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On the Non-Smooth Optimal Control Problem for a Parametrized Dynamic System under Conditions of Uncertainty

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Abstract. In the paper, we consider a class dynamic control system with a discrete parameter and under conditions of uncertainty in the initial data. The optimal control problem of the minimax type is formulated for a non-smooth terminal functional using the principle of the best-guaranteed result. This problem is studied by methods of multivalued and convex analysis. For this non-smooth control problem the necessary and sufficient conditions for optimality are obtained.

Keywords: control system, discrete parameter, terminal functional, minimax, non-smooth control problem, conditions for optimality.

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1. Introduction

The issues of decision making in the economic planning, in the design of technical devices and control processes lead to various optimization problems. Non-smooth optimization problems constitute a special class of mathematical models [1]-[3]. They are often represented with non-smooth objective functionals. As a result of studies of non-smooth optimization models, special methods for solving them have been developed, and sections of non-smooth and multi-valued analysis are being developed [1]-[6].

One of the approaches used in decision-making in conditions of incomplete information about the initial data of the system and external influences is the principle of optimization by the minimax criterion, which assumes obtaining a guaranteed value of the quality criterion [7],[8]. This usually leads to non-smooth optimal control problems in form minimax or maximin [9]-[14]. They are closely related to the problems of controlling an ensemble of trajectories dynamic systems[15]-[17].

In this paper, we consider a dynamic control system with a discrete parameter under conditions of incomplete information about the initial state. The goal of management is to achieve a guaranteed result under such conditions of inaccuracy of information. A terminal functional of the minimum function type is considered as a criterion for evaluating the quality of management. Necessary and sufficient optimality conditions are obtained. They develop research of work [12]-[13].

2. Statement of the problem

Let $\mathbb{R}^n$ be an n- dimensional Euclidean space; (x, y) is the scalar product of vectors $x, y \in \mathbb{R}^n$,$||x||$ is the norm of the vector $x \in \mathbb{R}^n$; $\sigma(X, \psi) = \sup\{(x, \psi) : x \in X\}$ is the support function of a limited set $X \subset \mathbb{R}^n$.

Consider a dynamic control system of the form

$$\dot{x} = A(t, y)x + B(t, u, y), t \in T, u \in U, \tag{2.1}$$

where x is state n-vector, u is control m -vector, y is k - dimensional parameter of external influences, $A(t, y)$ is $n \times n$ - matrix, $b(t, u, y) \in \mathbb{R}^n$, $T = [t_0, t_1]$ is given time interval.The initial state of the system is inaccurate, that is, only set possible values are known: $x(t_0) \in D$, where D is convex compact subset of space $\mathbb{R}^n$; U is compact set of space $\mathbb{R}^m$; the parameter y accepts discrete values, i.e. $y \in Y = \{y_1, y_2, ..., y_v\} \subset \mathbb{R}^k$.

We assume that the following conditions are met:

1) the elements of the matrix $A(t, y)$ are summable on variable $t \in T = [t_0, t_1]$ for each $y \in Y$;

2) the mapping $(t, u, y) \to b(t, u, y)$ is measurable on $t \in T$ and continuous on $u \in U$ for every $y \in Y$, moreover $||b(t, u, y)|| \leq \beta(t), \beta(\cdot) \in L_1(T)$.

The admissible controls for system (2.1) are measurable and bounded m -vector function $u = u(t)$, $t \in T$, which accepts values from U almost everywhere on $T = [t_0, t_1]$.

Let U_T be the set of all admissible controls $u(\cdot)$, and $H_T(u(\cdot),y)$ be the set of all absolutely continuous solutions $x = x(t,u(\cdot),x_0,y)$ of equation (2.1) with the initial condition $x(t_0) = x_0 \in D$ for given admissible control $u(\cdot) \in U_T(y)$ and discrete parameter $y \in Y$. Under given conditions, $H_T(u(\cdot),y)$ is compact set in the space of continuous n -vector functions $C^n(T)$ [17].

Let the quality of control of a dynamical system be evaluated by a non-smooth terminal functional

$$J(u(\cdot),x_0) = \min_{l\in L} \sum_{y\in Y} \big(P(y)x(t_1,u(\cdot),x_0,y),l\big), \tag{2.2}$$

where $P(y)$ is an $s \times n$ -matrix, L is a bounded and closed set of $\mathbb{R}^s$. Typically, with a given initial state of the control system, optimal control is determined from the conditions for minimizing terminal functionality. Since the initial state of the control system (2.1) is inaccurate, then the goal of management can be considered to achieve a guaranteed value of the quality criterion $J(u(\cdot),x_0)$ in form (2.2). The best guaranteed values of the functional (2.2) we will assume the minimum value of the following functional:

$$J(u(\cdot)) = \max_{x(\cdot)\in H_T(u(\cdot),y),y\in Y} \min_{l\in L} \sum_{y\in Y} \big(P(y)x(t_1),l\big). \tag{2.3}$$

So, for the system (2.1), we will consider the non-smooth optimal control problem of the minimax type:

$$\max_{x(\cdot)\in H_T(u(\cdot),y),y\in Y} \min_{l\in L} \sum_{y\in Y} \big(P(y)x(t_1),l\big) \to \min, u(\cdot) \in U_T. \tag{2.4}$$

From the view of the posed minimax is clear that it is a control problem of terminal state of the ensemble of trajectories of a dynamical system (2.1) under conditions of indeterminacy of initial data. We will study the necessary and sufficient optimality conditions for the minimax problem (2.4).

3. Conditions for optimality

Consider the set $X_T(t_1,u(\cdot),y) = \{\xi \in \mathbb{R}^n | \xi = x(t_1,u(\cdot),x(\cdot),y), x_0 \in D\}$. It is clear that

$$X_T(t_1,u(\cdot),y) = \{\xi \in \mathbb{R}^n \mid \xi = x(t_1),\ x(\cdot) \in H_T(u(\cdot),y)\}. \tag{3.1}$$

Due to the results of [13],[17], $X_T(t_1,u(\cdot),y)$ is a convex compact set of $\mathbb{R}^n$ and for the set $X_T(t_1,u(\cdot),y)$ following formula is valid

$$X_T(t_1,u(\cdot),y) = \left\{\xi : \xi = F(t_1,t_0,y)x_0 + \int_{t_0}^{t_1} F(t_1,t,y)b(t,u(t),y)\,dt, \quad x_0 \in D\right\}, \tag{3.2}$$

here $F(t,\tau,y)$ is the fundamental matrix of solutions to the differential equation

$$\frac{dx}{dt} = A(t,y)x,$$

i.e. $\frac{\partial F(t,\tau,y)}{\partial t} = A(t,y)F(t,\tau,y), t \in T, \tau \in T, F(\tau,\tau,y) = E, E$ is an identity $n \times n$- matrix.

Using the formulas (3.1) and (3.2) we have:

$$J(u(\cdot)) = \max_{x(\cdot)\in H_T(u(\cdot),y),y\in Y} \min_{l\in L} \sum_{y\in Y} (P(y)x(t_1),l) = \max_{\xi\in X_T(t_1,u(\cdot),y),y\in Y} \min_{l\in L} \sum_{y\in Y} (P(y)\xi,l) =$$

$$\max_{x_0\in D} \min_{l\in L} \left(\sum_{y\in Y} P(y)\Big(F(t_1,t_0,y)x_0 + \int_{t_0}^{t_1} F(t_1,t,y)\,b(t,u(t),y)\,dt\Big),l\right) = \max_{\xi\in \sum\limits_{y\in Y} P(y)\,X_T(t_1,u(\cdot),y)} \min_{l\in L}(\xi,l). \tag{3.3}$$

where

$$\sum_{y\in Y} P(y)X_T(t_1,u(\cdot),y)=\sum_{y\in Y} P(y)F(t_1,t_0,y)D+\int_{t_0}^{t_1}\sum_{y\in Y} P(y)F(t_1,t,y)b(t,u(t),y)\,dt. \tag{3.4}$$

Due to the properties of the set $X_T(t_1,u(\cdot),y)$, the set in form (3.4) is closed compact set of space $\mathbb{R}^s$. Now, using the minimax theorem known from convex analysis [18],we obtain that the equality is valid

$$\max_{\xi\in\sum\limits_{y\in Y} P(y)\,X_T(t_1,u(\cdot),y)}\min_{l\in L}(\xi,l)=\min_{l\in \operatorname{co} L}\max_{\xi\in\sum\limits_{y\in Y} P(y)\,X_T(t_1,u(\cdot),y)}(\xi,l)=\min_{l\in \operatorname{co} L}\sigma\left(\sum_{y\in Y} P(y)\,X_T(t_1,u(\cdot),y),\,l\right), \tag{3.5}$$

here $\operatorname{co} L$ is convex hull of the set L , $\sigma(\sum\limits_{y\in Y} P(y)X(t_1,u(\cdot),y),l)$ is the support function of the set $\sum\limits_{y\in Y} P(y)X_T(t_1,u(\cdot),y)$. So, from (3.3) (3.5) we obtain following statement:

Lemma 3.1. *The minimax problem (2.4) can be written as follows:*

$$\min_{l\in \operatorname{co} L}\sigma\left(\sum_{y\in Y} P(y)X(t_1,u(\cdot),y),\,l\right)\to\min,\quad u(\cdot)\in U_T. \tag{3.6}$$

In that way, the minimax problem (2.4) is reduced to the problem of repeated minimize (3.6). By virtue (3.3), the flowing formula is valid:

$$\sigma\left(\sum_{y\in Y} P(y)X_T(t_1,u,y),l\right)=\sigma(D,\psi^0(l))+\int_{t_0}^{t_1}\sum_{y\in Y}\langle b(t,u(t),y),\psi(t,y,l)\rangle\,dt, \tag{3.7}$$

where $\psi(t,y,l)=F'(t_1,t,y)P'(y)l,\ \psi^0(l)=\sum\limits_{y\in Y}\psi(t_0,y,l)$.

Consider the function

$$\gamma(l)=\sigma(D,\psi^0(l))+\int_{t_0}^{t_1}\min_{u\in U}\sum_{y\in Y}\langle b(t,u,y),\psi(t,y,l)\rangle dt.$$

Theorem. 3.1.The existence of the point $l^0\in\operatorname{co} L$ of the global minimum of the function $\gamma(l), l\in\operatorname{co} L$ and the fulfillment almost everywhere on T the conditions of a minimum

$$\min_{u\in U}\sum_{y\in Y}\left(b(t,u,y),\psi(t,y,l^0)\right)=\sum_{y\in Y}\left(b(t,u^0(t),y),\psi(t,y,l^0)\right), \tag{3.8}$$

is necessary and sufficient for optimality of the admissible control $u^0(t), t\in T$.

Proof. Necessity. According to Lemma 3.1 and formula (3.7), minimax problem (2.4) can be written as problem of minimizing of the following functional

$$\mu(u(\cdot))=\min_{l\in \operatorname{co} L}\left[\sigma(D,\psi^0(l))+\int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u(t),y),\psi(t,y,l))\,dt\right],u(\cdot)\in U_T(y).$$

Therefore, if $u^0(\cdot)$ is optimal control in problem (2.4), then

$$\mu(u^0(\cdot))=\min_{u(\cdot)\in U_T}\mu(u(\cdot)).$$

We have:

$$\min_{u(\cdot)\in U_T}\mu(u(\cdot))=\min_{u(\cdot)\in U_T}\min_{l\in L}\left[\sigma(D,\psi^0(l))+\int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u(t),y),\psi(t,y,l))\,dt\right]=$$

$$= \min_{l\in \operatorname{co} L}\left[\sigma(D,\psi^0(l)) + \min_{u(\cdot)\in U_T}\int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u(t),y),\psi(t,y,l))\,dt\right] = \min_{l\in \operatorname{co} L}\gamma(l).$$

Therefore, $\mu(u^0(\cdot)) = \min\limits_{u(\cdot)\in U_T}\mu(u(\cdot)) = \min\limits_{l\in \operatorname{co} L}\gamma(l)$.

Let $l^0 \in coL$ be a point of global minimum of continuous function

$$\eta^0(l) = \sigma(D,\psi^0(l)) + \int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u^0(t),y),\psi(t,y,l))\,dt, \quad l\in \operatorname{co} L.$$

Then we have:

$$\gamma(l^0) = \sigma(D,\psi^0(l^0)) + \int_{t_0}^{t_1}\min_{u\in U}\left[\sum_{y\in Y}(b(t,u,y),\psi(t,y,l^0))\right]dt \geq$$

$$\geq \min_{l\in \operatorname{co} L}\left[\sigma(D,\psi^0(l)) + \int_{t_0}^{t_1}\min_{u\in U}\sum_{y\in Y}(b(t,u,y),\psi(t,y,l))\,dt\right] = \min_{l\in coL}\gamma(l) = \min_{u(\cdot)\in U_T}\mu(u(\cdot)) = \mu(u^0(\cdot)) =$$

$$= \min_{l\in \operatorname{co} L}\left[\sigma(D,\psi^0(l)) + \int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u^0(t),y),\psi(t,y,l))\,dt\right] = \min_{l\in \operatorname{co} L}\eta^0(l) = \eta(l^0) = \sigma(D,\psi^0(l^0)) +$$

$$+\int_{t_0}^{t_1}\sum_{y\in Y}\big(b(t,u^0(t),y),\psi(t,y,l^0)\big)\,dt \geq \sigma(D,\psi^0(l^0)) + \int_{t_0}^{t_1}\min_{u\in U}\sum_{y\in Y}\big(b(t,u,y),\psi(t,y,l^0)\big)\,dt = \gamma(l^0).$$

From these ratios it follows that

$$\gamma(l^0) = \min_{l\in \operatorname{co} L}\gamma(l), \tag{3.9}$$

i.e. the point $l^0 \in \operatorname{co} L$ is point of the global minimum of the function $\gamma(l), l\in \operatorname{co} L$, and

$$\int_{t_0}^{t_1}\min_{u\in U}\sum_{y\in Y}(b(t,u,y),\psi(t,y,l^0))dt = \int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u^0(t),y),\psi(t,y,l^0))dt.$$

By virtue of the properties of the Lebesgue integral from the latter we get that the condition (3.8) is true for almost everyone $t\in T$.

Sufficiently. Let $l^0 \in \operatorname{co} L$ be the point of the global minimum of the function $\gamma(l), l\in \operatorname{co} L$, and the conditions of a minimum (3.8) be true almost everywhere on T.

Then:

$$\mu(u^0(\cdot)) = \min_{l\in \operatorname{co} L}\left[\sigma(D,\psi^0(l)) + \int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u^0(t),y),\psi(t,y,l))dt\right] \leq \sigma(D,\psi^0(l^0)) +$$

$$+\int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u^0(t),y),\psi(t,y,l^0))dt = \sigma(D,\psi^0(l^0)) + \int_{t_0}^{t_1}\min_{u\in U}\sum_{y\in Y}(b(t,u,y),\psi(t,y,l^0))dt = \gamma(l^0)$$

$$= \min_{l\in \operatorname{co} L}\gamma(l) \leq \mu(u^0(\cdot)) = \min_{l\in \operatorname{co} L}\left[\sigma(D,\psi^0(l)) + \int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u(t),y),\psi(t,y,l))dt\right] = \mu(u(\cdot))\ \forall u(\cdot)\in U_T.$$

Therefore, $\mu(u^0(\cdot)) = \min\limits_{u(\cdot)\in U_T}\mu(u(\cdot))$. So, $u^0(\cdot)$ is the optimal control in the minimax problem (2.4). The theorem is proven.

4. Conclusion

In this paper, we studied the problem of controlling an ensemble of trajectories of a system (2.1) with discrete parameter and under conditions of uncertainty in the initial data. The problem formulated in the form of a non-smooth control problem of the minimax type. Using methods from multivalued and convex analysis, we derived the necessary and sufficient optimality conditions. They make up the theoretical basis for the method of constructing a solution to problem (2.4) by solving finite-dimensional problems of the form (3.8) and (3.9).

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