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A free boundary problem with a Stefan condition for a ratio-dependent predator-prey model

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Abstract. In this paper we study a ratio-dependent predator-prey model with a free boundary caused by predator-prey interaction over a one dimensional habitat. We study the long time behaviors of the two species and prove a spreading-vanishing dichotomy; namely, as t goes to infinity, both prey and predator successfully spread to the whole space and survive in the new environment, or they spread within a bounded area and eventually die out. The criteria governing spreading and vanishing are obtained.

Keywords: free boundary, a prior bounds, existence and uniqueness, ratio-dependent model, spreading-vanishing dichotomy.

MSC (2020): 35K20, 35R35

1. Introduction

In this paper, we consider the following ratiodependent predatorprey model,

$$\begin{cases} u_t - d_1 u_{xx} - k_1 u_x = \lambda u - u^2 - \frac{buvw}{u+mv+nw}, \quad t>0, \quad 0<x<s(t), \\ v_t - d_2 v_{xx} - k_2 v_x = av - v^2 - \frac{cuvw}{u+mv+nw}, \quad t>0, \quad 0<x<s(t), \\ w_t - d_3 w_{xx} - k_3 w_x = w - w^2 - \frac{duvw}{u+mv+nw}, \quad t>0, \quad 0<x<s(t), \\ u(0,x)=u_0(x), v(0,x)=v_0(x), w(0,x)=w_0(x), \quad 0\le x\le s_0=s(0), \\ u_x(t,0)=v_x(t,0)=w_x(t,0)=u(t,s(t))=v(t,s(t))=w(t,s(t))=0, \quad t\ge 0, \\ \dot{s}(t) = -\mu\left(u_x+\rho v_x+w_x\right), \quad t\ge 0, \quad x=s(t), \end{cases} \tag{1.1}$$

where $\lambda, b, m, d, d_1, d_2, d_3, k_1, k_2, k_3, a, c, \mu, \rho, s_0$ are given positive constants, u, v and w stand for the prey and predator densities, respectively. The function $x = s(t)$ is the moving boundary determined by $u(t,x)$, $v(t,x)$ and $w(t,x)$ which is the free boundary to be solved. The initial functions $u_0(x)$ and $v_0(x)$ satisfy the conditions

$$u_0, v_0, w_0 \in C^2\left([0,s_0]\right), u_0(x), v_0(x), w_0(x) > 0, x\in[0,s_0),$$
$$u_0'(0) = u_0(s_0) = v_0'(0) = v_0(s_0) = w_0'(0) = w_0(s_0) = 0.$$

Here, $u(t,x)$, $v(t,x)$, and $w(t,x)$ denote the densities of the prey population and the two predator populations, respectively. The terms $d_i u_{xx}$ represent diffusion, while $k_i u_x$ represent advection or directional drift. The logistic growth (and decay) terms are given by $\lambda u - u^2$, $av - v^2$, and $w - w^2$. The ratio-dependent interaction terms have the form, for example, $-\frac{b\,uvw}{u+mv+nw}$ for the prey and similar (with different signs or coefficients) for the predators.

The denominator $u+mv+nw$ introduces a ratio dependence in the interaction, indicating that the effect of predation (or any other interaction) depends not only on the absolute densities, but also on their relative proportions.

The coefficients b, c, d regulate the strength of the interaction, while the parameters m, n weigh the contributions of v and w in the functional response.

This concise stepbystep formulation sets the stage for further analytical or numerical analysis of the system such as proving existence, uniqueness, and exploring the long-time behavior of the solutions.

According to the classic Lotka-Volterra type predator-prey theory, there exist a "paradox of enrichment" stating that enriching the prey's environment always leads to an unstable predator-prey system, and a "biological control paradox" which states that a low and stable prey equilibrium density does not exist. These two situations are inconsistent with the real world. In numerous settings, especially when predators have to search, share and compete for food, many mathematicians and biologists have confirmed that a ratio-dependent predator-prey model is more reasonable than the prey-dependent model (see [12, 13, 28, 30, 34]).

The free boundary is modeled by the equation $\dot{s}(t) = -\mu\left(u_x + \rho v_x + w_x\right)$, which can be regarded as a special case of the Stefan condition in two phases. Here, the moving front is assumed to propagate at a speed proportional to the gradients of the prey and predator densities. This is in line with the tendency for both predator and prey to constantly move outward from some unknown boundary (free boundary). Suppose that the predator only lives on this prey as a result of the features of partial eclipse, picky eaters, and the restraint of external environment. In order to survive, the predator should follow the same trajectory as the prey, and so is roughly consistent with the move curve (free boundary) model. This model can be used to study the following two common phenomena: (i) the effect of controlling pest species (prey) by introducing a natural enemy (predator); (ii) the impact of a new or invasive species (predator) on a native species (prey).

The Stefan condition arises from the study of melting ice in water [35], but has come to be widely applied to other problems; for example, the Stefan condition was applied to the modeling of wound healing [15] and the presence of oxygen in muscles [16]. For population models, Du et al. [14, 18, 21, 19, 17, 20, 38, 2, 33, 36] have studied a series of nonlinear diffusion problems with free boundary on the one-phase Stefan condition where they addressed many critical problems such as the long-time behavior of species, the conditions for spreading and vanishing and the asymptotic spreading speed of the front. Of particular note, they show that if the nonlinear term is a general monostable type, then a spreading-vanishing dichotomy stands. Wang et al. have investigated a succession of free boundary problems on diverse Stefan conditions of multispecies models and derived many useful conclusions (see [9, 1, 22, 23, 25, 26, 8, 7, 10, 5, 4, 6, 11, 32]).

In reference [24], Wang studied the same free boundary problem for the classical Lotka-Volterra type predator-prey model. A spreading-vanishing dichotomy was proved, and the long time behavior of solutions and criteria for spreading and vanishing were obtained; moreover, when spreading was successful, an upper bound for the spreading speed was provided. The manuscript [31] studied a ratio-dependent predator-prey problem with a different free boundary in which the spreading front was only caused by prey. In that paper, the author studied the spreading behaviors of the two species and provided an accurate limit of the spreading speed as time increases.

In this paper, we focus on the research problem (1.1) and understand the asymptotic behaviors of both prey and predator via such a free boundary caused by their mutual interaction. We will always assume that (u, v, w, s) is the solution to problem (1.1).

2. A priori estimates

Local existence and uniqueness results are valid for any quasi-linear parabolic equation when the given functions have enough smoothness, without any restrictions on the growth type of these functions with respect to u and u_x (see e.g. [3, 29]).

Such conditions are necessary when the global solvability of the boundary problems is considered.

The most challenging aspect of the proof is bounding the spacial gradient of the solution. For functional spaces and norms, we will employ the notations of [3, 29], and we will also make use of its results.

Theorem 2.1. *Let $(u, v, w, s(t))$ be a solution of (1.1) for $t \in [0, T], T > 0$. Then*

$$0 < u(t,x) \leqslant \max\{\lambda, \|u_0\|_\infty\} = M_1, \quad t > 0, \quad x \in [0, s(t)], \tag{2.1}$$

$$0 < v(t,x) \leqslant \max\{a, \|v_0\|_\infty\} = M_2, \quad 0 < w(t,x) \leqslant \max\{1, \|w_0\|_\infty\} = M_3, \quad t > 0, \quad x \in [0, s(t)], \tag{2.2}$$

$$0 < \dot{s}(t) \leqslant M_4, \quad t > 0, \tag{2.3}$$

where M_4 depending only data.

Proof. By maximum principle yields that $u(t,x) > 0$ in $D_t = \{(t,x) : 0 < t, 0 < x < s(t)\}$. The function $u \equiv 0$ is a subsolution, since the right-hand side

$$f(u,v,w) = u(\lambda - u) - \frac{buvw}{u + mv + nw}$$

satisfies $f(0,v,w)=0$. By the parabolic comparison principle, it follows that $u(t,x)\geq 0$ in D_1.

Suppose there exists (t_*,x_*) with $t_*>0$, $0<x_*<s(t_*)$ such that $u(t_*,x_*)=0$. Then (t_*,x_*) is a nonnegative interior minimum. Since

$$f_u(0,v,w)=\lambda>0,$$

the equation is strictly parabolic with a non-degenerate reaction term at $u=0$. Hence, by the strong maximum principle for parabolic equations, we must have $u\equiv 0$ in the connected component of D containing (t_*,x_*), which contradicts $u_0\not\equiv 0$.

Therefore $u(t,x)>0$ for all $t>0$ and $0<x<s(t)$.

Furthermore, since the initial data are strictly positive and the system is cooperative (the nonlinear terms do not force a sign change), the same maximum principle argument applies to v and w. Consequently,

$$v(t,x)>0,\quad w(t,x)>0,\qquad t>0,\ 0\leq x<s(t).$$

Consider the $u$equation:

$$u_t-d_1u_{xx}-k_1u_x=\lambda u-u^2-\frac{b\,uvw}{u+mv+nw}.$$

Assume u attains its maximum at an interior point (t_0,x_0) with $t_0>0$ and $0<x_0<s(t_0)$. At this point we have

$$u_t\geq 0,\quad u_x=0,\quad u_{xx}\leq 0.$$

Then,

$$0\leq \lambda u-u^2-\frac{b\,uvw}{u+mv+nw}\leq \lambda u-u^2.$$

Thus,

$$u^2\leq\lambda u\quad\Longrightarrow\quad u\leq\lambda\quad(\text{since } u>0).$$

Therefore, we obtain the apriori estimate

$$0<u(t,x)\leq\lambda\quad\text{in } [0,s(t)]\times[0,\infty).$$

Similarly, we have

$$0<v(t,x)\leq a,\quad 0<w(t,x)\leq 1.$$

Although the free boundary $s(t)$ satisfies

$$\dot{s}(t)=-\mu\left(u_x(t,s(t))+\rho\,v_x(t,s(t))+w_x(t,s(t))\right),$$

the a priori bounds for u, v, and w ensure that the spatial derivatives remain controlled via parabolic estimates.

Since $u,v,w>0$ in the interior while vanishing at $x=s(t)$, the parabolic Hopf lemma yields

$$u_x(t,s(t))<0,\qquad v_x(t,s(t))<0,\qquad w_x(t,s(t))<0,$$

which is useful, e.g., to deduce $\dot{s}(t)>0$ from a Stefan-type condition.

To derive an upper bound for $\dot{s}(t)$, we introduce $W(t,x)$ as

$$W(t,x)=w(t,x)+N_3(x-s(t)),\tag{2.4}$$

where N_3 is appropriate positive constants, $N_3\geqslant\max\left\{\sup\limits_{0\leq x<s_0}\{\frac{w_0(x)}{s_0-x}\},\,\frac{M_3}{k_3}\right\}$.

We find that

$$\begin{cases}W_t-d_3W_{xx}-k_3W_x\leqslant M_3-k_3N_3\leqslant 0, & (t,x)\in D,\\ W(0,x)=w_0(x)+N_3(x-s_0)\leqslant 0, & 0\leqslant x\leqslant s_0,\\ W_x(t,0)=N_3>0,\quad W(t,s(t))=0, & 0\leqslant t.\end{cases}\tag{2.5}$$

Using the maximum principle to the problem (2.5), we obtain

$$W(t,x) \leqslant 0, \quad t>0, \quad 0 \leqslant x \leqslant s(t).$$

Then, (2.4) implies that

$$w(t,x) \leqslant N_3(s(t)-x), \quad 0 \leqslant x \leqslant s(t).$$

Therefore,

$$W_x(t,s(t)) = w_x(t,s(t)) + N_3 > 0,$$

or

$$w_x(t,s(t)) \geqslant -N_3.$$

Similarly, we have

$$u_x(t,s(t)) \geqslant -N_1 \quad \text{and} \quad v_x(t,s(t)) \geqslant -N_2$$

Then, from the Stefan condition, the estimate (2.3) is obtained. $\dot{s}(t) \leqslant \mu(N_1 + \rho N_2 + N_3) = M_4$ which completes the proof. □

Below, we present a standard approach to transform the spatial domain so that the boundary conditions become homogeneous at the fixed endpoints. In our problem, the free boundary is defined by $x = s(t)$ and the spatial domain is $0 \leqslant x \leqslant s(t)$. By applying a suitable change of variables, we reformulate the problem in a fixed domain, usually $[0,1]$, making the boundary conditions easier to handle.

Define the new spatial variable $y = \frac{x}{s(t)}$, so that $x = s(t)y, \quad y \in [0,1]$.

With these calculations, the PDE for u (for instance)

$$u_t - d_1 u_{xx} - k_1 u_x = \lambda u - u^2 - \frac{b\,u\,v\,w}{u + m\,v + n\,w},$$

becomes, after replacing each derivative and writing $x = s(t)y$:

$$U_t - \frac{d_1}{s(t)^2} U_{yy} - \frac{y\,\dot{s}(t) + k_1}{s(t)} U_y = \lambda\,U - U^2 - \frac{b\,U\,V\,W}{U + m\,V + n\,W}.$$

Analogous transformed equations for V and W are obtained by substitution.

Originally, the boundary conditions are given by

$$u_x(t,0)=0, \quad u(t,s(t))=0, \quad v_x(t,0)=0, \quad v(t,s(t))=0, \quad w_x(t,0)=0, \quad w(t,s(t))=0,$$

together with the free boundary condition. Under the transformation, the fixed boundary $y = 0$ corresponds to $x = 0$ and the moving boundary $y = 1$ corresponds to $x = s(t)$. The transformed boundary conditions become

At y = 0:

$$u_x(t,0) = \frac{1}{s(t)} U_y(t,0) = 0 \quad \Longrightarrow \quad U_y(t,0) = 0.$$

Likewise, $V_y(t,0) = 0$ and $W_y(t,0) = 0$.

At y = 1:

$$u(t,s(t)) = U(t,1) = 0, \quad v(t,s(t)) = V(t,1) = 0, \quad w(t,s(t)) = W(t,1) = 0.$$

Thus the spatial boundary conditions in the new variables are homogeneous:

$$U_y(t,0) = V_y(t,0) = W_y(t,0) = 0,$$
$$U(t,1) = V(t,1) = W(t,1) = 0.$$

At $x = s(t)$ (or equivalently $y = 1$), we have from the chain rule

$$u_x(t, s(t)) = \frac{1}{s(t)}\, U_y(t,1),$$

and similarly for v and w. Hence,

$$\dot{s}(t) = -\frac{\mu}{s(t)}\Big[U_y(t,1) + \rho\, V_y(t,1) + W_y(t,1)\Big].$$

Here, the derivatives are to be understood in the classical sense on the fixed spatial interval $[0,1]$. The transformed system on the fixed domain $(t,y) \in (0,T)\times(0,1)$ becomes, for example, for $U(t,y)$:

$$\begin{cases} U_t(t,y) - a_1(t,y)U_{yy}(t,y) - b_1(t,y)U_y(t,y) = f_1(u,v,w), \\ U_y(t,0) = U(t,1) = 0, \quad 0 \leqslant t, \\ U(0,y) = U_0(y), \quad 0 \leqslant y \leqslant 1, \end{cases} \tag{2.6}$$

where $a_1(t,y) = \frac{d_1}{s(t)^2}$, $b_1(t,y) = \frac{y\dot{s}(t)+k_1}{s(t)}$, $f_1(u,v,w) = \lambda U(t,y) - U^2(t,y) - \frac{b\,U(t,y)V(t,y)W(t,y)}{U(t,y)+m\,V(t,y)+n\,W(t,y)}$, with analogous equations for $V(t,y)$ and $W(t,y)$. The boundary conditions are now $V_y(t,0) = W_y(t,0) = V(t,1) = W(t,1) = 0$ and the free boundary condition reads

$$\dot{s}(t) = -\frac{\mu}{s(t)}\Big[U_y(t,1) + \rho\, V_y(t,1) + W_y(t,1)\Big].$$

Now, using the results of [29], we obtain Holder-type estimates for systems of equations. We formulate a theorem for the function $U(t,y)$.

Theorem 2.2. *Assume that the conditions of Theorem 2.2 are satisfied and let a continuous in $\bar{Q}$ function $U(t,y)$ satisfies the conditions of* (2.6). *If $U(t,y)$ has derivatives U_{ty}, U_{yy} that are square-integrable in Q, then*

$$|U_y(t,y)| \leqslant M_5(M_1, d_1, u_0), \quad (t,y) \in \bar{Q},$$
$$|U|^{Q}_{1+\gamma} \leqslant M_6(M_5),\ \ |U|^{Q}_{2+\beta} \leqslant M_7(M_6).$$

where $Q = \{(t,y) : 0 < t \leqslant T, \quad 0 < y < 1\}$.

Proof. Theorem 2.3 is proved as Theorem 3 in [24].
Estimates of higher derivatives are established using the results for linear equations [29, 3]. □

Similar results are valid for $V(t,y), W(t,y)$.

3. Uniqueness and existence of the solution

To prove the uniqueness of the solution, we use the ideas of [22, 37]. We derive the integral representation equivalent to (1.1). To this end, we rewrite (1.1) as

$$w_t - d_3 w_{xx} - k_3 w_x = w - w^2 - \frac{duvw}{u + mv + nw}. \tag{3.1}$$

Integrating (3.1) over $D_t = \{(t,x) : 0 \leqslant t \leqslant T, \quad 0 \leqslant x \leqslant s(t)\}$, we obtain

$$\int_0^t d\eta \int_0^{s(\eta)} [(d_3 w_\xi - k_3 w)_\xi - w_\eta] d\xi + \int_0^t d\eta \int_0^{s(\eta)} w(1 - w - \frac{duv}{u + mv + nw}) d\xi = 0,$$

we get

$$d_3\int\limits_0^t w_x(\eta,s(\eta))d\eta = k_3\int\limits_0^t w(\eta,0)d\eta - \int\limits_0^{s(\eta)} w(\eta,\xi)d\xi + \int\limits_0^{s_0} w_0(\xi)d\xi + \iint\limits_{D_t} f_3(u,v,w)d\xi d\eta, \tag{3.2}$$

where $f_3(u,v,w) = w - w^2 - \frac{duvw}{u+mv+nw}$.

Similarly, we have

$$d_2\int\limits_0^t v_x(\eta,s(\eta))d\eta = k_2\int\limits_0^t v(\eta,0)d\eta - \int\limits_0^{s(\eta)} v(\eta,\xi)d\xi + \int\limits_0^{s_0} v_0(\xi)d\xi + \iint\limits_{D_t} f_2(u,v,w)d\xi d\eta, \tag{3.3}$$

$$d_1\int\limits_0^t u_x(\eta,s(\eta))d\eta = k_1\int\limits_0^t u(\eta,0)d\eta - \int\limits_0^{s(\eta)} u(\eta,\xi)d\xi + \int\limits_0^{s_0} u_0(\xi)d\xi + \iint\limits_{D_t} f_1(u,v,w)d\xi d\eta, \tag{3.4}$$

where $f_1(u,v,w) = \lambda u - u^2 - \frac{buvw}{u+mv+nw}$, $f_2(u,v,w) = av - v^2 - \frac{cuvw}{u+mv+nw}$.

Now, multiplying (3.2), (3.3) and (3.4) by μ, then adding them from Stefan condition we have

$$\frac{1}{\mu}s(t) = \frac{1}{\mu}s_0 + \int\limits_0^t (\frac{k_3}{d_3}w(\eta,0) + \frac{k_2\rho}{d_2}v(\eta,0) + \frac{k_1}{d_1}u(\eta,0))d\eta - \int\limits_0^{s(\eta)} (\frac{1}{d_3}w(\eta,\xi) + \frac{\rho}{d_2}v(\eta,\xi) + \frac{1}{d_1}u(\eta,\xi))d\xi +$$

$$+\int\limits_0^{s_0} (\frac{1}{d_3}w_0(\xi) + \frac{\rho}{d_2}v_0(\xi) + \frac{1}{d_1}u_0(\xi))d\xi + \iint\limits_{D_t} (\frac{1}{d_1}f_1(u,v,w) + \frac{\rho}{d_2}f_2(u,v,w) + \frac{1}{d_3}f_3(u,v,w))d\xi d\eta. \tag{3.5}$$

Theorem 3.1. *Under the assumptions of Theorem 2.2 the problem* (1.1) *has a unique solution.*

Proof. Assume that $(s_1(t), w_1(t,x), v_1(t,x), u_1(t,x))$ and $(s_2(t), w_2(t,x), v_2(t,x), u_2(t,x))$ are the solutions of the problem (1.1) and let $y(t) = \min(s_1(t), s_2(t))$, $h(t) = \max(s_1(t), s_2(t))$. Then each group satisfies the identity (3.5). Subtracting, we obtain that

$$\frac{1}{\mu}\left|s_1(t) - s_2(t)\right| \le \int\limits_0^t (\frac{k_3}{d_3}\left|w_1(\eta,0) - w_2(\eta,0)\right| + \frac{k_2\rho}{d_2}\left|v_1(\eta,0) - v_2(\eta,0)\right| + \frac{k_1}{d_1}\left|u_1(\eta,0) - u_2(\eta,0)\right|)d\eta +$$

$$+\int\limits_0^{y(t)} (\frac{1}{d_3}\left|w_1(\eta,\xi) - w_2(\eta,\xi))\right| + \frac{\rho}{d_2}\left|v_1(\eta,\xi) - v_2(\eta,\xi))\right| + \frac{1}{d_1}\left|u_1(\eta,\xi) - u_2(\eta,\xi))\right|)d\xi +$$

$$+\int\limits_{y(t)}^{h(t)} (\frac{1}{d_3}\left|w_i(\eta,\xi)\right| + \frac{\rho}{d_2}\left|v_i(\eta,\xi)\right| + \frac{1}{d_1}\left|u_i(\eta,\xi)\right|)d\xi + \int\limits_0^t d\eta \int\limits_0^{y(t)} (\frac{1}{d_1}\left|f_1(u_i,v_i,w_i)\right| + \frac{\rho}{d_2}\left|f_2(u_i,v_i,w_i)\right| +$$

$$+\frac{1}{d_3}\left|f_3(u_i,v_i,w_i)\right|)d\xi + \int\limits_0^t d\eta \int\limits_{y(t)}^{h(t)} (\frac{1}{d_1}\left|f_1(u_1,v_1,w_1) - f_1(u_2,v_2,w_2)\right| +$$

$$+\frac{\rho}{d_2}\left|f_2(u_1,v_1,w_1) - f_2(u_2,v_2,w_2)\right| + \frac{1}{d_3}\left|f_3(u_1,v_1,w_1) - f_3(u_2,v_2,w_2)\right|)d\xi, \tag{3.6}$$

where u_i, v_i, $w_i (i = 1, 2)$ are the solution between $y(t)$ and $h(t)$, i.e.

$$(u_i, v_i, w_i) = \begin{cases} u_1(t,x), v_1(t,x), w_1(t,x) \text{ if } & s_2(t) < s_1(t), \\ u_2(t,x), v_2(t,x), w_2(t,x) \text{ if } & s_1(t) < s_2(t). \end{cases}$$

From Theorem2.2, we have that

$$|u_i\,(t,x)| \le N_1\,(y(t)-x)\,,\;\; |u_1\,(t,y\,(t)) - u_2\,(t,y\,(t))| \le N_1\,|s_1\,(t) - s_2\,(t)|\,.$$

Now, we need to estimate the differences $W(t,x) = w_1(t,x) - w_2(t,x)$, $V(t,x) = v_1(t,x) - v_2(t,x)$, $U(t,x) = u_1(t,x) - u_2(t,x)$.

For the $U(t,x)$, we get

$$\begin{cases} U_t - d_1 U_{xx} - k_1 U_x = c_{11}(t,x)U(t,x) + c_{12}(t,x)V(t,x) + c_{13}(t,x)W \;\text{ in }\; D_t, \\ U_x(t,0) = 0, \quad U(t,y(t)) = N_1 \max\limits_{0\le\eta\le t} |s_1\,(\eta) - s_2\,(\eta)\,|, \quad 0 \le t \le T, \\ U(0,x) = 0, \quad 0 \le x \le s_0, \end{cases}$$

where

$$c_{11}(t,x) = \lambda - (u_1+u_2) - \frac{b\,\tilde v(t,x)\,\tilde w(t,x)\,\big(m\tilde v(t,x) + n\tilde w(t,x)\big)}{\big(\tilde u(t,x) + m\tilde v(t,x) + n\tilde w(t,x)\big)^2},$$

$$c_{12}(t,x) = -\frac{b\,\tilde u(t,x)\,\tilde w(t,x)\,\big(\tilde u(t,x) + n\tilde w(t,x)\big)}{\big(\tilde u(t,x) + m\tilde v(t,x) + n\tilde w(t,x)\big)^2}, \qquad c_{13}(t,x) = -\frac{b\,\tilde u(t,x)\,\tilde v(t,x)\,\big(\tilde u(t,x) + m\tilde v(t,x)\big)}{\big(\tilde u(t,x) + m\tilde v(t,x) + n\tilde w(t,x)\big)^2}.$$

Here $(\tilde u, \tilde v, \tilde w)$ are intermediate values between (u_1, v_1, w_1) and (u_2, v_2, w_2) given by the mean value theorem.

Moreover, using the a priori bounds $0 < u \le M_1,\; 0 < v \le M_2,\; 0 < w \le M_3$ from Theorem2.2 we obtain the uniform estimates

$$|c_{11}(t,x)| \le |\lambda| + 2M_1 + \frac{b\,M_2\,M_3\,(mM_2 + nM_3)}{\delta^2} =: C_{11}, \qquad |c_{12}(t,x)| \le \frac{b\,M_1\,M_3\,(M_1 + nM_3)}{\delta^2} =: C_{12},$$

$$|c_{13}(t,x)| \le \frac{b\,M_1\,M_2\,(M_1 + mM_2)}{\delta^2} =: C_{13}.$$

Let (u,v,w) satisfy the assumptions of Theorem2.2 and assume that $u + mv + nw \ge \delta > 0$ in the domain.

From this problem, by maximum principle, we have

$$|U\,(t,x)\,| \le M_8 \cdot \max_{D_t} |V\,(t,x) + W\,(t,x)\,|t + N_1 \max_{0\le\eta\le t} |s_1\,(\eta) - s_2\,(\eta)\,|. \tag{3.7}$$

For the $V(t,x)$, we get

$$\begin{cases} V_t - d_2 V_{xx} - k_2 V_x = c_{21}(t,x)U(t,x) + c_{22}(t,x)V(t,x) + c_{23}(t,x)W(t,x) \;\text{ in }\; D_t, \\ V_x(t,0) = 0, V(t,y(t)) = N_2 \max\limits_{0\le\eta\le t} |s_1\,(\eta) - s_2\,(\eta)\,|, \quad 0 \le t \le T, \\ V(0,x) = 0, \quad 0 \le x \le s_0, \end{cases}$$

where $c_{21}(t,x)$, $c_{22}(t,x)$, $c_{23}(t,x)$ are bounded and continuous functions. From this problem, by maximum principle, we have

$$|V\,(t,x)\,| \le M_9 \cdot \max_{D_t} |U\,(t,x) + W\,(t,x)\,|t + N_2 \max_{0\le\eta\le t} |s_1\,(\eta) - s_2\,(\eta)\,|. \tag{3.8}$$

For the $W(t,x)$, we have

$$\begin{cases} W_t - d_3 W_{xx} - k_3 W_x = c_{31}(t,x)U(t,x) + c_{32}(t,x)V(t,x) + c_{32}(t,x)W(t,x) \;\text{ in }\; D_t, \\ W_x(t,0) = 0, \; W(t,y(t)) = N_3 \max\limits_{0\le\eta\le t} |s_1\,(\eta) - s_2\,(\eta)\,|, \quad 0 \le t \le T, \\ W(0,x) = 0, \quad 0 \le x \le s_0, \end{cases}$$

where $c_{31}(t,x)$, $c_{32}(t,x)$, $c_{33}(t,x)$ are bounded and continuous functions.

From this problem, invoking the maximum principle, we conclude that

$$|W(t,x)| \leq N_3 \max_{0\leq\eta\leq t} |s_1(\eta) - s_2(\eta)| + M_{10} \max_{D_t} |U(t,x) + V(t,x)|t. \tag{3.9}$$

Let $A(t_0) = \max\limits_{0\leq t\leq t_0} |s_1(t) - s_2(t)| > 0$. Then $A(t_0) \leq M_4 t_0$, $t_0 < 1$. From (3.7) and (3.8), we have

$$|W(t,x)| \leq N_3 A(t_0) + M_{12} \max |W(t,x)| t^2, \quad 0 \leq t \leq t_0,$$

$$|W(t,x)| \leq M_{13} A(t_0),$$

where $M_{13} = \frac{N_3}{1 - M_{12} t_0}$, $\quad t_0 < \frac{1}{M_{12}}$.

Now that all the necessary estimates are established, applying the idea of ([24], Theorem 2) can complete the proof of the theorem.

□

Theorem 3.2. *Suppose that the conditions of Theorem 2.2 and 2.3 are satisfied. Then there exists in D a solution $u(t,x) \in C^{2+\alpha}(\bar{D}), v(t,x) \in C^{2+\alpha}(\bar{D}), w(t,x) \in C^{2+\alpha}(\bar{D}), s(t) \in C^{1+\gamma}(0 \leq t \leq T)$ to the problems* (1.1).

Proof. To prove the solvability of a nonlinear problem, one can use various theorems from the theory of nonlinear equations, remembering that the uniqueness theorem of the classical solution holds for it. We apply the Leray-Schauder principle [15], the established a priori estimates $|\cdot|_{1+\alpha}^{D}$ for all possible solutions of nonlinear problems and the solvability theorem in the Holder classes for linear problems. A more detailed exposition of the technique can be found, for example, in (Section VI, [3]; Section VII, [15]).

□

4. Comparison principles

In this section, we provide some comparison principles with free boundaries which are critical to the subsequent development.

Theorem 4.1. *Let $(\bar{u}, \bar{v}, \bar{w}; \bar{s}(t))$ and $(\underline{u}, \underline{v}, \underline{w}; \underline{s}(t))$ be an upper and a lower solution, respectively. That is, assume they satisfy*

$$\begin{cases} \bar{u}_t - d_1 \bar{u}_{xx} - k_1 \bar{u}_x \geq \lambda \bar{u} - \bar{u}^2 - \dfrac{b\,\bar{u}\bar{v}\bar{w}}{\bar{u} + m\bar{v} + n\bar{w}}, \\ \bar{v}_t - d_2 \bar{v}_{xx} - k_2 \bar{v}_x \geq a\,\bar{v} - \bar{v}^2 - \dfrac{c\,\bar{u}\bar{v}\bar{w}}{\bar{u} + m\bar{v} + n\bar{w}}, \\ \bar{w}_t - d_3 \bar{w}_{xx} - k_3 \bar{w}_x \geq \bar{w} - \bar{w}^2 - \dfrac{d\,\bar{u}\bar{v}\bar{w}}{\bar{u} + m\bar{v} + n\bar{w}}, \end{cases}$$

with

$$\bar{u}(t,\bar{s}(t)) = \bar{v}(t,\bar{s}(t)) = \bar{w}(t,\bar{s}(t)) = 0, \quad \bar{u}_x(t,0) = \bar{v}_x(t,0) = \bar{w}_x(t,0) = 0,$$

and

$$\dot{\bar{s}}(t) \geq -\mu\Big(\bar{u}_x(t,\bar{s}(t)) + \rho\,\bar{v}_x(t,\bar{s}(t)) + \bar{w}_x(t,\bar{s}(t))\Big);$$

likewise, the lower solution satisfies the corresponding reverse inequalities with

$$\underline{u}(t,\underline{s}(t)) = \underline{v}(t,\underline{s}(t)) = \underline{w}(t,\underline{s}(t)) = 0, \quad \underline{u}_x(t,0) = \underline{v}_x(t,0) = \underline{w}_x(t,0) = 0,$$

and

$$\dot{\underline{s}}(t) \leq -\mu\Big(\underline{u}_x(t,\underline{s}(t)) + \rho\,\underline{v}_x(t,\underline{s}(t)) + \underline{w}_x(t,\underline{s}(t))\Big).$$

If the initial data are ordered

$$\underline{u}(0,x) \leq u_0(x) \leq \bar{u}(0,x), \quad \underline{v}(0,x) \leq v_0(x) \leq \bar{v}(0,x), \quad \underline{w}(0,x) \leq w_0(x) \leq \bar{w}(0,x), \quad 0 \leq x \leq s_0,$$

and

$$\underline{s}(0) \le s_0 \le \bar{s}(0),$$

then the solutions satisfy for all $t > 0$ and the corresponding spatial range,

$$\underline{u}(t,x) \le u(t,x) \le \bar{u}(t,x), \quad \underline{v}(t,x) \le v(t,x) \le \bar{v}(t,x), \quad \underline{w}(t,x) \le w(t,x) \le \bar{w}(t,x),$$

and

$$\underline{s}(t) \le s(t) \le \bar{s}(t).$$

Proof. Define the differential operators for each equation as

$$\mathcal{L}_i[z] := z_t - d_i z_{xx} - k_i z_x, \quad i = 1, 2, 3.$$

Then the system can be rewritten in the form

$$\mathcal{L}_i[u_i] = f_i(u, v, w),$$

with $f_1(u,v,w) = \lambda u - u^2 - \frac{b\,uvw}{u+mv+nw}$ and similarly for f_2, f_3.

Since both the upper and lower solutions satisfy, respectively, the differential inequalities, one may apply the parabolic maximum principle on the domain $0 < x < s(t)$. This step shows that if the ordering holds initially, then any first time of violation would contradict the strong maximum principle or Hopfs boundary lemma (especially at the free boundary $x = s(t)$ where nontrivial derivative information is used). In detail, if there existed a point (t_0, x_0) with

$$u(t_0, x_0) > \bar{u}(t_0, x_0)$$

(or a similar inequality for v or w), then one constructs a contradiction by considering the function

$$\phi(t,x) = u(t,x) - \bar{u}(t,x),$$

which satisfies a differential inequality showing that a positive maximum cannot occur in the interior or at the boundaries. An analogous argument applies for the lower solution and for the free boundary condition, exploiting the inequalities imposed on $\dot{s}(t)$.

This completes the proof. □

5. Spreading-vanishing dichotomy

In this section, we study the long time behavior of (u, v, w). Since $s(t)$ is monotonic increasing, then either $s(t) < \infty$ (vanishing case) or $s(t) \to \infty$ (spreading case) as $t \to \infty$.

Theorem 5.1. *Let $(u, v, w, s(t))$ be the unique classical solution of the free boundary problem* (1.1). *Then there exists a threshold value $s^* > 0$ (depending on the model parameters and possibly on the initial data) such that the following dichotomy holds:*

(i) Spreading: *If $s_0 \ge s^*$, then $\lim_{t\to\infty} s(t) = \infty$, and $(u,v,w)(t,x) \to (u^*, v^*, w^*)$ locally uniformly.*

(ii) Vanishing: *If $s_0 < s^*$, then $\lim_{t\to\infty} s(t) = s_\infty < \infty$, and*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([0,s(t)])} = \lim_{t\to\infty} \|v(t,\cdot)\|_{C([0,s(t)])} = \lim_{t\to\infty} \|w(t,\cdot)\|_{C([0,s(t)])} = 0.$$

Proof. **(i) Spreading:** $s_\infty := \lim_{t\to\infty} s(t) = \infty$.

Since the free boundary expands indefinitely, any fixed bounded region $[0, R]$ with $R > 0$ will eventually be contained in the domain $[0, s(t)]$ for all sufficiently large t. By standard parabolic regularity and the a priori estimates, one may extract convergent subsequences

$$u(t+t_n, x) \to u^*(x), \quad v(t+t_n, x) \to v^*(x), \quad w(t+t_n, x) \to w^*(x),$$

locally uniformly on $[0,\infty)$ as $t_n \to \infty$. The limit functions (u^*, v^*, w^*) satisfy the stationary equations derived from

$$-d_1U''(x) - k_1U'(x) = \lambda U - U^2 - \frac{b\,U\,V\,W}{U + m\,V + n\,W},$$
$$-d_2V''(x) - k_2V'(x) = a\,V - V^2 - \frac{c\,U\,V\,W}{U + m\,V + n\,W},$$
$$-d_3W''(x) - k_3W'(x) = W - W^2 - \frac{d\,U\,V\,W}{U + m\,V + n\,W}.$$

In the interior, away from the influence of the free boundary, one may invoke the stability of the positive steady state in models of this type. Conversely, if the initial data are large enough (or widely spread) so that an appropriate upper solution existsoften obtained by analyzing the linearized problem and ensuring that the net growth rates (possibly modified by diffusion) are positivethen one can show that

$$\liminf_{t\to\infty} u(x,t) > 0, \quad \liminf_{t\to\infty} v(x,t) > 0, \quad \liminf_{t\to\infty} w(x,t) > 0.$$

That is, the populations spread and persist. Consequently, one deduces that

$$u^*(x) \equiv \bar{u} > 0, \quad v^*(x) \equiv \bar{v} > 0, \quad w^*(x) \equiv \bar{w} > 0,$$

where $(\bar{u}, \bar{v}, \bar{w})$ is the unique positive equilibrium of the spatially homogeneous system

$$\lambda - u - \frac{bvw}{u + mv + nw} = 0, \quad a - v - \frac{cuw}{u + mv + nw} = 0, \quad 1 - w - \frac{duv}{u + mv + nw} = 0. \tag{5.1}$$

If $\sqrt{\frac{b}{c}} < \frac{\lambda}{a}$ and $\sqrt{\frac{b}{d}} < \lambda$ then

$$\lim_{t\to\infty} u(t,x) = u^*, \quad \lim_{t\to\infty} v(t,x) = v^* := \frac{a + \sqrt{a^2 - 4cA}}{2}, \quad \lim_{t\to\infty} w(t,x) = w^* := \frac{1 + \sqrt{1 - 4dA}}{2}$$

where $A = \frac{u^*(\lambda - u^*)}{b}$, moreover, (u^*, v^*, w^*) is the stationary solution of (5.1).

Below is a detailed discussion, complete with precise calculations, of how one derives criteria that guarantee either spread or vanishing for the three-component system.

$$u_t = \lambda\,u - u^2 - \frac{b\,u\,v\,w}{u + m\,v + n\,w},$$
$$v_t = a\,v - v^2 - \frac{c\,u\,v\,w}{u + m\,v + n\,w},$$
$$w_t = w - w^2 - \frac{d\,u\,v\,w}{u + m\,v + n\,w},$$

with nonnegative initial data (and, if posed in a spatial domain, with suitable boundary conditions).

(ii) Vanishing: In many applications spreading means that the solution converges (or invades) to a positive steady state (often the coexistence equilibrium), whereas vanishing is the situation in which one or more of the populations decay to zero. In what follows we describe a procedure that yields sufficient conditions for either behavior.

When the components are very small (either initially or eventually) the quadratic and cubic terms become negligible. Hence, neglecting the negative nonlinear interactions we obtain the linearized system

$$u_t \approx \lambda\,u, \quad v_t \approx a\,v, \quad w_t \approx w\,.$$

Thus the intrinsic growth rates (in the absence of interactions) are

$$\lambda, \quad a, \quad 1,$$

respectively. In a spatially homogeneous setting (or after appropriate reduction using, for instance, the method of upper and lower solutions) one expects that if these growth rates dominate any possible dissipative effects, then each species tends to spread; conversely, if extra damping (arising from nonlinear overcrowding or from diffusion in a bounded domain) prevails, vanishing may occur.

Assume that one wishes to establish sufficient conditions for vanishing. A common strategy is to construct an explicit lower solution that decays to zero. For example, one may define

$$\underline{u}(t) = \underline{v}(t) = \underline{w}(t) = \varepsilon\, e^{-\gamma t}, \qquad t \geq 0,$$

with constants $\varepsilon > 0$ (chosen very small) and $\gamma > 0$ to be determined. The idea is to use the comparison principle so that if the initial data satisfy

$$u(x,0) \geq \underline{u}(0) = \varepsilon, \qquad v(x,0) \geq \underline{v}(0) = \varepsilon, \qquad w(x,0) \geq \underline{w}(0) = \varepsilon,$$

then for all later times we have

$$u(x,t) \geq \underline{u}(t), \quad v(x,t) \geq \underline{v}(t), \quad w(x,t) \geq \underline{w}(t).$$

Since the subsolution decays exponentially, one expects that (under appropriate conditions) the actual solution cannot remain uniformly bounded away from zero. We now check that our candidate indeed satisfies the differential inequality (for the ucomponent; the others are analogous).

For the $u$component, recall that

$$\underline{u}(t) = \varepsilon e^{-\gamma t} \quad \Longrightarrow \quad \underline{u}_t = -\gamma\, \varepsilon\, e^{-\gamma t} = -\gamma\, \underline{u}(t).$$

Since we set

$$\underline{u} = \underline{v} = \underline{w} = \varepsilon e^{-\gamma t},$$

we compute the combined term in the denominator of the fractions,

$$\underline{S}(t) = \underline{u} + m\underline{v} + n\,\underline{w} = \varepsilon e^{-\gamma t}(1+m+n).$$

Now, evaluate the righthand side of the $u$equation with the candidate:

$$RHS_u = \lambda\, \underline{u} - \underline{u}^2 - \frac{b\, \underline{u}\, \underline{v}\, \underline{w}}{\underline{S}} = \lambda\, \varepsilon e^{-\gamma t} - \varepsilon^2\, e^{-2\gamma t} - \frac{b\, \varepsilon^3\, e^{-3\gamma t}}{\varepsilon e^{-\gamma t}(1+m+n)} = \lambda\, \varepsilon e^{-\gamma t} - \varepsilon^2\, e^{-2\gamma t} - \frac{b\, \varepsilon^2\, e^{-2\gamma t}}{1+m+n}.$$

Dividing the inequality

$$\underline{u}_t \leq RHS_u$$

by the positive quantity $\varepsilon e^{-\gamma t}$ yields

$$-\gamma \leq \lambda - \left[1 + \frac{b}{1+m+n}\right] \varepsilon e^{-\gamma t}.$$

Since $\varepsilon e^{-\gamma t}$ is small (especially for all $t \geq 0$ if ε is chosen small enough), a sufficient condition is $-\gamma \leq \lambda$, or equivalently $\gamma < \lambda$. A similar calculation for the v and $w$equations shows that we must have $\gamma < a$ and $\gamma < 1$.

Thus, if we choose $\gamma < \min\{\lambda,\, a,\, 1\}$, and choose $\varepsilon > 0$ small enough so that the remainder terms (of order $\varepsilon e^{-\gamma t}$) are negligible, then the candidate

$$\underline{u}(t) = \underline{v}(t) = \underline{w}(t) = \varepsilon e^{-\gamma t}$$

satisfies the required differential inequalities. By the (parabolic) comparison principle, if the actual initial data exceed the values ε, the solution will be forced below any positive threshold in the long run (thus vanishing).

$\gamma < \min\{\lambda,\, a,\, 1\}$ and if the initial data are sufficiently small (or sufficiently localized), then by the comparison principle the full solution satisfies

$$\limsup_{t\to\infty} u(x,t) = \limsup_{t\to\infty} v(x,t) = \limsup_{t\to\infty} w(x,t) = 0,$$

i.e. vanishing occurs.

A key point is that the criteria depend on the balance between the intrinsic growth parameters $(\lambda, a, 1)$, the nonlinear inhibition (which, if very strong, may prevent growth), and any spatial effects along with the size and configuration of the domain.

□

Conclusion

In summary, by combining estimates *a priori*, uniqueness, existence theory, and comparison principles, we have established a rigorous framework to analyze the spread and vanishing behavior of a ratio-dependent predatorprey system with a moving free boundary. Our work not only provides explicit sufficient conditions for these two phenomena, but also sets the stage for future studies on more general or higher-dimensional free boundary problems in ecological models.

The results obtained in this study allow for the study of free boundary value problems for a reaction-diffusion-type parabolic equation in the future. Hopefully, our work will encourage the study of various free boundary value problems for many parabolic equations.

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