } f \in L^0(\Omega)_+\}$ is a Banach-Kantorovich space with an $L^0(\Omega)$-valued norm

$$\|x\|_{\infty,L^0(\Omega)} = \inf\{f \in L^0(\Omega)_+ : |x| \leq f\},\quad x \in L^\infty(B, L^0(\Omega)).$$

As an application, the notion of a symmetric Banach-Kantorovich space $(E, \|\cdot\|_E)$ over $L^0(\Omega)$ is introduced, where $E \subset L^0(B)$, and $\|\cdot\|_E$ is the $L^0(\Omega)$-valued norm in E, endowing it with the structure of the Banach-Kantorovich space. Examples of symmetric Banach-Kantorovich spaces are given, which are vector-valued analogues of classical L^p-spaces, $1 \leq p \leq \infty$, associated with a numerical σ-finite measure.

Throughout the paper, we use the terminology and notation of the theory of Boolean algebras [11], an order complete vector lattice [12], the theory of vector integration and the theory of Banach-Kantorovich spaces [1], as well as the terminology of the general theory of symmetric spaces [7].

2. Preliminaries

Let (Ω, Σ, μ) be a measurable space with σ-finite measure μ, and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the algebra of all real measurable functions on (Ω, Σ, μ) (functions coinciding almost everywhere are identified). $L^0(\Omega)$ is an order complete vector lattice with respect to the natural partial order $(f \leq g \Leftrightarrow g - f \geq 0$ almost everywhere). The weak unit is $\mathbf{1}(\omega) \equiv 1$, and the set $B(\Omega)$ of all idempotents in $L^0(\Omega)$ is a complete Boolean algebra. Denote $L^0(\Omega)_+ = \{f \in L^0(\Omega) : f \geq 0\}$.

Let X be a vector space over the field $\mathbb{R}$ of real numbers. A mapping $\|\cdot\| : X \to L^0(\Omega)$ is called an $L^0(\Omega)$-valued norm on X if the following relations hold for any $x, y \in X$ and $\lambda \in \mathbb{R}$:

(1) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$;
(2) $\|\lambda x\| = |\lambda|\ \|x\|$;
(3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a lattice-normed space over $L^0(\Omega)$. A lattice-normed space X is said to be d-decomposable if for any $x \in X$ and any decomposition $\|x\| = f_1 + f_2$ into a sum of nonnegative disjoint elements $f_1, f_2 \in L^0(\Omega)$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$, $\|x_1\| = f_1$ and $\|x_2\| = f_2$.

A net $\{x_\alpha\}_{\alpha\in A}$ of elements of $(X, \|\cdot\|)$ is said to (bo)-converge to $x \in X$ if the net $\{\|x - x_\alpha\|\}_{\alpha\in A}$ (o)-converges to zero in $L^0(\Omega)$, that is, there exists a decreasing net $\{f_\gamma\}_{\gamma\in\Gamma}$ in $L^0(\Omega)$ such that $f_\gamma \downarrow 0$ and for each $\gamma \in \Gamma$ there is $\alpha(\gamma) \in A$ with $\|x - x_\alpha\| \leq f_\gamma$ $(\alpha \geq \alpha(\gamma))$ [1, 1.3.4] (note, that the o-convergence of a net in $L^0(\Omega)$ is equivalent to its convergence almost everywhere). A net $\{x_\alpha\}_{\alpha\in A} \subset X$ is called (bo)-fundamental if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero. A lattice normed space is called (bo)-complete if every (bo)-fundamental network in it (bo)-converges to an element of this space.

The Banach-Kantorovich space over $L^0(\Omega)$ is defined as a (bo)-complete d-decomposable lattice-normed space over $L^0(\Omega)$. If a Banach Kantorovich space $(X, \|\cdot\|)$ is in addition a vector lattice and the norm $\|\cdot\|$ is monotone (i.e. the condition $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in X$) then it is called a Banach-Kantorovich lattice over $L^0(\Omega)$ (see [1],[2]). Useful examples of Banach-Kantorovich lattices are constructed using vector integration theory. Let us recall some basic notions of the theory of vector integration (see [1],[2]).

Let B be a arbitrary complete Boolean algebra with zero $\mathbf{0}$ and unit $\mathbf{1}$. A mapping $m : B \to L^0(\Omega)$ is called a $L^0(\Omega)$*-valued measure* if it satisfies the following conditions: $m(e) \geq 0$ for all $e \in B$; $m(e \vee g) = m(e) + m(g)$ for any $e, g \in B$ with $e \wedge g = \mathbf{0}$; $m(e_\alpha) \downarrow 0$ for any net $e_\alpha \downarrow \mathbf{0}$, $\{e_\alpha\} \subset B$.

A measure m is said to be *strictly positive*, if $m(e) = 0$ implies $e = \mathbf{0}$. A strictly positive $L^0(\Omega)$-valued measure m is said to be *decomposable*, if for any $e \in B$ and a decomposition $m(e) = f_1 + f_2$, $f_1, f_2 \in L^0(\Omega)_+$ there exist $e_1, e_2 \in B$, such that $e = e_1 \vee e_2$, $m(e_1) = f_1$ and $m(e_2) = f_2$. A measure m is decomposable if and only if it is a Maharam measure, that is, the measure m is strictly positive and for any $e \in B$, $0 \leq f \leq m(e)$, $f \in L^0(\Omega)$, there exist $q \in B$, $q \leq e$, such that $m(q) = f$ [13].

The following statement shows that, in the case of the Maharam measure m, there is a natural embedding of the Boolean algebra $B(\Omega)$ into the Boolean algebra B.

Proposition 2.1. [10, Proposition 2.3]. *For each $L^0(\Omega)$-valued Maharam measure $m : B \to L^0(\Omega)$, there exists a unique injective completely additive homomorphism $\varphi : B(\Omega) \to B$ such that $\varphi(B(\Omega))$ is a regular Boolean subalgebra of B, and $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$.*

Let $Q(B)$ be the Stone compact of a complete Boolean algebra B, and let $L^0(B) := C_\infty(Q(B))$ be the algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(B)$. Denotes by $C(Q(B))$ the Banach algebra of all continuous real functions on $Q(B)$ with the uniform norm $\|x\|_\infty = \sup\limits_{t \in Q(B)} |x(t)|$.

We denote by $s(x) := \sup\limits_{n \geq 1}\{|x| > n^{-1}\}$, the support of an element $x \in L^0(B)$, where $\{|x| > \lambda\} \in B$ is the characteristic function χ_{E_λ} of the set E_λ which is the closure in $Q(B)$ of the set $\{t \in Q(B) : |x(t)| > \lambda\}$, $\lambda \in \mathbb{R}$.

Let $m : B \to L^0(\Omega)$ be a Maharam measure. We identify B with the complete Boolean algebra of all idempotents in $L^0(B)$, i.e., we assume $B \subset L^0(B)$. By Proposition 1, there exists a regular Boolean subalgebra $\nabla(m)$ in B and a Boolean isomorphism φ from $B(\Omega)$ onto $\nabla(m)$ such that $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$. In this case, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) = C_\infty(Q(\nabla(m)))$ (the corresponding isomorphism will also be denoted by φ), and the algebra $C_\infty(Q(\nabla(m)))$ itself can be considered as a subalgebra and as a regular vector sublattice in $L^0(B) = C_\infty(Q(B))$ (this means that the exact upper and lower bounds for bounded subsets of $L^0(\nabla(m))$ are the same in $L^0(B)$ and in $L^0(\nabla(m))$. In addition, $L^0(B)$ is an $L^0(\nabla(m))$-module.

We now specify the vector integral of the [1] for elements of some abstract σ-Dedekind complete vector lattice. Take as a extended σ-Dedekind complete vector lattice the algebra $L^0(B)$. Consider in $L^0(B)$ the vector sublattice $\mathcal{S}(B)$ of all B-simple elements of $x = \sum\limits_{i=1}^{n} \alpha_i e_i$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $e_1, \ldots, e_n \in B$ are pairwise disjoint. Let $m : B \to L^0(\Omega)$ be a $L^0(\Omega)$-valued measure on B. If $x \in \mathcal{S}(B)$ then we put by definition

$$I_m(x) := \int x\, dm := \sum_{k=1}^{n} \alpha_k m(e_k).$$

As was described in [1], the integral I_m can be extended to the spaces of m-integrable elements $\mathcal{L}^1(B, m)$. On identifying equivalent elements, we obtain the K_σ-space $L^1(B, m)$. For each $x \in L^1(B, m)$ (the entry $x \in L^1(B, m)$ means that an equivalence class with a representative of x is considered) the formula

$$\|x\|_{1,m} := \int |x| dm$$

defines an $L^0(\Omega)$-valued norm, that is $(L^1(B, m), \|x\|_{1,m})$ is a lattice-normed space over $L^0(\Omega)$ (see [1, 6.1.3]). Moreover, in the case when $m : B \to L^0(\Omega)$ is a Maharam measure, the pair $(L^1(B, m), \|x\|_{1,m})$ is a Banach-Kantorovich space. In addition, $L^0(\nabla(m)) \cdot L^1(B, m) \subset L^1(B, m)$, $\int (\varphi(\alpha)x) dm = \alpha \int x dm$ for all $x \in L^1(B, m)$, $\alpha \in L^0(\Omega)$ [1, theorem 6.1.10].

Let $p \in [1, \infty)$, and let

$$L^p(B, m) = \{x \in L^0(B) : |x|^p \in L^1(B, m)\},$$

$$\|x\|_{p,m} := \Big[\int |x|^p dm\Big]^{\frac{1}{p}}, \quad x \in L^p(B,m).$$

It is known that for the Maharam measure m the pair $(L^p(B,m), \|x\|_{p,m})$ is the Banach-Kantorovich space [2, 4.2.2]. In addition,

$$\varphi(\alpha)x \in L^p(B,m) \ \ \forall \ \ x \in L^p(B,m), \ \alpha \in L^0(\Omega), \ 1 \le p < \infty,$$

and $\|\varphi(\alpha)x\|_{p,m} = |\alpha|\|x\|_{p,m}$.

In what follows we identify $\varphi(L^0(\Omega))$ and $L^0(\nabla(m))$, and instead of $\varphi(f)$ we will write $f \in L^0(\Omega)$.

The element $x \in L^0(B)$ is called $L^0(\Omega)$-*bounded,* if there exists an element $f \in L^0(\Omega)_+$ such that $|x| \le f$. Denote by $L^\infty(B, L^0(\Omega))$ the set of all $L^0(\Omega)$-bounded elements from $L^0(B)$. It is clear that $L^\infty(B, L^0(\Omega))$ is a subalgebra in $L^0(B)$, as well as order complete vector sublattice in $L^0(B)$, moreover, $L^0(\Omega) \subset L^\infty(B, L^0(\Omega))$, $C(Q(B)) \subset L^\infty(B, L^0(\Omega))$.

For each $x \in L^\infty(B, L^0(\Omega)$ put

$$\|x\|_{\infty, L^0(\Omega)} = \inf\{f \in L^0(\Omega)_+ : |x| \le f\}.$$

It follows directly from the definition of element $\|x\|_{\infty,L^0(\Omega)} \in L^0(\Omega)_+$ that $|x| \le \|x\|_{\infty,L^0(\Omega)}$. The following results follow from the work of [14].

Theorem 2.2. $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ *is the Banach-Kantorovich lattice over* $L^0(\Omega)$.

3. Distribution functions and decreasing rearrangements, associated with a Maharam measure

Let m be $L^0(\Omega)$-valued Maharam measure on a complete Boolean algebra B. In the rest of this section we assume that $m(\mathbf{1}) = \mathbf{1}$.

Denote by $L^0_{++}(\Omega)$ the set of all positive elements $\lambda \in L^0_+(\Omega)$ such that $s(\lambda) = \mathbf{1}$. It is clear that for any $\lambda \in L^0_{++}(\Omega)$ there exists $\lambda^{-1} \in L^0_{++}(\Omega)$ such that $\lambda \cdot \lambda^{-1} = \mathbf{1}$.

The following notation is used below for the elements $x, y \in L^0(B)$
$\{x > y\} := s((x-y)_+)$, $\{x < y\} := s((x-y)_-)$, $\{x \ge y\} := \mathbf{1} - s((x-y)_-)$,
$\{x \le y\} := \mathbf{1} - s((x-y)_+)$.

Definition 3.1. Let $0 \le x \in L^0(B)$ and $h \in L^0_{++}(\Omega)$. The $L^0(\Omega)$-valued distribution function $d(.;x) : L^0_{++}(\Omega) \to L^0(\Omega)_+$ is defined by

$$d(h;x) := m(\{x > h\}),$$

where $\{x > h\} \in B$ is the idempotent in the algebra $L^0(B)$, which is the characteristic function $\chi_{E_h(x)}$ of the closure $E_h(x)$ of the set $\{s \in Q(B) : x(s) > h(s)\}$.

Note that $L^0(\Omega)$-valued distribution function $d(x)$ is also given by

$$d(h;x) = \int \chi_{E_h(x)} dm, \ h \in L^0_{++}(\Omega).$$

A mapping $d(.,x)$ is decreasing, and right-continuous, that is, if $h_n \in L^0_{++}(\Omega), n = 0, 1, ...,$ and $h_n \downarrow h_0$, then $d(h_0;x) = \sup\limits_{n \ge 1} d(h_n;x)$.

Proposition 3.2. *Suppose* $x, y, x_n\,(n = 1, 2, \ldots)$ *belong to* $L^0(B)$, *and let* $h, g \in L^0_{++}(\Omega)$. *Then*
(i). If $|x| \le |y|$, *then* $d(h;|x|) \le d(h;|y|)$;
(ii). $d(h; g|x|) = d(\frac{h}{g}; |x|)$;
(iii.) If $x \ge 0, y \ge 0$, $h_1, h_2 \in L^0_{++}(\Omega)$, *then* $d(h_1 + h_2; x + y) \le d(h_1;x) + d(h_2;y)$;
(iv). If $|x_n| \uparrow |x|$, *then* $d(h;|x_n|) \uparrow d(h;|x|)$ *for every* $h \in L^0_{++}(\Omega)$.

Proof. (i). If $|x| \le |y|$, then $\{|x| > h\} \le \{|y| > h\}$. Consequently, $d(h;|x|) = m(\{|x| > h\}) \le m(\{|y| > h\}) = d(h;|y|)$.

(ii). $d(h;g|x|) = m(\{g|x| > h\}) = m(\{|x| > \frac{h}{g}\}) = d(\frac{h}{g};|x|)$.

(iii). If $s \in Q(B)$ and $x(s) + y(s) > h_1(s) + h_2(s)$, then either $x(s) > h_1(s)$ or $y(s) > h_2(s)$. Therefore $\{x + y > h_1 + h_2\} \le \{x > h_1\} \vee \{y > h_2\}$. Consequently,

$$d(h_1 + h_2; x + y) = m(\{x + y > h_1 + h_2\}) \le m(\{x > h_1\}) + m(\{y > h_2\}) = d(h_1;x) + d(h_2;y).$$

(iv). We fix $h \in L^0_{++}(\Omega)$ and put $G_h(x) = \{s \in Q(B) : |x(s)| > h(s)\}$, $G_h(x_n) = \{s \in Q(B) : |x_n(s)| > h(s)\}$, $(n = 1, 2, ...)$. Since $|x_n| \le |x_{n+1}|$, then $G_h(x_n) \subset G_h(x_{n+1})$. Furthermore, the condition $|x_n| \uparrow |x|$ imply that $G_h(x) = \bigcup\limits_{n=1}^{\infty} G_h(x_n)$. Hence, by the monotone convergence property of measure m, we have

$$d(h;|x_n|) = m(\{|x_n| > h\}) \uparrow m(\{|x| > h\}) = d(h;|x|).$$

□

Example 3.3. Let $x = e \in B$ and $h \in L^0_{++}(\Omega)$. Then $d(h;e) = m(\{e > h\}) = m(e)$, if $h < \mathbf{1}$, and $d(h;e) = \mathbf{0}$, if $h \ge \mathbf{1}$.

It will be worthwhile to formally compute the $L^0(\Omega)$-valued distribution function $d(x)$ of a positive B-simple element $x \in \mathcal{S}(B)$:

Example 3.4. Let $x \in \mathcal{S}(B)_+$, i.e.

$$x = \sum_{k=1}^{n} \alpha_k e_k, \tag{1}$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+ = (0, \infty)$, and $e_1, \dots, e_n$ are pairwise disjoint elements of B. Without loss of generality we may assume that $\alpha_1 > \alpha_2 > \dots > \alpha_n > 0$. Then

$$d(h;x) = \sum_{i=1}^{k} m(e_i) \quad \text{if} \ \ \alpha_{k+1} \cdot \mathbf{1} \le h < \alpha_k \cdot \mathbf{1} \quad (h \in L^0_{++}(\Omega)), \tag{2}$$

where $k = 1, 2, \dots, n$, and $\alpha_{n+1} = 0$.

Definition 3.5. Two positive elements $x, y \in L^0(B)$ are said to be m-equimeasurable, if they have the same distribution function, that is, if $d(h;x) = d(h;y)$ for all $h \in L^0_{++}(\Omega)$.

Example 3.6. Two idempotents $e_1, e_2 \in B$ are m-equimeasurable if and only if $m(e_1) = m(e_2)$ (see Example 3.3).

Example 3.7. Let $x, y \in \mathcal{S}(B)_+$, $x = \sum\limits_{k=1}^{n} \alpha_k e_k$ and $y = \sum\limits_{k=1}^{n} \beta_k g_k$, where $\alpha_k, \beta_k \in \mathbb{R}_+$, $\alpha_1 > \alpha_2 > \dots > \alpha_n > 0$, $\beta_1 > \beta_2 > \dots > \beta_n > 0$, and $\{e_k\}$, respectively, $\{g_k\}$ are pairwise disjoint elements of B. By Example 3.4, we have

$$d(h;x) = \sum_{i=1}^{k} m(e_i), \quad \text{if} \ \ \alpha_{k+1} \cdot \mathbf{1} \le h < \alpha_k \cdot \mathbf{1},$$

$$d(h;y) = \sum_{i=1}^{k} m(g_i), \quad \text{if} \ \ \beta_{k+1} \cdot \mathbf{1} \le h < \beta_k \cdot \mathbf{1} \ (h \in L^0_{++}(\Omega)),$$

where $k = 1, 2, \dots, n$, and $\alpha_{n+1} = \beta_{n+1} = 0$.

From equality $d(h;x) = d(h;y)$ we get $\alpha_k = \beta_k$ and $\sum\limits_{i=1}^{k} m(e_i) = \sum\limits_{i=1}^{k} m(g_i)$ for all $k = 1, \dots, n$. Of the last equalities by $k = 1$ we have $m(e_1) = m(g_1)$. Further, if $k = 2$ the equality $m(e_1) + m(e_2) = m(g_1) + m(g_2)$ is true, thus $m(e_2) = m(g_2)$, etc., when $k = n$ we get $m(e_n) = m(g_n)$.

Thus the elements x and y m-equimeasurable if and only if $\alpha_k = \beta_k$ and $m(e_k) = m(g_k)$ for all $k = 1, \dots, n$.

Note that the statement from Example 4 holds for simple elements $a, b \in \mathcal{S}(B)_+$, $a = \sum_{k=1}^{n} t_k p_k$ and $b = \sum_{k=1}^{n} s_k q_k$, where $t_k, s_k \in \mathbb{R}_+$, $0 < t_1 < t_2 < \ldots < t_n$, $0 < s_1 < s_2 < \ldots < s_n$, and $\{p_k\}$, respectively, $\{q_k\}$ are pairwise disjoint elements of B. It is clear that $a = x = \sum_{k=0}^{n-1} t_{n-k} p_{n-k}$ and $b = y = \sum_{k=0}^{n-1} s_{n-k} g_{n-k}$, where $t_k, s_k \in \mathbb{R}_+$ and $t_n > t_{n-1} > \ldots > t_1 > 0$, $s_n > s_{n-1} > \ldots > s_1 > 0$. According to the above, we have that the elements x and y m-equimeasurable if and only if $t_k = s_k$ and $m(p_k) = m(q_k)$ for all $k = 1, \ldots, n$.

Definition 3.8. The $L^0(\Omega)$-valued decreasing rearrangement of $x \in L^0(B)$ is the mapping $m(.;x) : (0, \infty) \to L^0(\Omega)_+$, defined by

$$m(t;x) = \inf\{h \in L^0_{++}(\Omega) : d(h;|x|) \leq t \cdot \mathbf{1}\},\ t > 0. \tag{3}$$

It is clear that $m(t;x) \leq m(s;x)$ for $s < t$. In addition, the map $m(t;x)$ has the following useful continuity property.

Proposition 3.9. *If* $t_n, t > 0, n = 1, 2, ...,$ *and* $t_n \downarrow t$, *then* $m(t;x) = \sup_{n \geq 1} m(t_n;x)$.

Proof. Let $t_n, t > 0, n = 1, 2, ...,$ and $t_n \downarrow t$. Then

$$m(t;x) = \inf\{h \in L^0_{++}(\Omega) : d(h;|x|) \leq \inf_{n\geq 1} t_n\} = \inf\left\{h \in L^0_{++}(\Omega) : d(h;|x|) \leq t_n \text{ for all } n \geq 1\right\}$$

$$= \sup_{n \geq 1} \left\{ \inf\{h \in L^0_{++}(\Omega) : d(h;|x|) \leq t_n\}\right\} = \sup_{n\geq 1} m(t_n;x),$$

i.e. $m(t;x) = \sup_{n \geq 1} m(t_n;x)$. □

Example 3.10. Now we compute the $L^0(\Omega)$-valued decreasing rearrangement of the B-simple element x given by (1). Let $h \in L^0_{++}(\Omega)$ and let $\lambda_k = \sum_{i=1}^{k} m(e_i), (k = 1, 2, \ldots, n)$. Then according to (2) we have

$$d(h;x) = \begin{cases} \mathbf{0}, & h \geq \alpha_1 \cdot \mathbf{1}; \\ \lambda_k, & \alpha_{k+1} \cdot \mathbf{1} \leq h < \alpha_k \cdot \mathbf{1},\ \ 1 \leq k \leq n-1; \\ \lambda_n, & \mathbf{0} < h \leq \alpha_n \mathbf{1}. \end{cases}$$

Referring to (3), we see that $m(t;x) = \mathbf{0}$ if $t \cdot \mathbf{1} \geq \lambda_n$. Also, if $\lambda_n > t \cdot \mathbf{1} \geq \lambda_{n-1}$, then $m(t;x) = \alpha_n \cdot \mathbf{1}$, and if $\lambda_{n-1} > t \cdot \mathbf{1} \geq \lambda_{n-2}$, then $m(t;x) = \alpha_{n-1} \cdot \mathbf{1}$, and so on. Hence,

$$m(t;x) = \begin{cases} \alpha_1 \cdot \mathbf{1}, & \mathbf{0} < t \cdot \mathbf{1} \leq \lambda_1; \\ \alpha_k \cdot \mathbf{1}, & \lambda_k > t \cdot \mathbf{1} \geq \lambda_{k-1},\ \ 2 \leq k \leq n-1; \\ \mathbf{0}, & t \cdot \mathbf{1} > \lambda_n. \end{cases} \tag{4}$$

Example 3.11. Let $x, y \in \mathcal{S}(B)_+$, $x = \sum_{k=1}^{n} \alpha_k e_k$ and $y = \sum_{k=1}^{n} \beta_k g_k$ (see Example 3.7). Let $m(t;x) = m(t;y)$ for all $t > 0$. Then the elements x and y are m-equimeasurable.

Indeed, if $\mathbf{0} < t \cdot \mathbf{1} \leq \lambda_1$, then by (4), $m(t;x) = \alpha_1 \cdot \mathbf{1}$. Hence $m(t;y) = \alpha_1 \cdot \mathbf{1}$ if and only if $\beta_1 = \alpha_1$ and $\mu_1 = m(g_1) = m(e_1) = \lambda_1$. Also, if $\lambda_2 > t \cdot \mathbf{1} \geq \lambda_1$, then $m(t;x) = \alpha_2 \cdot \mathbf{1} = m(t;y)$. Because, $\beta_2 = \alpha_2$ and $\mu_2 = m(g_1) + m(g_2) = m(e_1) + m(e_2) = \lambda_2$, i.e. $m(g_2) = m(e_2)$, and so on. For $t \cdot \mathbf{1} > \lambda_n$ we have $\beta_n = \alpha_n$ and $m(g_n) = m(e_n)$. Therefore, the elements x and y are m-equimeasurable (see Example 3.7).

For fixed $x \in L^0(B)$, $t > 0$, we put

$$\xi(t;x) = \inf\{\|xe\|_{\infty, L^0(\Omega)} : e \in B,\ xe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\}. \tag{5}$$

Proposition 3.12. *For any* $x \in L^0(B)$, $t > 0$ *the equality*

$$\xi(t;x) = m(t;x),$$

is true, in addition, $m\{|x| > m(t;x)\} \leq t \cdot \mathbf{1}$.

Proof. Fix $t > 0$, and put

$$H(x) = \{h \in L^0_{++}(\Omega) : d(h;|x|) \leq t \cdot \mathbf{1}\}.$$

If $h_1, h_2 \in H(x)$, $e = \{h_1 < h_2\}$, then $h = h_1 \wedge h_2 = h_1 \cdot e + h_2 \cdot (\mathbf{1} - e) \in L^0_{++}(\Omega)$, in this case, by virtue of Proposition 2.1, we have

$$d(h;|x|) = m(\{|x| > h\}) = m(\{|x| > h_1\}) \cdot e + m(\{|x| > h_2\}) \cdot (\mathbf{1} - e) \leq t \cdot e + t \cdot (\mathbf{1} - e) = t \cdot \mathbf{1},$$

i.e. $h_1 \wedge h_2 \in H(x)$. Using mathematical induction, we obtain that for any finite set $\{h_i\}_{i=1}^n \subset H(x)$, the inclusion $\bigwedge\limits_{i=1}^{n} h_i \in H(x)$ is true. Since (Ω, Σ, μ) is a measurable space with σ-finite measure μ, the Boolean algebra $B(\Omega)$ has a countable type, i.e. any family of pairwise disjunct elements from $B(\Omega)$ is at most countable. Hence there exists a sequence $\{h_k\}_{k=1}^{\infty} \subset H(x)$ for which $h_k \downarrow g$(see, for example, [12, Chapter VI, §3]), where

$$g = m(t;x) = \inf\{h \in L^0_{++}(\Omega) : d(h;|x|) \leq t \cdot \mathbf{1}\} \in L^0(\Omega)_+.$$

Since $h_k \downarrow g$ and for all $k \in \mathbb{N}$ the inequality $d(h_k;|x|) \leq t \cdot \mathbf{1}$ is true, then

$$t \cdot \mathbf{1} \geq d(h_k;|x|) \uparrow d(g;|x|).$$

In particular, the inequality

$$m\{|x| > g\} = d(g;|x|) \leq t \cdot \mathbf{1} \quad \text{i.e.} \quad m(\mathbf{1} - e) \leq t \cdot \mathbf{1},$$

is true, where $e = \{|x| \leq g\} \in B$. Since $|x|e \leq g \in L^0(\Omega)_+$, then $xe \in L^\infty(B, L^0(\Omega))$, in addition, the inequality $\|xe\|_{\infty, L^0(\Omega)} \leq g$ is true. Hence, $\xi(t;x) \leq g = m(t;x)$.

To prove the reverse inequality, we set

$$E(x) = \{e \in B : xe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\}.$$

By (5), we have $\xi(t;x) = \inf\{\|xe\|_{\infty, L^0(\Omega)} : e \in E(x)\}$. Suppose that the inequality $m(t;x) = g \leq \xi(t;x)$ is not satisfied. Therefore, there exists an $e \in E(x)$ for which the inequality $g \leq \|xe\|_{\infty, L^0(\Omega)}$ does not hold. In particular, this means that there exist $0 \neq q \in B(\Omega)$, $\varepsilon > 0$, for which the inequalities

$$|xeq| \leq \|xeq\|_{\infty, L^0(\Omega)} = \|xe\|_{\infty, L^0(\Omega)} \cdot q \leq qg + \varepsilon q$$

are true.

Let $r = \{|xe| > qg + 2\varepsilon \cdot q\} \in B$. From the relations

$$|xe|rq \geq (qg + 2\varepsilon \cdot q)q = qg + 2\varepsilon \cdot q > qg + \varepsilon \cdot q \geq |xeq| = |xe|q$$

it follows that $|xe|rq > |xe|q$, which is impossible.

Thus, the inequality $m(t;x) \leq \xi(t;x)$ is satisfied. Consequently, the equality $\xi(t;x) = m(t;x)$ is true. $\square$

In the next proposition, some basic properties of $L^0(\Omega)$-valued decreasing rearrangements are collected.

Proposition 3.13. *Let* $x, y \in L^0(B)$, $t > 0$. *Then*

(*i*). $\xi(t; x) = \inf\{\|x - z\|_{\infty, L^0(\Omega)} : z \in R_t,\ (x - z) \in L^\infty(B, L^0(\Omega))\}$, *where* $R_t = \{z \in L^0(B) : m(s(|z|)) \leq t \cdot \mathbf{1}\}$, $s(|z|)$ *the support of* $|z|$.

(*ii*). $m(t; x) = m(t; |x|)$ *and* $m(t; \lambda x) = |\lambda| m(t; x)$ *for all* $\lambda \in \mathbb{R}$;

(*iii*). *If* $m(s(x)) \leq t \cdot \mathbf{1}$, *then* $m(t; x) = \mathbf{0}$;

(*iv*). *If* $|x| \leq |y|$, *then* $m(t; x) \leq m(t; y)$;

(*v*). $m(t_1 + t_2; x + y) \leq m(t_1; x) + m(t_2; y)$ *for all* $t_1, t_2 > 0$.

(*vi*). *If* $x \in L^\infty(B, L^0(\Omega))$, $t_n > 0$, $n = 1, 2, ...,$ *and* $t_n \downarrow 0$, *then*

$$\|x\|_{\infty, L^0(\Omega)} = \sup_{n \geq 1} m(t_n; x).$$

(*vii*). *If* $x_n, x \in L^0(B)$, $n \in \mathbb{N}$ *and* $0 \leq x_n \uparrow x$, *then* $m(t, x_n) \uparrow m(t, x)$ *for all* $t > 0$.

Proof. (*i*). We put

$$h = \inf\{\|x - z\|_{\infty, L^0(\Omega)} : z \in R_t,\ (x - z) \in L^\infty(B, L^0(\Omega))\}.$$

If $z \in R_t$, $x - z \in L^\infty(B, L^0(\Omega))$ and $e = \mathbf{1} - s(|z|)$, then $e \in B$, $(x - z)e = xe$ and $|xe| \leq |x - z|$. Hence $\|xe\|_{\infty, L^0(\Omega)} \leq \|x - z\|_{\infty, L^0(\Omega)}$. Since $z \in R_t$, then $m(\mathbf{1} - e) = m(s(|z|)) \leq t \cdot \mathbf{1}$. Therefore,

$$\xi(t; x) = \inf\{\|xe\|_{\infty, L^0(\Omega)} : e \in B,\ xe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\} \leq h.$$

To obtain the reverse inequality, we fix $t > 0$ and for elements $u = s(x_+) - s(x_-) \in L^0(B)$, $p = \{|x| > \xi(t; x)\} \in B$ put $z = up|x|$, $e = \mathbf{1} - p$. Then $x - z = u|x| - up|x| = u(\mathbf{1} - p)|x| = uex \in L^\infty(B, L^0(\Omega))$, moreover,

$$\|x - z\|_{\infty, L^0(\Omega)} = \|ue|x|\,\|_{\infty, L^0(\Omega)} = \|ex\|_{\infty, L^0(\Omega)} = \|e|x|\|_{\infty, L^0(\Omega)} \leq \xi(t; x).$$

By Proposition 3.12, we have

$$m(s(|z|)) = m\{|x| > \xi(t; x)\} \leq t \cdot \mathbf{1},$$

i.e. $z \in R_t$. Therefore,

$$\inf\{\|x - z\|_{\infty, L^0(\Omega)} : z \in R_t,\ (x - z) \in L^\infty(B, L^0(\Omega))\} \leq \xi(t; x).$$

(*ii*). The equality $m(t; x) = m(t; |x|)$ follows directly from the definition of the mapping $m(t; x)$. If $0 \neq \lambda \in \mathbb{R}$, then

$$m(t; \lambda x) = \inf\{h \in L^0_{++}(\Omega) : m\{|\lambda x| > h\} \leq t \cdot \mathbf{1}\} =$$
$$= \inf\{h \in L^0_{++}(\Omega) : m\{|x| > |\lambda|^{-1} h\} \leq t \cdot \mathbf{1}\} =$$
$$= \inf\{|\lambda| g \in L^0_{++}(\Omega) : m\{|x| > g\} \leq t \cdot \mathbf{1}\} =$$
$$= |\lambda| \inf\{g \in L^0_{++}(\Omega) : m\{|x| > g\} \leq t \cdot \mathbf{1}\} = |\lambda| m(t; x).$$

(*iii*). Since $m(s(x)) \leq t \cdot \mathbf{1}$ and $\{|x| > \frac{1}{n} \cdot \mathbf{1}\} \leq s(x)$, then $m\{|x| > \frac{1}{n} \cdot \mathbf{1}\} \leq t \cdot \mathbf{1}$ for all $n \in \mathbb{N}$. Therefore, $m(t; x) = \mathbf{0}$.

(*iv*). If $|x| \leq |y|$, then $d(h; |x|) \leq d(h; |y|)$ for all $h \in L^0_{++}(\Omega)$, and therefore $m(t; x) \leq m(t; y)$.

(*v*). For all $h_1, h_2 \in L^0(\Omega)_+$ the following inequality holds

$$\{|x + y| > h_1 + h_2\} \leq \{|x| > h_1\} \vee \{|y| > h_2\}.$$

Hence $m\{|x + y| > h_1 + h_2\} \leq m\{|x| > h_1\} + m\{|y| > h_2\}$.

We fix $\varepsilon > 0$ and set $h_1 = m(t_1; x)$, $h_2 = m(t_2; y) + \varepsilon \cdot \mathbf{1}$. Using the inequality $m\{|y| > h_2\} \leq m\{|y| > m(t_2; y)\}$ and Proposition 3.12, we have

$$m\{|x+y| > m(t_1; x) + m(t_2; y) + \varepsilon \cdot \mathbf{1}\} \leq m\{|x| > m(t_1; x)\} + m\{|y| > m(t_2; y) + \varepsilon \cdot \mathbf{1}\} \leq$$

$$\leq t_1 \cdot \mathbf{1} + m\{|y| > m(t_2; y)\} \leq (t_1 + t_2) \cdot \mathbf{1},$$

i.e. $m\{|x+y| > m(t_1; x) + m(t_2; y) + \varepsilon \cdot \mathbf{1}\} \leq (t_1 + t_2) \cdot \mathbf{1}$.

Since $m(t_1; x) + m(t_2; y) + \varepsilon \cdot \mathbf{1} \in L^0_{++}(\Omega)$, then from the definition of the mapping $m(t; x)$ the following inequality follows

$$m((t_1 + t_2); x + y) \leq m(t_1; x) + m(t_2; y) + \varepsilon \cdot \mathbf{1}.$$

From here, at $\varepsilon \downarrow 0$, we obtain the required inequality

$$m((t_1 + t_2); x + y) \leq m(t_1; x) + m(t_2; y).$$

(vi). First we show that for all $q \in B(\Omega)$, $x \in L^\infty(B, L^0(\Omega))$, $t > 0$ the equality $\xi(t; qx) = q\xi(t; x)$ is true. Since

$$\xi(t; qx) = \inf\{\|qxe\|_{\infty, L^0(\Omega)} : e \in B,\ qxe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\} =$$

$$= \inf\{q\|qxe\|_{\infty, L^0(\Omega)} : e \in B,\ qxe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\} = q\xi(t; qx),$$

then from the inequality $|qx| \leq |x|$ and Proposition 3.12 follows the inequality $q\xi(t; qx) \leq q\xi(t; x)$ (see property (iv)).

On the other hand, if $e \in B$, $qxe \in L^\infty(B, L^0(\Omega))$ and $m(\mathbf{1} - e) \leq t \cdot \mathbf{1}$, then

$$\xi(t; x) = q\xi(t; x) + (\mathbf{1} - q)\xi(t; x) \leq q\|qxe\|_{\infty, L^0(\Omega)} + (\mathbf{1} - q)\xi(t; x).$$

Therefore,

$$q\xi(t; x) \leq q\|qxe\|_{\infty, L^0(\Omega)} \leq \|qxe\|_{\infty, L^0(\Omega)},$$

and hence $q\xi(t; x) \leq \xi(t; qx)$. Thus, the equality $\xi(t; qx) = q\xi(t; x)$ is true.

If $x \in L^\infty(B, L^0(\Omega))$, then $x \cdot \mathbf{1} \in L^\infty(B, L^0(\Omega))$, and it follows directly from the definition of the mapping $\xi(t; x)$ that $\xi(t; x) \leq \|x\|_{\infty, L^0(\Omega)}$ for all $t > 0$. Moreover, the inequality $\xi(t_1; x) \leq \xi(t_2; x)$ is true at $0 < t_2 < t_1$.

Thus, $\xi(t_n; x) \uparrow z \leq \|x\|_{\infty, L^0(\Omega)}$ at $t_n \downarrow 0$ for some $z \in L^0(\Omega)_+$.

If $z \neq \|x\|_{\infty, L^0(\Omega)}$, then for any $\varepsilon > 0$ exist $q_\varepsilon \in B(\Omega)$, that

$$\xi(t_n; xq_\varepsilon) = \xi(t_n; x)q_\varepsilon \leq zq_\varepsilon < q_\varepsilon \cdot \|x\|_{\infty, L^0(\Omega)}$$

for all $t_n \in (0, \varepsilon)$. Hence, by virtue of proposition 3.12, we get that

$$m(t_n; xq_\varepsilon) = \xi(t_n; xq_\varepsilon) \leq zq_\varepsilon \text{ for all } t_n \in (0, \varepsilon).$$

Again using proposition 3.12, we have,

$$m(\{|xq_\varepsilon| > zq_\varepsilon\}) \leq m(\{|xq_\varepsilon| > m(t_n, xq_\varepsilon)\}) \leq t_n \cdot \mathbf{1}$$

for all $t_n \in (0, \varepsilon)$. Therefore, $m(\{|xq_\varepsilon| > zq_\varepsilon\}) = 0$. This means that $|xq_\varepsilon| \leq zq_\varepsilon$, in particular, $\|xq_\varepsilon\|_{\infty, L^0(\Omega)} \leq zq_\varepsilon$, which is not the case. Thus,

$$\|x\|_{\infty, L^0(\Omega)} = \sup_{n \geq 1} \xi(t_n; x).$$

(vii). Since $m(t, x_n) \leq m(t, x_{n+1}) \leq m(t, x)$ for all $n = 1, 2, \ldots$ and $t > 0$, it is clear that $m(t, x_n) \uparrow_n$ and that

$$\sup_{n \geq 1} m(t, x_n) \leq m(t, x)$$

for all $t > 0$. For the proof of the reverse inequality, it may be assumed that $t > 0$ is such that $m(t, x_n) < h$, for some $h \in L^0_{++}(\Omega)$. Hence $d(h, x_n) \leq t \cdot \mathbf{1}$ for all n. Since $d(h, x_n) \uparrow d(h, x)$ (Proposition 3.2(iv)), it follows that $d(h, x) \leq t \cdot \mathbf{1}$. Thus $m(t, x) < h$. This suffices to show that $m(t, x) \leq \sup_{n \geq 1} m(t, x_n)$. Consequently $m(t, x) = \sup_{n \geq 1} m(t, x_n)$. □

Corollary 3.14. *For any* $x, y \in L^\infty(B, L^0(\Omega))$, $t > 0$ *the inequality holds*

$$|\xi(t; x) - \xi(t; y)| \leq \|x - y\|_{\infty, L^0(\Omega)}.$$

Proof. By proposition 3.13 (v) we have that

$$\xi(t_1 + t_2; x) = \xi(t_1 + t_2; y + (x - y)) \leq \xi(t_1; y) + \xi(t_2; x - y) \leq \xi(t_1; y) + \|x - y\|_{\infty, L^0(\Omega)}.$$

Similarly,

$$\xi(t_1 + t_2; y) = \xi(t_1 + t_2; x + (y - x)) \leq \xi(t_2; x) + \xi(t_1; y - x) \leq \xi(t_2; x) + \|x - y\|_{\infty, L^0(\Omega)}.$$

Assuming $t_1 = t$, $t_2 = 0$, in these inequalities, we obtain

$$\xi(t; x) \leq \xi(t; y) + \|x - y\|_{\infty, L^0(\Omega)} \text{ and } \xi(t; y) \leq \xi(t; x) + \|x - y\|_{\infty, L^0(\Omega)},$$

from which it follows that $|\xi(t; x) - \xi(t; y)| \leq \|x - y\|_{\infty, L^0(\Omega)}$. □

4. Symmetric Banach-Kantorovich spaces

In this section, a class of symmetric Banach-Kantorovich spaces is introduced and examples of such spaces are given. We will need the following useful property about the equality of integrals for integrable m-equimeasurable elements.

Theorem 4.1. *Let* $0 \leq x \in L^0(B)$ *and* $0 \leq y \in L^1(B, m)$. *If* x *and* y m*-equimeasurable, then* $x \in L^1(B, m)$ *and* $\int x dm = \int y dm$.

Proof. Let $x, y \in \mathcal{S}(B)_+$, $x = \sum\limits_{k=1}^{n} \alpha_k e_k$ and $y = \sum\limits_{k=1}^{n} \beta_k g_k$. Then by m-equimeasurable x and y (see Example 4),

$$\int x dm = \sum_{k=1}^{n} \alpha_k m(e_k) = \sum_{k=1}^{n} \beta_k m(g_k) = \int y dm.$$

Let now $x \in L^0(B)_+$, $0 \leq y \in L^1(B, m)$ and $d(.; x) = d(.; y)$. Let us, first assume that $y \in C(Q(B))$. Recall that by assumption $m(\mathbf{1}) = \mathbf{1}$, and therefore $C(Q(B)) \subset L^1(B, m))$, in this case, $\|y\|_{1,m} \leq \|y\|_\infty \mathbf{1}$. Without loss of generality we may assume that $\|y\|_\infty \leq 1$ (see Proposition 3.7(ii)). Since $d(.; x) = d(.; y)$, then $m\{x > \mathbf{1}\} = m\{y > \mathbf{1}\} = \mathbf{0}$, that is $\|x\|_\infty \leq 1$, and therefore $x \in L^1(B, m)$.

Note that from the identities

$$m\{x \leq t \cdot \mathbf{1}\} = m(\mathbf{1}) - m\{x > t \cdot \mathbf{1}\} = m(\mathbf{1}) - m\{y > t \cdot \mathbf{1}\} = m\{y \leq t \cdot \mathbf{1}\}$$

using the equalities

$$\{s \cdot \mathbf{1} < x \leq t \cdot \mathbf{1}\} = \{x > s \cdot \mathbf{1}\} - \{x > t \cdot \mathbf{1}\}, \{s \cdot \mathbf{1} < y \leq t \cdot \mathbf{1}\} = \{y > s \cdot \mathbf{1}\} - \{y > t \cdot \mathbf{1}\}$$

for any $0 < s < t$, we obtain

$$m\{s \cdot \mathbf{1} < x \leq t \cdot \mathbf{1}\} = m\{x > s \cdot \mathbf{1}\} - m\{x > t \cdot \mathbf{1}\} = m\{y > s \cdot \mathbf{1}\} - m\{y > t \cdot \mathbf{1}\} = m\{s \cdot \mathbf{1} < y \leq t \cdot \mathbf{1}\}.$$

We now show that there are increasing sequences of positive simple elements $x_n \in C(Q(B))$ and $y_n \in C(Q(B))$ $(n = 1, 2, \ldots)$, such that $x_n \uparrow x$ and $y_n \uparrow y$ and the equalities $d(t; x_n) = d(t; y_n)$ are true for all $n = 1, 2, \ldots$.

Consider the following two sequences

$$x_n = (\sum_{k=1}^{2^n} \frac{k-1}{2^n} e_k) \uparrow x, \quad y_n = (\sum_{k=1}^{2^n} \frac{k-1}{2^n} g_k) \uparrow y,$$

where $e_k = \{\frac{k-1}{2^n} \cdot \mathbf{1} < x \leq \frac{k}{2^n} \cdot \mathbf{1}\}$, $g_k = \{\frac{k-1}{2^n} \cdot \mathbf{1} < y \leq \frac{k}{2^n} \cdot \mathbf{1}\}$. Since $d(.;x) = d(.;y)$, then $m(e_k) = m(g_k)$, and therefore $d(t;x_n) = d(t;y_n)$ (see Example 4). Hence,

$$\int x_n dm = \sum_{k=1}^{2^n} \frac{k-1}{2^n} m(e_k) = \sum_{k=1}^{2^n} \frac{k-1}{2^n} m(g_k) = \int y_n dm \uparrow \int y dm.$$

Thus,

$$\int x dm = (o)\text{-} \lim_{n\to\infty} \int x_n dm = \int y dm.$$

Now let y be an arbitrary positive element $L^1(B,m)$. Consider two increasing sequences of positive elements of $C(Q(B))$

$$x_n = xp_n \uparrow x, \quad y_n = yq_n \uparrow y,$$

where $p_n = \{x \leq n \cdot \mathbf{1}\}$, $q_n = \{y \leq n \cdot \mathbf{1}\}$. It is clear that

$$m\{x_n > t \cdot \mathbf{1}\} = m\{xp_n > t \cdot \mathbf{1}\} = m\{t \cdot \mathbf{1} < x \leq n \cdot \mathbf{1}\} = m\{t \cdot \mathbf{1} < y \leq n \cdot \mathbf{1}\} =$$

$$= m\{yq_n > t \cdot \mathbf{1}\} = m\{y_n > t \cdot \mathbf{1}\}$$

for any $t \in \mathbb{R}^+$. Since y_n is an integrable element of $C(Q(B))$, it follows from the above, that $\int x_n dm = \int y_n dm$. At the same time, there is a limit

$$(o)\text{-} \lim_{n\to\infty} \int x_n dm = (o)\text{-} \lim_{n\to\infty} \int y_n dm = \int y dm.$$

Hence $x \in L^1(B,m)$ and $\int x dm = \int y dm$. □

Corollary 4.2. *Let $0 \leq x \in L^0(B)$ and $0 \leq y \in L^p(B,m)$, $p > 1$. If x and y m-equimeasurable, then $x \in L^p(B,m)$ and $\|x\|_{p,m} = \|y\|_{p,m}$.*

Proof. Since $y^p \in L^1(B,m)$ and

$$m\{x^p > t \cdot \mathbf{1}\} = m\{x > t^{\frac{1}{p}} \cdot \mathbf{1}\} = m\{y > t^{\frac{1}{p}} \cdot \mathbf{1}\} = m\{y^p > t \cdot \mathbf{1}\}$$

for any $t \in \mathbb{R}^+$, $p > 1$, then for the elements x^p and y^p the proof of Theorem 4.1 is preserved, by virtue of which we obtain

$$x^p \in L^1(B,m) \text{ and } \int x^p dm = \int y^p dm,$$

i.e. $x \in L^p(B,m)$ and $\|x\|_{p,m} = \|y\|_{p,m}$. □

Definition 4.3. Let E be a nonzero linear subspace in $L^0(B)$ with the property of ideality, i.e. for $x \in L^0(B)$ and $y \in E$, from $|x| \leq |y|$ it follows that $x \in E$. Consider the $L^0(\Omega)$-valued norm $\|\cdot\|_E$ on E, which endows E with the structure of a Banach-Kantorovich lattice. We say that E is a symmetric Banach-Kantorovich space over $L^0(\Omega)$, if m-equimeasurablity of the elements x and y, where $x \in L^0(B)_+$, $0 \leq y \in E$, implies that $x \in E$ and $\|x\|_E = \|y\|_E$.

The main and most important examples of symmetric Banach-Kantorovich spaces are the spaces $L^p(B,m)$, $1 \leq p < \infty$, and $L^\infty(B, L^0(\Omega))$.

Theorem 4.4. (i). *$(L^p(B,m), \|\cdot\|_{p,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$ for every $1 \leq p < \infty$.*

(ii). *$(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty, L^0(\Omega)})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.*

Proof. (*i*). According to [1, Section 6.1], linear subspace $L^1(B,m) \subset L^0(B)$ has the ideality property, moreover, the norm $\|\cdot\|_{1,m}$ is monotone, and the space $L^1(B,m)$, equipped with this norm, is a Banach-Kantorovich lattice. It remains to apply theorem 2, by virtue of which the pair $(L^1(B,m), \|\cdot\|_{1,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

Now let $|x| \le |y|,\ x \in L^0(B),\ y \in L^p(B,m)$, where $1 < p < \infty$. Since $|x|^p \le |y|^p \in L^1(B,m)$, then $|x|^p \in L^1(B,m)$ and

$$\|x\|_{p,m}^p = \||x|^p\|_{1,m} \le \||y|^p\|_{1,m} = \|y\|_{p,m}^p,$$

and therefore $\|x\|_{p,m} \le \|y\|_{p,m}$, i.e. $\|\cdot\|_{p,m}$ is $L^0(\Omega)$-valued monotone norm on $L^p(B,m)$, which endows $L^p(B,m)$ with the structure of a Banach-Kantorovich lattice over $L^0(\Omega)$. It remains to apply Corollary 2, by virtue of which the pair
$(L^p(B,m), \|\cdot\|_{p,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

(*ii*). By Theorem 1, the pair $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a Banach-Kantorovich lattice, moreover, it is clear that $L^\infty(B, L^0(\Omega))$ has the ideality property and the norm $\|\cdot\|_{\infty,L^0(\Omega)}$ is monotone on $L^\infty(B, L^0(\Omega))$.

Let $x \in L^0(B),\ y \in L^\infty(B, L^0(\Omega))$, and let x and y be m-equimeasurable. Assign $h(\varepsilon) = \|y\|_{\infty,L^0(\Omega)} + \varepsilon \cdot \mathbf{1},\ \varepsilon > 0$. Since $h(\varepsilon) \in L^0_{++}(\Omega)$, then

$$m\{|x| > h(\varepsilon)\} = m\{|y| > h(\varepsilon)\} = \mathbf{0}.$$

Hence, $|x| \le h(\varepsilon)$, and therefore $x \in L^\infty(B, L^0(\Omega)$, moreover, $\|x\|_{\infty,L^0(\Omega)} \le h(\varepsilon)$ for every $\varepsilon > 0$. From this it follows that $\|x\|_{\infty,L^0(\Omega)} \le \|y\|_{\infty,L^0(\Omega)}$.

Let's put now $h_1(\varepsilon) = \|x\|_{\infty,L^0(\Omega)} + \varepsilon \cdot \mathbf{1} \in L^0_{++}(\Omega),\ \varepsilon > 0$. Using equalities

$$m\{|y| > h_1(\varepsilon)\} = m\{|x| > h_1(\varepsilon)\} = \mathbf{0},$$

we get that $\|y\|_{\infty,L^0(\Omega)} \le h_1(\varepsilon)$ for every $\varepsilon > 0$. This means that $\|y\|_{\infty,L^0(\Omega)} \le \|x\|_{\infty,L^0(\Omega)}$. Thus, $\|x\|_{\infty,L^0(\Omega)} = \|y\|_{\infty,L^0(\Omega)}$.

Consequently, $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$. □

Following the general theory of functional symmetric spaces, consider a space $L^1(B,m) \cap L^\infty(B, L^0(\Omega)$ with a norm

$$\|x\|_{L^1 \cap L^\infty} = \|x\|_{1,m} \vee \|x\|_{\infty,L^0(\Omega)}, x \in L^1(B,m) \cap L^\infty(B, L^0(\Omega)).$$

Proposition 4.5. $(L^1(B,m) \cap L^\infty(B, L^0(\Omega)), \|\cdot\|_{L^1 \cap L^\infty})$ *is a symmetric Banach-Kantorovich space over* $L^0(\Omega)$.

Proof. Since $m(\mathbf{1}) = \mathbf{1}$, and for every $x \in L^\infty(B, L^0(\Omega))$ the inequality $|x| \le \|x\|_{\infty,L^0(\Omega)}$ is true, then $L^\infty(B, L^0(\Omega)) \subset L^1(B,m)$, moreover, $\|x\|_{1,m} \le \|x\|_{\infty,L^0(\Omega)}$. Hence, $L^1(B,m) \cap L^\infty(B, L^0(\Omega) = L^\infty(B, L^0(\Omega))$ and $\|x\|_{1,m} \vee \|x\|_{\infty,L^0(\Omega)} = \|x\|_{\infty,L^0(\Omega)}$. Thus, the pair

$$(L^1(B,m) \cap L^\infty(B, L^0(\Omega)), \|\cdot\|_{L^1 \cap L^\infty}(B, L^0(\Omega)) = (L^\infty(B, L^0(\Omega), \|\cdot\|_{\infty,L^0(\Omega)})$$

is a symmetric Banach-Kantorovich space over $L^0(\Omega)$ (see Theorem 4.4 (*ii*)). □

5. Conclusion.

In the present work we consider vector-valued Maharam measures m defined on a complete Boolean algebra B with values in the algebra $L^0(\Omega)$ of all measurable real functions defined on a measurable space (Ω, Σ, μ) with a σ-finite numerical measure μ. We define and study the $L^0(\Omega)$-valued decreasing rearrangements of functions from $C_\infty(Q(B))$ associated with the measure m and taking values in the algebra $L^0(\Omega)$ (here $C_\infty(Q(B))$ is the commutative unital algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets from the Stone compact $Q(B)$ of B). A class of symmetric Banach-Kantorovich spaces associated with a measure m is introduced and examples of such spaces are given.

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