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On confluent forms of the one hypergeometric function of three variables and their applications

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Abstract. Hypergeometric functions are divided into complete and confluent functions. For the first time, Srivastava and Karlsson described a method for constructing a set of all possible triple Gaussian hypergeometric series and compiled a table showing definitions and areas of convergence for 205 different complete series (Srivastava-Karlsson List) depending on three variables. Several authors subsequently obtained various integral representations and transformations for the functions proposed by Srivastava and Karlsson. In this work we compile integral representations and transformation formulas for all confluent forms of one complete hypergeometric function in three variables from the Srivastava-Karlsson List. To prove integral representations for 8 confluent hypergeometric functions of three variables, properties of beta and gamma functions are used. The Boltz formula allows us to derive transformation formulas for the confluent functions under consideration.

Keywords: Euler type integral representation, transformation formula, confluent hypergeometric functions in three variables, beta function, gamma function.

MSC (2020): 33C15, 33C20, 33C65, 44A20

1. Introduction

The great interest in the theory of hypergeometric functions (including functions of one, two or more variables) is primarily due to the fact that hypergeometric functions allow us to find solutions to various applied problems related to thermal conductivity and dynamic processes, electromagnetic oscillations, aerodynamics, quantum mechanics and potential theory. These functions, which relate to higher and special (or transcendental) functions [3, 19, 20], are often called special functions of mathematical physics.

Basically, hypergeometric functions of two variables, as the corresponding functions of one variable, can be represented either by the Euler-Laplace type or by the Mellin-Barnes type of definite integrals. Integral representations are useful in connection with the analytic continuation of hypergeometric functions in two variables, their transformation theory, and also for the integration of hypergeometric systems of partial differential equations. An exposition of these results for double hypergeometric series of the second order together with references to the original literature are to be found in the monograph [7, Chap. 5, Sect. 5.7]. When the order of the hypergeometric function exceeds two, analogous results for the Kampé de Fériet function in two variables are found in [11, 12, 24].

It is known that there are 205 hypergeometric functions of three variables of second order, of which regions of convergence have been given in the literature [22, Chap. 3]. Hypergeometric functions of three variables can be expressed either by integrals of the Laplace type or by integrals of the Euler type [5, 14]. The list of hypergeometric functions of three variables is too extensive, moreover, Ergashev [9] recently announced 395 confluent hypergeometric functions of three variables, and it is impossible to give a complete list of integral representations here. Integral representations of the hypergeometric functions with three variables are helpful for the analytical continuation [16]. In addition, an analytic continuation of the Horn hypergeometric function with an arbitrary number of variables is given in [4]. Therefore, the integral representations are mainly in the theory of transformation, as well as the integration of hypergeometric systems of partial differential equations [9, 15]. In the attending work, the authors aim to obtain some new integral representations, reduction and transformation formulas for all possible confluent forms of the hypergeometric function of three variables F_{4b} from the Srivastava-Karlsson List, which are designated in [9] by from E_{153} to E_{160}.

2. Preliminaries

In this section, the definition and some helpful relations will concern.

The hypergeometric function ${}_2F_1$ is defined by

$$ {}_2F_1\left(a,b;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!},\ |x|<1, $$

where $(\lambda)_n$ is the Pochhammer symbol defined by $(\lambda)_n=\Gamma(\lambda+n)/\Gamma(\lambda)$, with Γ is the gamma function. Hereinafter, as usual, if the Pochhammer symbol $(\lambda)_n$ occurs in the denominator, then $\lambda\neq 0,-1,-2,\dots$.

The Kummer function is defined by the series

$$ {}_1F_1\left(a;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m}{(c)_m}\frac{x^m}{m!}. $$

The Bessel-Clifford function is defined by

$$ \bar{J}_\alpha(z)=\sum_{m=0}^{\infty}\frac{\left(-z^2/4\right)^m}{(\alpha+1)_m m!}. $$

The two complete Appell F_2 [1] and Horn H_2 [17] functions are, respectively, defined by

$$ F_2\left(a,b,b';c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_m\,(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|+|y|<1, $$

$$ H_2\left(a,b,b',b'';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_m\,(b')_n\,(b'')_n}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ (1+|x|)|y|<1. $$

Horn's confluent functions of two variables [17] , namely, $\mathrm{H}_2-\mathrm{H}_5$, H_{11}, are given as

$$ \mathrm{H}_2\left(a,b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_m\,(b')_n}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<1,|y|<\infty, \tag{2.1} $$

$$ \mathrm{H}_3\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_m}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<1,|y|<\infty, $$

$$ \mathrm{H}_4\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_n}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,|y|<\infty, $$

$$ \mathrm{H}_5\left(a;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,|y|<\infty, $$

$$ \mathrm{H}_{11}\left(a,b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_n\,(b')_n}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,|y|<1. $$

The two two-variables hypergeometric functions, called confluent Humbert functions [18], are defined by

$$ \Psi_1\left(a,b;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m}{(c)_m\,(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<1,|y|<\infty, $$

$$ \Psi_2\left(a;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(c)_m\,(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,\ |y|<\infty. $$

This paper uses standard definitions and notations, including the Pochhammer's symbol $(\lambda)_n$, the beta function $\mathrm{B}(x,y)$, the gamma function $\Gamma(z)$, the Gauss hypergeometric function and its generalization ${}_pF_q$ [7], the Appell functions [1], the Humbert functions [18], the Horn functions [17], and the complete [22] and confluent [9] hypergeometric functions of three variables (see also [21]).

3. Confluent hypergeometric functions in three variables

One of the three-variable hypergeometric functions most commonly used in applications is the hypergeometric function in three variables defined by Erdélyi [6]

$$F_{4b}\left(a_1,a_2,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(a_4)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\tag{3.1}$$

for $\{\,|x|+|y|<1\}\cap\{|z|<1/\left(1+|x|+|y|\right)\}$. For the definition and properties of this function the reader is referred to [22, Chap. 3].

The following triple confluent hypergeometric series [9] (for regions of convergence, see [21])

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\quad |x|+|y|<1,|z|<\infty,\tag{3.2}$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_3)_p(a_4)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\quad \left(1+|x|\right)|z|<1,|y|<\infty,\tag{3.3}$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\quad |x|+|y|<1,|z|<\infty,\tag{3.4}$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_3)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<1,|y|<\infty,|z|<\infty,\tag{3.5}$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<1,|y|<\infty,|z|<\infty,\tag{3.6}$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_3)_p(a_4)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<\infty,|y|<\infty,|z|<1,\tag{3.7}$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<\infty,|y|<\infty,|z|<\infty,\tag{3.8}$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<\infty,|y|<\infty,|z|<\infty,\tag{3.9}$$

can be obtained from (3.1) by limit (confluence) formulas

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a_1,a_2,a_3,\frac{1}{\varepsilon},b;c_1,c_2;x,y,\varepsilon z\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a_1,\frac{1}{\varepsilon},a_3,a_4,b;c_1,c_2;x,\varepsilon y,z\right),$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a_1,a_2,\frac{1}{\varepsilon},\frac{1}{\varepsilon},b;c_1,c_2;x,y,\varepsilon^2 z\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a_1,\frac{1}{\varepsilon},a_3,\frac{1}{\varepsilon},b;c_1,c_2;x,\varepsilon y,\varepsilon z\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a,\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},b;c_1,c_2;x,\varepsilon y,\varepsilon^2 z\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to 0}F_{4b}\left(\frac{1}{\varepsilon},\frac{1}{\varepsilon},a_3,a_4,b;c_1,c_2;\varepsilon x,\varepsilon y,z\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to 0}F_{4b}\left(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},a,b;c_1,c_2;\varepsilon x,\varepsilon y,\varepsilon z\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to 0}F_{4b}\left(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},b;c_1,c_2;\varepsilon x,\varepsilon y,\varepsilon^2 z\right).$$

Note, in [13] the functions E_{153} and E_{155} are first defined and designated as A_1 and A_2, respectively. Analogues of the function E_{155} with four and more variables are found in [2, 8].

Using the series manipulation technique, we can obtain the following equivalent forms of (3.1) – (3.9):

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{(a_2)_n\,(a_3)_p\,(b)_{n-p}}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}\;{}_2F_1\left(a_1,b+n-p;c_1;x\right), \tag{3.10}$$

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m\,(a_2)_n\,(b)_{m+n}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\;{}_1F_1\left(a_3;1-b-m-n;-z\right),$$

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p\,(a_3)_p}{(1-b)_p}\frac{z^p}{p!}\;F_2\left(b-p,a_1,a_2;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{(a_2)_n\,(b)_n}{(c_2)_n}\frac{y^n}{n!}\;\mathrm{H}_2\left(b+n,a_1,a_3;c_1;x,z\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{(a_3)_p\,(a_4)_p\,(b)_{n-p}}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}\;{}_2F_1\left(a_1,b+n-p;c_1;x\right), \tag{3.11}$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,p=0}^{\infty}\frac{(a_1)_m\,(a_3)_p\,(a_4)_p\,(b)_{m-p}}{(c_1)_m}\frac{x^m}{m!}\frac{z^p}{p!}\;{}_1F_1\left(b+m-p;c_2;y\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m\,(b)_{m+n}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\;{}_2F_1\left(a_3,a_4;1-b-m-n;-z\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(-1)^p\,(a_3)_p\,(a_4)_p}{(1-b)_p}\frac{z^p}{p!}\;\Psi_1\left(b-p,a_1;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{(b)_n}{(c_2)_n}\frac{y^n}{n!}\;H_2\left(b+n,a_1,a_3,a_4;c_1;x,z\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m=0}^{\infty}\frac{(a_1)_m\,(b)_m}{(c_1)_m}\frac{x^m}{m!}\;\mathrm{H}_{11}\left(b+m,a_3,a_4;c_2;y,z\right),$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{(b)_{n-p}\,(a_2)_n}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}\;{}_2F_1\left(a_1+n-p,b;c_1;x\right), \tag{3.12}$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{(b)_{m+n}\,(a_1)_m\,(a_2)_n}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\;\bar{J}_{-b-m-n}\left(2\sqrt{z}\right),$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p}{(1-b)_p}\frac{z^p}{p!}\;F_2\left(b-p,a_1,a_2;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{(b)_n\,(a_2)_n}{(c_2)_n}\frac{y^n}{n!}\;\mathrm{H}_3\left(b+n,a_1;c_1;x,z\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(a_3\right)_p\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\ {}_2F_1\left(a_1,b+n-p;c_1;x\right), \tag{3.13}$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,p=0}^{\infty}\frac{\left(a_1\right)_m\left(a_3\right)_p\left(b\right)_{m-p}}{\left(c_1\right)_m}\frac{x^m}{m!}\frac{z^p}{p!}\ {}_1F_1\left(b+m-p;c_2;y\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(a_1\right)_m\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ {}_1F_1\left(a_3;1-b-m-n;-z\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p\left(a_3\right)_p\left(b\right)_{-p}}{\left(1-b\right)_p}\frac{z^p}{p!}\ \Psi_1\left(b+n-p,a_1;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{\left(b\right)_n}{\left(c_2\right)_n}\frac{y^n}{n!}\ \mathrm{H}_2\left(b+n,a_1,a_3;c_1;x,z\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m=0}^{\infty}\frac{\left(a_1\right)_m\left(b\right)_m}{\left(c_1\right)_m}\frac{x^m}{m!}\ \mathrm{H}_4\left(b+m,a_3;c_2;y,z\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\ {}_2F_1\left(a,b+n-p;c_1;x\right), \tag{3.14}$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,p=0}^{\infty}\frac{\left(a\right)_m\left(b\right)_{m-p}}{\left(c_1\right)_m}\frac{x^m}{m!}\frac{z^p}{p!}\ {}_1F_1\left(b+m-p;c_1;y\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(a\right)_m\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ \bar{J}_{-b-m-n}\left(2\sqrt{z}\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p}{\left(c_2\right)_n\left(1-b\right)_p}\frac{z^p}{p!}\ \Psi_1\left(b-p,a;c_1;x,y\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,p=0}^{\infty}\frac{\left(b\right)_n}{\left(c_2\right)_n}\frac{y^n}{n!}\ \mathrm{H}_3\left(b+n,a;c_1;x,z\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m=0}^{\infty}\frac{\left(a\right)_m\left(b\right)_m\left(b+m\right)_{n-p}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\ \mathrm{H}_5\left(b+m;c_2;y,z\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(a_3\right)_p\left(a_4\right)_p\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\ {}_1F_1\left(b+n-p;c_1;x\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ {}_2F_1\left(a_3,a_4;1-b-m-n;-z\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p\left(a_3\right)_p\left(a_4\right)_p}{\left(1-b\right)_p}\frac{z^p}{p!}\ \Psi_2\left(b-p;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{\left(b\right)_n}{\left(c_2\right)_n}\frac{y^n}{n!}\ \mathrm{H}_{11}\left(b+n,a_3,a_4;c_1;x,z\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(a\right)_p\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\ {}_1F_1\left(b+n-p;c_1;x\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ {}_1F_1\left(a;1-b-m-n;-z\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{\left(-1\right)^p\left(a\right)_p}{\left(1-b\right)_p}\frac{z^p}{p!}\;\;\Psi_2\left(b-p;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{\left(b\right)_n}{\left(c_2\right)_n}\frac{y^n}{n!}\;\;\mathrm{H}_4\left(b+n,a;c_1;x,z\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\;\;{}_1F_1\left(b+n-p;c_1;x\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\;\;\bar{J}_{-b-m-n}\left(2\sqrt{z}\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{\left(-1\right)^p}{\left(1-b\right)_p}\frac{z^p}{p!}\;\;\Psi_2\left(b-p;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{\left(b\right)_n}{\left(c_2\right)_n}\frac{y^n}{n!}\;\;\mathrm{H}_5\left(b+n;c_1;x,z\right).$$

4. Integral representations

Theorem 4.1. *If $Re(\beta)>Re(\alpha)>0$, then the following integral representations hold true:*

$$\begin{aligned}&\mathrm{E}_{153}\left(\alpha,a_2,a_3,b;\beta,c;x,y,z\right)\\&=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_2\left(b,a_2,a_3;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\mathrm{E}_{154}\left(\alpha,a_2,a_3,b;\beta,c;x,y,z\right)\\&=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_{11}\left(b,a_2,a_3;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\mathrm{E}_{155}\left(\alpha,a,b;\beta,c;x,y,z\right)\\&=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_3\left(b,a;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\mathrm{E}_{156}\left(\alpha,a,b;\beta,c;x,y,z\right)\\&=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_4\left(b,a;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\mathrm{E}_{157}\left(\alpha,b;\beta,c;x,y,z\right)\\&=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_5\left(b;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.5}$$

$$\mathrm{E}_{158}\left(\alpha,a,b;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{158}\left(\beta,a,b;c_1,c_2;x,y,z\xi\right)d\xi,\tag{4.6}$$

$$\mathrm{E}_{159}\left(\alpha,b;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{159}\left(\beta,b;c_1,c_2;x,y,z\xi\right)d\xi,\tag{4.7}$$

$$\mathrm{E}_{160}\left(b;\beta,c;x,y,z\right)=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta-\alpha\right)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{160}\left(b;\alpha,c;x,y,z\xi\right)d\xi.\tag{4.8}$$

Proof. To prove the relation (4.1) conformed in Theorem 4.1, let I denotes its right-hand side. Then, from the definition of Horn's function H_2 in (2.1), we get

$$I=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{n,p=0}^{\infty}\frac{(b)_{n-p}(a_2)_n(a_3)_p}{(c)_n}\frac{y^n}{n!}\frac{z^p}{p!}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}(1-x\xi)^{-b-n+p}d\xi.$$

By applying the following integral representation of the Gaussian function [7, p. 59, Eq. (10)]

$${}_2F_1(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\xi^{a-1}(1-\xi)^{c-a-1}(1-x\xi)^{-b}d\xi,\ Re(c)>Re(a)>0,$$

in (4.9), we obtain

$$I=\sum_{n,p=0}^{\infty}\frac{(a_2)_n(a_3)_p(b)_{n-p}}{(c)_n}\frac{y^n}{n!}\frac{z^p}{p!}F(\alpha,b+n-p;\beta;x). \tag{4.9}$$

Now, by virtue of the expansion (3.10), we get the required result. Using the same manner, we find the identities (4.2) to (4.5).

Let J denotes right-hand side of (4.6). By applying the following integral representation of the beta function [7, p. 9, Eq. (1)]

$$\mathrm{B}(\alpha,\beta)=\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}d\xi,\ Re(\alpha)>0,\ Re(\beta)>0,$$

in

$$J=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{m,n,p=0}^{\infty}\frac{(\beta)_p(a)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\int_0^1\xi^{\alpha-1+p}(1-\xi)^{\beta-\alpha-1}d\xi,$$

we obtain

$$J=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{m,n,p=0}^{\infty}\frac{(\beta)_p(a)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\mathrm{B}(\alpha+p,\beta-\alpha)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{4.10}$$

Now, using the well-known expression for the beta function in terms of the gamma function [7, p. 9, Eq. (5)]

$$\mathrm{B}(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

in (4.10), we get the required integral representation (4.6). Using the same manner, we find the identities (4.7) and (4.8). □

Using a similar demonstration as the previous proof, we can give the following theorem.

Theorem 4.2. *If $Re(\alpha)>0$ and $Re(\beta)>0$, then the following integral representations hold true:*

$$\mathrm{E}_{153}(\alpha,a,\beta,b;c_1,c_2;x,y,z)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{176}(\alpha+\beta,a,b;c_2,c_1;y,x\xi,z(1-\xi))d\xi,$$

$$\mathrm{E}_{153}(\alpha,\beta,a,b;c_1,c_2;x,y,z)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{178}(\alpha+\beta,a,b;c_1,c_2;x\xi,y(1-\xi),z)d\xi,$$

$$\mathrm{E}_{154}(\alpha,\beta,a,b;c_1,c_2;x,y,z)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{177}(\alpha+\beta,a,b;c_2,c_1;y,x\xi,z(1-\xi))d\xi,$$

$$\mathrm{E}_{155}(\alpha,\beta,b;c_1,c_2;x,y,z)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{173}(\alpha+\beta,b;c_1,c_2;x\xi,y(1-\xi),z)d\xi,$$

$$\mathrm{E}_{156}\left(\alpha,\beta,b;c_1,c_2;x,y,z\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{174}\left(\alpha+\beta,b;c_2,c_1;y,x\xi,z\left(1-\xi\right)\right)d\xi,$$

where E_{173}*,* E_{174}*,* E_{176}*,* E_{177} *and* E_{178} *are the confluent functions defined in* [9]*:*

$$\mathrm{E}_{173}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \sqrt{|x|}+\sqrt{|y|}<1,\ |z|<\infty,$$

$$\mathrm{E}_{174}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{n+p}(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ |x|<\infty,\ \ |y|<1,\ |z|<\infty,$$

$$\mathrm{E}_{176}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|+|y|<1,\ |z|<\infty,$$

$$\mathrm{E}_{177}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|<\infty,|z|+2\sqrt{|yz|}<1,$$

$$\mathrm{E}_{178}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \sqrt{|x|}+\sqrt{|y|}<1,\ \ |z|<\infty.$$

Theorem 4.3. *The following double integral representations hold true:*

$$\mathrm{E}_{153}\left(\alpha,\gamma,a,b;\beta,\delta;x,y,z\right)=\frac{\Gamma\left(\beta\right)\Gamma\left(\delta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\gamma\right)\Gamma\left(\beta-\alpha\right)\Gamma\left(\delta-\gamma\right)}\int_0^1\int_0^1 \xi^{\alpha-1}\eta^{\gamma-1}\left(1-\xi\right)^{\beta-\alpha-1}\times$$

$$\times\left(1-\eta\right)^{\delta-\gamma-1}\left(1-x\xi-y\eta\right)^{-b}{}_1F_1\left(a;1-b;-z\left(1-x\xi-y\eta\right)\right)d\xi d\eta, \tag{4.11}$$

$$Re(\beta)>Re(\alpha)>0,\ \ Re(\delta)>Re(\gamma)>0,\ \ b\neq 0,\pm1,\pm2,\pm3,...;$$

$$\mathrm{E}_{153}\left(\alpha,\beta,\gamma,b;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(\alpha+\beta+\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma\right)}\int_0^1\int_0^1 \xi^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\xi\right)^{\alpha-1}\times$$

$$\times\left(1-\eta\right)^{\gamma-1}\mathrm{E}_{184}\left(\alpha+\beta+\gamma,b;c_1,c_2;x\left(1-\xi\right),y\xi\eta,z\xi\left(1-\eta\right)\right)d\xi, \tag{4.12}$$

$$Re(\alpha)>0,\ \ Re(\beta)>0,\ \ Re(\gamma)>0,$$

where E_{184} *is the confluent hypergeometric function defined in* [9]*:*

$$\mathrm{E}_{184}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \sqrt{|x|}+\sqrt{|y|}<1,\ |z|<\infty.$$

Proof. To prove the relation (4.11) it is necessary to use the well-known integral representation for the Appell function F_2 (see [7, p.230, Eq.(2)]):

$$F_2\left(a,b,b';c,c';x,y\right)=\frac{\Gamma\left(c\right)\Gamma\left(c'\right)}{\Gamma\left(b\right)\Gamma\left(b'\right)\Gamma\left(c-b\right)\Gamma\left(c'-b'\right)}\times$$

$$\times\int_0^1\int_0^1 \xi^{b-1}\eta^{b'-1}\left(1-\xi\right)^{c-b-1}\left(1-\eta\right)^{c'-b'-1}\left(1-x\xi-y\eta\right)^{-a}d\xi d\eta,$$

$$Re(c)>Re(b)>0,\ \ Re(c')>Re(b')>0.$$

$\square$

5. Reduction formulas

Under exceptional circumstances, hypergeometric functions of three variables can be expressed in terms of simpler functions, notably in terms of hypergeometric functions of one or two variables or in terms of elementary functions. In such cases we speak of reducible hypergeometric functions and of reduction formulas. The exceptional circumstances arise either if the parameters in a hypergeometric series satisfy one or several relations, or if the two variables are connected by a relation. In the latter case the relation is usually the equation of a singular curve of the system of partial differential equations associated with the series in question.

Certain trivial reduction formulas are obvious: if $a_1 = 0$ in (3.2) – (3.8), if $z = 0$ in any of the series, the hypergeometric series of three variables can be expressed in terms of series of two variables: such trivial reductions are disregarded in the sequel.

The following reduction formulas can be proved either by expanding in infinite series and comparing coefficients, or by manipulating integral representations:

$$\mathrm{E}_{153}\left(c_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_2\left(b, a_2, a_3; c_2; \frac{y}{1-x}, (1-x)\,z\right),$$

$$\mathrm{E}_{154}\left(c_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_{11}\left(b, a_2, a_3; c_2; \frac{y}{1-x}, (1-x)\,z\right),$$

$$\mathrm{E}_{155}\left(c_1, a_2, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_3\left(b, a_2; c_2; \frac{y}{1-x}, (1-x)\,z\right),$$

$$\mathrm{E}_{156}\left(c_1, a_2, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_4\left(b, a_2; c_2; \frac{y}{1-x}, (1-x)\,z\right),$$

$$\mathrm{E}_{157}\left(c_1, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_5\left(b, c_2; \frac{y}{1-x}, (1-x)\,z\right).$$

6. Transformations

Although there is essentially only one hypergeometric series of the second order in one variable (namely Gauss' series), its transformation theory is quite extensive (see, Sections 2.9 to 2.11 in [7]). With the considerable number of hypergeometric series of the second order in two and three variables, the complete set of transformations would run into the hundreds, and only a few examples can be given here. The best means for deriving these (and other) transformations is the integral representations of the functions concerned where a change of variables of integration, or a deformation of the contour of integration will often yield the desired results.

First we have transformations of a series into a series of the same type:

$$\mathrm{E}_{153}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{E}_{153}\left(c_1-a_1, a_2, a_3, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)\,z\right), \tag{6.1}$$

$$\mathrm{E}_{153}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x-y)^{-b}\times$$
$$\times\,\mathrm{E}_{153}\left(c_1-a_1, c_2-a_2, a_3, b; c_1, c_2; \frac{x}{x+y-1}, \frac{y}{x+y-1}, \frac{(1-x-y)^2}{1-x}z\right), \tag{6.2}$$

$$\mathrm{E}_{154}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{E}_{154}\left(c_1-a_1, a_2, a_3, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)\,z\right), \tag{6.3}$$

$$\mathrm{E}_{155}\left(a_1, a_2, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{E}_{155}\left(c_1-a_1, a_2, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)\,z\right), \tag{6.4}$$

$$\mathrm{E}_{156}\left(a_1, a_2, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{E}_{156}\left(c_1-a_1, a_2, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)\,z\right), \tag{6.5}$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\left(1-x\right)^{-b}\mathrm{E}_{157}\left(c_1-a,b;c_1,c_2;\frac{x}{x-1},\frac{y}{1-x},\left(1-x\right)z\right). \tag{6.6}$$

All these correspond to Euler's transformation of the ordinary hypergeometric series:

$${}_2F_1(a,b;c;x)=(1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)=(1-x)^{-b}{}_2F_1\left(c-a,b;c;\frac{x}{x-1}\right). \tag{6.7}$$

The transformations (6.1) – (6.6) can be proved by applying the Boltz formula (6.7) to the expansions (3.10) – (3.14). To give an example, by the expansion (3.10) for E_{153}, we have

$$\begin{aligned}&\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)\\&=\sum_{n,p=0}^{\infty}\frac{(a_2)_n\,(a_3)_p\,(b)_{n-p}}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}(1-x)^{-b-n+p}{}_2F_1\left(c_1-a_1,b+n-p;c_1;\frac{x}{x-1}\right)\\&=\left(1-x\right)^{-b}\mathrm{E}_{153}\left(c_1-a_1,a_2,a_3,b;c_1,c_2;\frac{x}{x-1},\frac{y}{1-x},\left(1-x\right)z\right).\end{aligned}$$

No simple transformations of this type seem to be known for any of the confluent hypergeometric functions E_{158}, E_{159}, E_{160}, defined in (3.7) – (3.9).

7. Applications

Confluent hypergeometric functions of three variables have important applications.

1) Fundamental solutions of the generalized bi-axially symmetric Helmholtz equation

$$\sum_{k=1}^{n}\frac{\partial^2u}{\partial x_k^2}+\frac{2\alpha}{x_1}\frac{\partial u}{\partial x_1}+\frac{2\beta}{x_2}\frac{\partial u}{\partial x_2}-\lambda^2u=0,\ 0<2\alpha,2\beta<1$$

in the domain $\{(x_1,...,x_n):x_1>0,x_2>0\}$ are expressed by a confluent function E_{155}, for instance, one of which has an explicit form (for details, see [10]):

$$q_1=k_1r^{-2\alpha-2\beta}\mathrm{E}_{155}\left(\alpha,\beta,\alpha+\beta;2\alpha,2\beta;-\frac{4x_1\xi_1}{r^2},-\frac{4x_2\xi_2}{r^2},-\frac{\lambda^2}{4}r^2\right),\ r^2=\sum_{k=1}^{n}\left(x_k-\xi_k\right)^2.$$

Moreover, to construct fundamental solutions of the equation

$$\sum_{k=1}^{n}\left(\frac{\partial^2u}{\partial x_k^2}+\frac{2\alpha_k}{x_k}\frac{\partial u}{\partial x_k}\right)-\lambda^2u=0,\ 0<2\alpha_k<1$$

a multivariable analogue of the function E_{155} defined as

$$\mathrm{H}_A^{(n,1)}\left[\begin{matrix}a,b_1,...,b_n;\\c_1,...,c_n;\end{matrix}\ x_1,...,x_n;y\right]=\sum_{k_1,...,k_n,l=0}^{\infty}(a)_{k_1+...+k_n-l}\prod_{j=1}^{n}\frac{(b_j)_{k_j}}{(c_j)_{k_j}}\frac{x_j^{k_j}}{k_j!}\cdot\frac{y^l}{l!},\ \sum_{j=1}^{n}|x_j|<1,$$

is used [23].

2) Consider the three-dimensional singular Helmholtz equation

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z-\lambda^2u=0,\ 0<2\alpha,2\beta,2\gamma<1 \tag{7.1}$$

in the first octant $\Omega:=\{(x,y,z):x>0,y>0,z>0\}$.

Dirichlet problem. Find a regular solution $u\left(x,y,z\right)\in C\left(\overline{\Omega}\right)\cap C^2(\Omega)$ to the singular Helmholtz equation (7.1) , satisfying the following conditions

$$u(x,y,0)=\tau_1(x,y),\quad 0\le x,y<\infty,\ u(x,0,z)=\tau_2(x,z),\quad 0\le x,z<\infty, \tag{7.2}$$

$$u(0,y,z)=\tau_3(y,z),\quad 0\le y,z<\infty,\ \lim_{R\to\infty}u(x,y,z)=0,\ \ R=\sqrt{x^2+y^2+z^2}, \tag{7.3}$$

where $\tau_{1,2,3}\left(t,s\right)$ are given functions.

Theorem 7.1. *The following function*

$$u(x,y,z) = (1-2\gamma)\, k_3 x^{1-2\alpha} y^{1-2\beta} z^{1-2\gamma} \int\limits_0^\infty \int\limits_0^\infty \frac{\tau_1(t,s)\, ts}{r_1^{2a}} \mathrm{E}_{155}(1-\alpha, 1-\beta, a; 2-2\alpha,\ 2-2\beta; X)\, dtds$$

$$+ (1-2\beta)\, k_3 x^{1-2\alpha} y^{1-2\beta} z^{1-2\gamma} \int\limits_0^\infty \int\limits_0^\infty \frac{\tau_2(t,s)\, ts}{r_2^{2a}} \mathrm{E}_{155}(1-\alpha, 1-\gamma, a; 2-2\alpha,\ 2-\gamma; Y)\, dtds$$

$$+ (1-2\alpha)\, k_3 x^{1-2\alpha} y^{1-2\beta} z^{1-2\gamma} \int\limits_0^\infty \int\limits_0^\infty \frac{\tau_3(t,s)\, ts}{r_3^{2a}} \mathrm{E}_{155}(1-\gamma, 1-\beta, a; 2-2\gamma,\ 2-2\beta; Z)\, dtds,$$

where $a = 7/2 - \alpha - \beta - \gamma$,

$$r_1^2 = (x-t)^2 + (y-s)^2 + z^2,\ r_2^2 = (x-t)^2 + y^2 + (z-s)^2\ r_3^2 = x^2 + (y-t)^2 + (z-s)^2;$$

$$X = \left(-\frac{4xt}{r_1^2}, -\frac{4ys}{r_1^2}, -\frac{1}{4}\lambda^2 r_1^2\right),\ Y = \left(-\frac{4xt}{r_2^2}, -\frac{4zs}{r_2^2}, -\frac{1}{4}\lambda^2 r_2^2\right),\ Z = \left(-\frac{4yt}{r_3^2}, -\frac{4zs}{r_3^2}, -\frac{1}{4}\lambda^2 r_3^2\right),$$

is a regular solution of equation (7.1) *in* Ω, *satisfying the conditions* (7.2) *and* (7.3).

Proof. The validity of the statements of theorem 7.1 is verified by direct calculation. □

References

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