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# Differential game of one evader and multiple pursuers with exponential integral constraints

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**Abstract.** We analyze an evasion differential game involving one evader and multiple pursuers in $\mathbb{R}^n$. The control functions of the players are subject to exponential integral constraints to ensure bounded energy consumption. Evasion is considered possible if, for any time $t$, the position of the evader differs from the positions of all the pursuers. In this work, we establish a sufficient condition for the possibility of evasion. We construct an admissible evasion strategy and demonstrate that, for any number of pursuers $m$, evasion is possible. Additionally, we show that the number of maneuvers required for evasion does not exceed $m$.

**Keywords:** Differential game, evasion, control function, exponential integral constraints, evasion strategy, evader, multiple pursuers.
**MSC (2020):** 91A23; 49N75.

## 1. Introduction

Pursuit-evasion games have been a significant topic in differential game theory, with various approaches and results developed over the years. An enormous amount of work has been devoted to studying problems (for example, Azamov [1], Azamov et al. [2, 3, 4, 5] Pontryagin [21], Petrosyan [19]).

Several studies considered pursuit-evasion differential games with many players such as Chen et al. [6], Garcia et al. [7], Ibragimov and Salimi [9], Ibragimov [11], Ibragimov and Tursunaliev [13], Kumkov et al. [16], Kuchkarov et al. [14], Petrov [20], Ruziboev et al. [22, 23], Salimi and Ferrara [31], and Von Moll et al. [34].

Further extensions of the pursuit-evasion problem have been considered in various works. Ibragimov et al. [8] studied an evasion differential game that involves one evader and many pursuers. The dynamics of the players are described by linear differential equations, with integral constraints applied to the control functions of the players. They demonstrated that evasion is possible for any positive integer $m$ by showing that the total energy of the pursuers does not exceed the energy of the evader. Ibragimov et al. [12], Pansera et al. [18], Sharifi et al. [30] and Mamadaliev et al. [17] contributed to previous results in pursuit-evasion games and extended the analysis by considering integral constraints on the motion capabilities of the players.

Many studies have considered different variations of the above problem. Kuchkarov et al. [15] analyzed a differential game of the approach of many pursuers and one evader described by linear systems of the same type. They obtained estimates for the payoff function of the game that players can ensure and provide an explicit description of strategies. Ibragimov et al. [10] explored admissible and adaptive strategies in multi-agent interactions.

Samatov and Soyibboev [25] studies a pursuit differential game in which players move under inertial dynamics controlled by acceleration vectors. Using the parallel approach strategy, optimal interception is ensured against any evader action. The capture set is shown to be a linear combination of two Apollonius sets defined by the players' initial positions and velocities.

In addition, comparisons with existing work help illustrate the novelty of the approach and its potential applications in real-world scenarios. Rilwan et al. [24], Satimov [28, 29], Samatov and Uralova [26, 27], Scott and Leonard [33], Shchelchkov [32], Zhao et al. [35], Zhang et al. [36], and Zhou et al. [37] provided further insight into related topics.

In many practical scenarios, the accumulated heat in a system depends on the control effort applied over time, but past inputs contribute less to the current thermal state due to heat dissipation. To model this behavior, we impose an exponentially weighted integral constraint on the control input

$$\int_0^t e^{-k(t-s)}|u(s)|^2 ds \le \rho^2, \quad \forall t \ge 0, \tag{1.1}$$

where $u(s)$ is the control input (e.g., power in a heating system), $k > 0$ is the thermal dissipation rate, which governs how fast past control inputs lose their impact due to heat dissipation, $\rho^2$ is a bound on the effective thermal load, the exponential weight $e^{-k(t-s)}$ ensures that older control inputs contribute less to the current heat state. If we multiply the inequality (1.1) by $e^{kt}$ and denote $e^{ks/2}u(s)$ by $\bar{u}(s)$, then the inequality (1.1) takes the form

$$\int_0^t |\bar{u}(s)|^2 ds \le \rho^2 e^{kt}, \quad \forall t \ge 0.$$

Clearly, the control $u(s)$ is uniquely defined by the control $\bar{u}(s)$. In the present paper, we consider thermal type (exponential) constraints on the control functions of players.

We show that evasion is possible from any initial position of the players. In addition, we construct an explicit strategy for the evader and then prove the admissibility of the strategy. To the best of our knowledge, no prior research has addressed the specific simple motion evasion differential game with exponential integral constraints. The main difficulties in solving the problem are constructing an evasion strategy and proving that the constructed strategy guarantees evasion.

In this work, the construction of strategy requires the identification of approach times $\theta_i$. Furthermore, our approach requires $\theta_i$ to be bounded, as well as new techniques to estimate the distance between a pursuer $x_p(t)$ and the evader. Note that according to the strategy constructed, the evader moves with a positive speed in a vicinity of the $y$-axis, for any control functions of the pursuers on the time interval $[0, T]$. The fact that each maneuvering interval of the evader is contained within $[0, T]$ plays a crucial role in establishing key estimates required for the proof of the main result.

## 2. Statement of problem

We consider a simple motion evasion differential game of one evader $y$ and $m$ pursuers $x_i$, $i = 1, ..., m$, in $\mathbb{R}^n$, $n \ge 2$. Game is described by the following equations:

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(0) = x_i^0, \quad i = 1, \ldots, m, \\ &\dot{y} = v, \quad y(0) = y^0, \end{aligned} \tag{2.1}$$

where $x_i, x_i^0, y, y^0, u_i, v \in \mathbb{R}^n$, $n \ge 2$, $x_i^0 \ne y^0$, $i = 1, ..., m$ and $u_1, \ldots, u_m$ are the control parameters of pursuers and $v$ is that of evader.

**Definition 2.1.** A measurable function $u_i(t)$, $t \ge 0$, is called an admissible control of the pursuer $x_i$ if

$$\int_0^t |u_i(s)|^2 ds \le \rho_i^2 e^{2kt}, \ i = 1, \ldots, m, \tag{2.2}$$

where $\rho_1, \rho_2, ..., \rho_m$ and $k$ are given positive numbers.

**Definition 2.2.** A measurable function $v(t)$, $t \ge 0$, is called an admissible control of the evader $y$ if

$$\int_0^t |v(s)|^2 ds \le \sigma^2 e^{2kt}, \tag{2.3}$$

where $\sigma$ is a given positive number.

**Definition 2.3.** A function $V : [0, \infty) \times \mathbb{R}^{(2m+1)n} \to \mathbb{R}^n$,

$$(t, y, x_1, ..., x_m, u_1, ..., u_m) \mapsto V(t, y, x_1, ..., x_m, u_1, ..., u_m),$$

is called a strategy of evader if the following initial value problem

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(0) = x_i^0, \quad i = 1, 2, \ldots, m, \\ &\dot{y} = V(t, y, x_1, ..., x_m, u_1, ..., u_m), \quad y(0) = y^0, \end{aligned} \tag{2.4}$$

has a unique solution $(x_1(t), \dots, x_m(t), y(t))$, $t \geq 0$, for any admissible controls of the pursuers $u_i = u_i(t)$, $i = 1, ..., m$, and along this solution the following inequality

$$\int_0^t |V(s, y(s), x_1(s), \dots x_m(s), u_1(s), \dots, u_m(s))|^2 ds \leq \sigma^2 e^{2kt}.$$

holds.

**Definition 2.4.** If there exists a strategy $V$ of the evader such that for any admissible controls of pursuers $x_i(t) \neq y(t)$ for all $t \geq 0$, and $i = 1, \dots, m$, then we say that evasion is possible.

**Problem 1.** Construct a strategy $V$ for the evader $y$ and find a condition for the parameters $\rho_i$, $i = 1, ..., m$, $\sigma$, which guarantees the evasion in game (2.1)-(2.3).

Note that the evader knows the values $y(t), x_1(t), \dots, x_m(t)$, $u_1(t), \dots, u_m(t)$ at the current time $t$. During the game, the pursuers apply arbitrary controls $u_1(t), \dots, u_m(t)$, $t \geq 0$, and attempt to realize the equation $x_i(t) = y(t)$ at least for one $i \in \{1, 2, ..., m\}$, whereas the evader strives to ensure the inequalities $x_i(t) \neq y(t)$ for all $i = 1, ..., m$ and $t \geq 0$.

## 3. The main result

We consider the evasion problem in the case where $n = 2$ and prove a theorem on evasion. The following presents the main result of this paper.

**Theorem 3.1.** *If*

$$\rho_1^2 + \dots + \rho_m^2 < \sigma^2, \tag{3.1}$$

*then evasion is possible in game* (2.1)-(2.3).

Without loss of generality, we assume that $y^0 = (0, 0)$, that is, the evader is at the origin at the initial time. Then, we construct a strategy for the evader which guarantees evasion. There is no restriction in assuming that $J = \{1, 2, ..., m\}$ meaning that only the first $m$ pursuers $x_1^0, x_2^0, ..., x_m^0$ are in the upper half plane. Let $J = \{i \mid x_{i2}^0 > 0,\ 1 \leq i \leq m\}$. In Fig. 3, this set of indices is $J = \{1, 2, 3, 4, 5\}$.

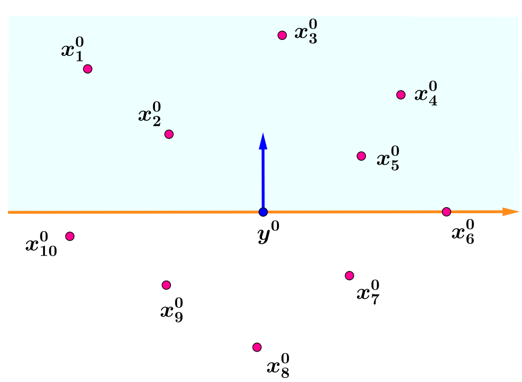


Figure 3. Example of initial states of players.

The solutions of the initial value problem (2.1) are given by

$$x_i(t) = x_i^0 + \int_0^t u_i(s) ds, \quad i = 1, ..., m, \quad y(t) = y^0 + \int_0^t v(s) ds. \tag{3.2}$$

We prove the theorem in several subsections.

3.1. **Notations.** Let $\alpha$ be any number satisfying the condition

$$0<\alpha<\frac{(\sigma-\rho)^2}{2(\max_{1\le i\le m}|y_2^0-x_{i2}^0|+1)},\quad \rho=(\rho_1^2+...+\rho_m^2)^{1/2}. \tag{3.3}$$

We choose a number $a_1$ from the condition

$$0<a_1<\min\left\{\frac{1}{2},\frac{(\sigma-\rho)^2}{4\alpha},\frac{\sigma^2}{8\alpha},\min_{1\le i\le m}|y^0-x_i^0|\right\}. \tag{3.4}$$

Let

$$T_0=\frac{1}{\alpha}\max_{1\le i\le m}|y_2^0-x_{i2}^0|,\quad T=T_0+\frac{2a_1}{\alpha},\quad \beta=\frac{\alpha^3}{4\cdot 6^4\sigma^6 e^{6kT}}. \tag{3.5}$$

We observe $\beta<\frac{1}{2}$ since $\alpha<\sigma,\ k>0$.

Let a sequence $\{a_k\}_{k=1}^{\infty}$ be defined by the formula $a_{k+1}=\beta\cdot a_k^4,\ k=1,2,...$ It is not difficult to prove that this sequence has the following:

**Property 3.2.** $\sum_{k=p+1}^{\infty}a_k\le 2a_{p+1}$ for any $p\ge 1$.

*Proof.* Since $\beta<\frac{1}{2},\ a_1<\frac{1}{2}$, we have $a_{k+1}<a_k^4,\ k=1,2,...$, and hence, $a_k<1,\ k=1,2,...$ Then

$$\sum_{k=p+1}^{\infty}a_k=a_{p+1}+a_{p+2}+...<a_{p+1}+a_{p+1}^4+a_{p+1}^{16}+...<a_{p+1}+a_{p+1}^2+a_{p+1}^3+...=\frac{a_{p+1}}{1-a_{p+1}}<2a_{p+1}.$$

The proof of the property is complete. □

3.2. **Definitions of approach times.** Let $\theta_0=0$ and $\theta_1>0$ be the first time at which

$(i)$ $|x_i(\theta_1)-y(\theta_1)|=a_1,$

$(ii)$ $x_{i2}(\theta_1)>y_2(\theta_1),$

for some $i\in J$. Note that such a time $\theta_1$ may not exist. If there are several pursuers $x_i$ that satisfy conditions $(i)$ and $(ii)$, (for example, $\theta_1$ is an $a_1$-approach time for both pursuers $x_1$ and $x_2$, but $\theta_1$ cannot be an $a_1$-approach time for both pursuers $x_3$ and $x_4$ because it does not satisfy condition $(ii)$ (see Fig. 4)), then we can assume, by relabeling if necessary, that one of such pursuers $x_i$ is $x_1$. We call $\theta_1$ the $a_1$-approach time. The times $\theta_i$ are unspecied and depend on the evaders strategy and the controls of the pursuers. It is important to note the fact that all the numbers $\theta_i$ will be in the interval $[0,T]$, which will be established in Subsection 3.4.

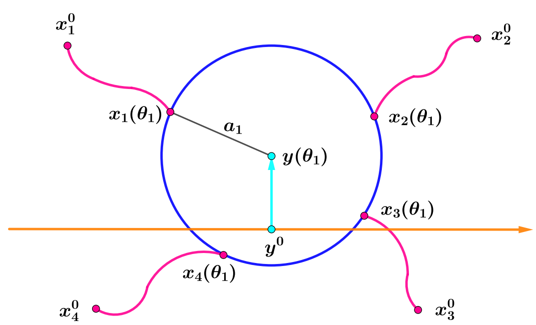


FIGURE 4. $a_1$-approach time.

If $\theta_1, \theta_2, ..., \theta_{k-1},\ \theta_1 < \theta_2 < ... < \theta_{k-1}$, are $a_1-, a_2-, ..., a_{k-1}-$ approach times, respectively, then we define $\theta_k > \theta_{k-1}$ to be the $a_k$-approach time if the following conditions are satisfied

$(i)$ $\quad |x_i(\theta_k) - y(\theta_k)| = a_k,$

$(ii)$ $\quad x_{i2}(\theta_k) > y_2(\theta_k),$

for some $i \in J$. If there are more than one such pursuers $x_i$, then we assume without loss of generality that one of them is $x_k$. In this way, we define $a_k$-approach times, $\theta_k,\ k \in J_0 = \{1, 2, ..., m_0\}$, i.e., $\theta_1, \theta_2, ..., \theta_{m_0}$, where $m_0$ is a positive integer. Note that $a_k$-approach times $\theta_k$ will not necessarily be defined for all pursuers $x_i, i \in J$. We will establish that at most one approach time will be defined for each pursuer $x_i, i \in J$, and therefore $m_0 \le m$.

Let

$$\theta'_k = \theta_k + \frac{2a_k}{\alpha}, \quad k = 1, 2, ..., m_0.$$

Note that we have defined $\theta_k$ and $\theta'_k$ only for $k = 1, 2, ..., m_0$.

3.3. **A function assigning a maneuver for the evader.** Denote $I_k = \cup_{j=k}^{m_0} [\theta_j, \theta'_j)$, $I_{m_0+1} = \emptyset$. We define a function $r : [0, T] \to \{0, 1, \dots, m_0\}$, which plays a key role in assigning a maneuver for the evader. Set

$$r(t) \quad = \quad \begin{cases} 0, & t \in [0, T] \setminus I_1, \\ k, & t \in [\theta_k, \theta'_k] \setminus I_{k+1},\ k = 1, ..., m_0. \end{cases} \tag{3.6}$$

The function $r(t)$ has the following property.

**Property 3.3.** Let $m_0 > 1$. Then, for $k = 1, 2, \dots, (m_0 - 1)$,

$(i)$ If $\theta'_k \le \theta_{k+1}$, then $r(t) = k$ for $\theta_k \le t < \theta'_k$,

$(ii)$ If $\theta_{k+1} \le \theta'_k$, then $r(t) = k$ for $\theta_k \le t < \theta_{k+1}$.

*Proof.* Assume that $\theta'_k \le \theta_{k+1}$. Then $[\theta_k, \theta'_k) \setminus I_{k+1} = [\theta_k, \theta'_k)$ since $\theta'_k \le \theta_{k+1} < \theta_{k+2} < ...$ and $[\theta_k, \theta'_k) \cap I_{k+1} = \emptyset$. Therefore, $r(t) = k$ for $t \in [\theta_k, \theta'_k)$. This proves item $(i)$.

To prove item $(ii)$, suppose that $\theta_{k+1} \le \theta'_k$. Since $\theta_k < \theta_{k+1} < \cdots < \theta_{m_0}$, we have $[\theta_k, \theta_{k+1}) \subset [\theta_k, \theta'_k) \setminus I_{k+1}$. Therefore, $r(t) = k$ for $t \in [\theta_k, \theta_{k+1})$ by the definition of $r(t)$ (3.6). □

**Example 3.4.** If

$$0 = \theta_0 < \theta_1 < \theta_2 < \theta'_2 < \theta'_1 < \theta_3 < \theta_4 < \theta'_3 < \theta'_4 < \theta_5 < \theta'_5,$$

then $r(t)$ has the graph shown in Fig. 5.

3.4. **Construction and admissibility of strategy for the evader.** We now construct a strategy for the evader. Let $u_i(t),\ i = 1, ..., m$, be arbitrary controls of pursuers. Set

$$v(t) \quad = \quad V_0(t) = \left(0, \alpha + \left(\sum_{i=1}^{m} |u_i(t)|^2\right)^{1/2}\right), \quad t \in [0, T] \setminus I_1, \tag{3.7}$$

$$v(t) \quad = \quad V_r(t) = (V_{r1}(t), U(t)),\ t \in [0, T] \cap I_1, \tag{3.8}$$

where $r = r(t)$, $V_k(t) = (V_{k1}(t), U(t))$, $\theta_k \le t < \theta'_k$, $k = 1, \dots, m_0$, is defined as follows

$$V_{k1}(t) \quad = \quad \begin{cases} \alpha + |u_{k1}(t)|, & y_1(\theta_k) \ge x_{k1}(\theta_k), \\ -(\alpha + |u_{k1}(t)|), & y_1(\theta_k) < x_{k1}(\theta_k), \end{cases} \tag{3.9}$$

$$U(t) \quad = \quad \alpha + \left(\sum_{i=1}^{m} u_{i2}^2(t)\right)^{1/2}.$$

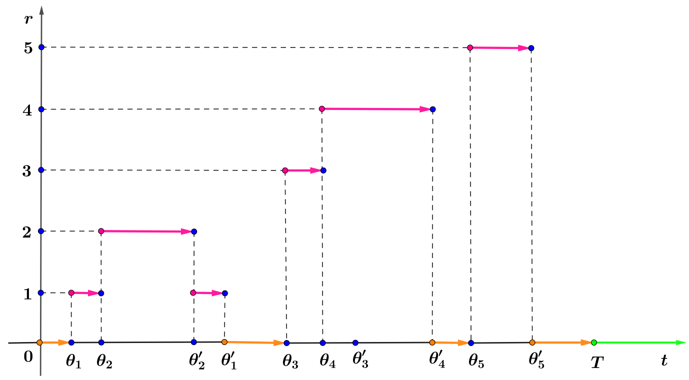


FIGURE 5. The graph of function $r(t)$.

Note that $U(t)$ doesn't depend on $k$. Finally, let

$$v(t) = \left(0, \left(\sum_{i=1}^{m} |u_i(t)|^2\right)^{1/2}\right), \quad t > T. \tag{3.10}$$

Equation (3.8) shows that the function $r = r(t)$ assigns the control $V_r(t)$ for $v(t)$.
We now show that the strategy defined by equations (3.7)-(3.10) is admissible. Indeed, let we denote

$$\varphi(t) = \begin{cases} (0, \alpha), & t \in [0, T) \setminus I_1 \\ (\alpha, \alpha), & t \in I_1 \\ (0, 0), & t > T \end{cases}, \quad \psi(t) = \begin{cases} \left(0, \left(\sum_{i=1}^{m} |u_i(t)|^2\right)^{1/2}\right), & t \in [0, T] \setminus I_1, \\ \left(|u_{r1}(t)|, \left(\sum_{i=1}^{m} u_{i2}^2(t)\right)^{1/2}\right), & t \in I_1, \\ \left(0, \left(\sum_{i=1}^{m} |u_i(t)|^2\right)^{1/2}\right), & t > T. \end{cases}$$

Note that

$$\int_0^t |\varphi(s)|^2 ds \le 2\alpha^2 T, \quad |\psi(t)|^2 \le \sum_{i=1}^{m} |u_i(t)|^2. \tag{3.11}$$

Clearly, for $v(t)$ defined by (3.7)-(3.10) we have $v_1^2(t) + v_2^2(t) = |\varphi(t) + \psi(t)|^2$. Therefore, using the Minkowskii inequality and (3.11) we obtain, for $t \ge 0$,

$$\begin{aligned} \left(\int_0^t |v(s)|^2 ds\right)^{1/2} &= \left(\int_0^t |\varphi(s) + \psi(s)|^2 ds\right)^{1/2} \le \left(\int_0^t |\varphi(s)|^2 ds\right)^{1/2} + \left(\int_0^t |\psi(s)|^2 ds\right)^{1/2} \\ &\le (2\alpha^2 T)^{1/2} + \left(\int_0^t \sum_{i=1}^{m} |u_i(s)|^2 ds\right)^{1/2} \le \alpha\sqrt{2T} + \left(\sum_{i=1}^{m} \rho_i^2 e^{2kt}\right)^{1/2} = \alpha\sqrt{2T} + \rho e^{kt} \le \sigma e^{kt}, \end{aligned}$$

since by definition of $T, T_0$ and $\alpha$

$$\begin{aligned} \alpha\sqrt{2T} &= \alpha\sqrt{2\left(T_0 + \frac{2a_1}{\alpha}\right)} = \sqrt{2\alpha\left(\max_{i=1,\dots,m} |y_2^0 - x_{i2}^0| + 2a_1\right)} \\ &\le \sqrt{2\alpha\left(\max_{i=1,\dots,m} |y_2^0 - x_{i2}^0| + 1\right)} \le \sigma - \rho. \end{aligned}$$

Here, in the last inequality we used (3.3). Thus, the evasion strategy (3.7)-(3.10) is admissible.
Next, we prove the following statement.

3.5. **One characteristics of the strategy.**

**Lemma 3.5.** *If the evader uses strategy* (3.7)-(3.10), *then*

(*a*) *For all* $k \in J_0 = \{1, ..., m_0\}$, *we have* (*i*) $\theta_k \le T_0$ *and* (*ii*) $\theta'_k \le T$.

(*b*) *If* $y_2^0 \ge x_{i2}^0$ *for some* $i \in \{1, ..., m\}$, *then* $y_2(t) > x_{i2}(t)$ *for all* $t > 0$.

*Proof.* We first show that $y_2(T_0) \ge x_{i2}(T_0)$ for all $i = 1, \ldots, m$. Indeed, by (3.7)-(3.9) we have

$$v_2(t) \ge \alpha + \left(\sum_{i=1}^{m} u_{i2}^2(t)\right)^{1/2} \ge \alpha + |u_{i2}(t)|,\ 0 \le t \le T, \tag{3.12}$$

and therefore,

$$\frac{d}{dt}(y_2(t) - x_{i2}(t)) = (v_2(t) - u_{i2}(t)) \ge (v_2(t) - |u_{i2}(t)|) \ge \alpha > 0. \tag{3.13}$$

Hence, $y_2(t) - x_{i2}(t)$, $0 \le t \le T$, increases strictly. Since $T_0 < T$, by (3.13) we have

$$y_2(T_0) - x_{i2}(T_0) = y_2^0 - x_{i2}^0 + \int_0^{T_0} (v_2(s) - u_{i2}(s))ds \ge y_2^0 - x_{i2}^0 + \alpha T_0 = y_2^0 - x_{i2}^0 + \max_{1 \le j \le m} |y_2^0 - x_{j2}^0| \ge 0.$$

Thus, $y_2(T_0) \ge x_{i2}(T_0)$ for all $i = 1, \ldots, m$.

Next, since $v_2(t) \ge \alpha + |u_{i2}(t)|$ for $T_0 \le t \le T$ (see Fig.4a), and $v_2(t) \ge |u_{i2}(t)|$ for $t > T$ (see Fig.4b), therefore for $t > T_0$ we have

$$\begin{aligned} y_2(t) - x_{i2}(t) &= y_2(T_0) - x_{i2}(T_0) + \int_{T_0}^{t} (v_2(s) - u_{i2}(s))ds \\ &\ge y_2(T_0) - x_{i2}(T_0) + \int\limits_{[T_0,t]\cap[T_0,T]} (\alpha + |u_{i2}(s)| - u_{i2}(s))ds \\ &> y_2(T_0) - x_{i2}(T_0) \ge 0. \end{aligned} \tag{3.14}$$

Thus, $y_2(t) > x_{i2}(t)$ for all $t > T_0$ and $i = 1, \ldots, m$.

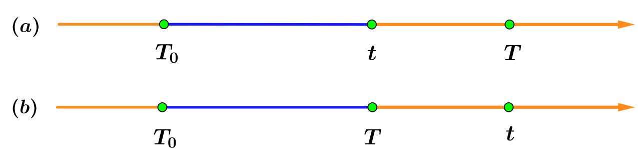


FIGURE 6. The location of $t$ relative to $T$.

In particular, we obtain that there is no $a_k$-approach time $\theta_k$ in the time interval $[T_0, \infty)$, since by definition of an $a_k$-approach time $\theta_k$, one has to have $y_2(\theta_k) < x_{k2}(\theta_k)$. This is impossible for $\theta_k \ge T_0$ since as proved above $y_2(t) \ge x_{k2}(t)$ for all $t \ge T_0$. Hence, $\theta_k \le T_0$ for all $k = 1, .., m_0$.

Next, by definition of $\theta'_k$ we have

$$\theta'_k = \theta_k + \frac{2a_k}{\alpha} \le T_0 + \frac{2a_1}{\alpha} = T, \tag{3.15}$$

and the proof of item (*a*) of Lemma 3.5 follows. In particular, (3.15) implies that $I_1 \subset [0, T]$.

**Remark 3.6.** Due to the inclusion $I_1 \subset [0, T]$ the set $[0, T] \cap I_1$ in (3.8) is equal to $I_1$.

To show item (*b*), using $y_2^0 \ge x_{i2}^0$ we observe that for $t > 0$

$$y_2(t) - x_{i2}(t) = y_2^0 - x_{i2}^0 + \int_0^t (v_2(s) - u_{i2}(s))ds \ge \int\limits_{[0,t]\cap[0,T]} (\alpha + |u_{i2}(s)| - u_{i2}(s))ds > 0.$$

The intersection $[0, t] \cap [0, T]$ in (3.16) is equal to either $[0, t]$ (see Fig.5a) or $[0, T]$ (see Fig.5b).

Thus, we have $y_2(t) > x_{i2}(t)$ for all $t > 0$ by (3.16). This completes the proof of Lemma 3.5. □

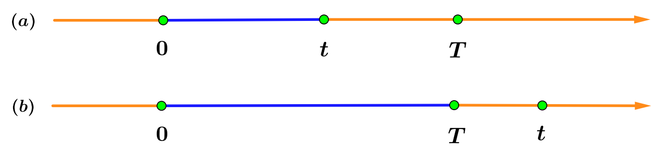


FIGURE 7. The location of $t$ relative to $T$.

3.6. **Fictitious evader $z_p$.** Take any integer $p \in \{1, \dots, m_0\}$ and assume that $\theta_p$ is the $a_p$-approach time of the pursuer $x_p$ to the evader $y$. We will estimate the distance between $x_p(t)$ and $y(t)$ on $[\theta_p, \theta_p']$. To this end, we introduce a fictitious evader (FE) $z_p$ whose motion is described by the following equation

$$\dot{z}_p = w_p, \quad z_p(\theta_p) = y(\theta_p),$$

where $w_p$ is the control parameter of $z_p$. The fictitious evader $z_p(t)$ is defined only on the interval $[\theta_p, \theta_p']$ and

$$w_p(t) = V_p(t) = (V_{p1}(t), U(t)), \quad \theta_p \le t \le \theta_p'. \tag{3.16}$$

The trajectory of FE

$$z_p(t) = y(\theta_p) + \int_{\theta_p}^{t} V_p(s)ds, \quad \theta_p \le t \le \theta_p'.$$

Since by (3.16) $v_2(t) = U(t)$, therefore

$$z_{p2}(t) = z_{p2}(\theta_p) + \int_{\theta_p}^{t} U(s)ds = y_2(\theta_p) + \int_{\theta_p}^{t} v_2(s)ds = y_2(t), \quad \theta_p \le t \le \theta_p'.$$

We now demonstrate that

$$\int_{\theta_p}^{\theta_p'} |V_p(s)|^2 ds \le \sigma^2 e^{2kT}. \tag{3.17}$$

By denoting

$$\varphi_1(t) = (\alpha, \alpha), \quad \psi_1(t) = \left( |u_{p1}(t)|, \left( \sum_{i=1}^{m} u_{i2}^2(t) \right)^{1/2} \right)$$

we have

$$|V_p(t)|^2 = V_{p1}^2(t) + U^2(t) = (\alpha + |u_{p1}(t)|)^2 + \left( \alpha + \left( \sum_{i=1}^{m} u_{i2}^2(t) \right)^{1/2} \right)^2 = |\varphi_1(t) + \psi_1(t)|^2, \quad \theta_p \le t \le \theta_p'.$$

Therefore, using the Minkowskii inequality we obtain

$$\begin{aligned} \left( \int_{\theta_p}^{\theta_p'} |V_p(s)|^2 ds \right)^{1/2} &= \left( \int_{\theta_p}^{\theta_p'} |\varphi_1(s) + \psi_1(s)|^2 ds \right)^{1/2} \le \left( \int_{\theta_p}^{\theta_p'} |\varphi_1(s)|^2 ds \right)^{1/2} + \left( \int_{\theta_p}^{\theta_p'} |\psi_1(s)|^2 ds \right)^{1/2} \\ &\le (2\alpha^2(\theta_p' - \theta_p))^{1/2} + \left( \int_{\theta_p}^{\theta_p'} \sum_{i=1}^{m} |u_i(s)|^2 ds \right)^{1/2}, \end{aligned} \tag{3.18}$$

since by (2.2) and (3.15) we have $\int_{\theta_p}^{\theta_p'} |u_i(s)|^2 ds \le \rho_i^2 e^{2k\theta_p'} \le \rho_i^2 e^{2kT}$, and then it follows from (3.18) that

$$\left( \int_{\theta_p}^{\theta_p'} |V_p(s)|^2 ds \right)^{1/2} \le 2\sqrt{\alpha a_p} + \left( \sum_{i=1}^{m} \rho_i^2 e^{2kT} \right)^{1/2} = 2\sqrt{\alpha a_p} + \left( \rho^2 e^{2kT} \right)^{1/2} = 2\sqrt{\alpha a_p} + \rho e^{kT} < \sigma e^{kT}.$$

since by (3.4) $a_p \le a_1 < \frac{(\sigma - \rho)^2}{4\alpha}$, and hence (3.17) is true.

3.7. **Distance between fictitious evader and pursuer.**

**Lemma 3.7.** *Let the pursuer $x_p$ apply an arbitrary admissible control $u_p(t)$ on $\theta_p \le t \le \theta'_p$. Then*

$$|z_p(t) - x_p(t)| > \frac{\alpha a_p^2}{6\sigma^2 e^{2kT}}, \quad \theta_p \le t \le \theta'_p, \quad \text{and} \quad y_2(t) - x_{p2}(t) \ge a_p, \quad t \ge \theta'_p. \tag{3.19}$$

*Proof.* Let $\theta_p \le t < \theta'_p$ and for definiteness assume that $x_{p1}(\theta_p) \le y_1(\theta_p)$. Then by (3.9) we have $V_{p1}(t) = \alpha + |u_{p1}(t)|$. Therefore,

$$|z_p(t) - x_p(t)| \ge z_{p1}(t) - x_{p1}(t) = y_1(\theta_p) - x_{p1}(\theta_p) + \int_{\theta_p}^t (V_{p1}(s) - u_{p1}(s))ds$$
$$\ge \int_{\theta_p}^t (\alpha + |u_{p1}(s)| - u_{p1}(s))ds \ge \alpha(t - \theta_p). \tag{3.20}$$

On the other hand,

$$|z_p(t) - x_p(t)| \ge |z_p(\theta_p) - x_p(\theta_p)| - \int_{\theta_p}^t |V_p(s) - u_p(s)|ds. \tag{3.21}$$

The integral in (3.21) can be estimated by using the Cauchy-Schwartz inequality as follows

$$\int_{\theta_p}^t |V_p(s) - u_p(s)|ds \le \left(\int_{\theta_p}^t 1^2 ds \int_{\theta_p}^t |V_p(s) - u_p(s)|^2 ds\right)^{1/2} \le \left((t-\theta_p)\int_{\theta_p}^t 2(|V_p(s)|^2 + |u_p(s)|^2)ds\right)^{1/2}. \tag{3.22}$$

Since $t - \theta_p < \theta'_p - \theta_p \le T$ by (3.15), we have

$$\int_{\theta_p}^t |V_p(s)|^2 ds \le \sigma^2 e^{2kT}, \quad \int_{\theta_p}^t |u_p(s)|^2 ds \le \rho_p^2 e^{2kT} \le \sigma^2 e^{2kT},$$

then it follows from (3.22) that

$$\int_{\theta_p}^t |V_p(s) - u_p(s)|ds \quad \le \quad (t-\theta_p)^{1/2}\left(4\sigma^2 e^{2kT}\right)^{1/2} = 2\sigma(t-\theta_p)^{1/2} e^{kT}.$$

By using (3.23) and the equation $|z_p(\theta_p) - x_p(\theta_p)| = a_p$, (3.21) yields that

$$|z_p(t) - x_p(t)| \ge a_p - 2\sigma(t-\theta_p)^{1/2} e^{kT}. \tag{3.23}$$

It is easily seen from (3.20) and (3.23) that

$$|z_p(t) - x_p(t)| \ge h(t) = \max\{h_1(t), h_2(t)\}, \quad t \ge \theta_p, \tag{3.24}$$

where

$$h_1(t) = \alpha(t - \theta_p), \quad h_2(t) = a_p - 2\sigma(t-\theta_p)^{1/2} e^{kT}.$$

Note that the funcion $h_1(t)$, $t \ge \theta_p$, is increasing, and the function $h_2(t)$, $t \ge \theta_p$, is decreasing, therefore, it is not difficult to see that the function $h(t)$, $t \ge \theta_p$, attains its minimum at the point $t = t_*$ where

$$h_1(t) = h_2(t), \quad t \ge \theta_p. \tag{3.25}$$

Let $(t - \theta_p)^{1/2} = d$. Then equation (3.25) takes the form

$$\alpha d^2 = a_p - 2\sigma d e^{kT},$$

or $\alpha d^2 + 2\sigma e^{kT} d - a_p = 0$. This equation has the following positive root

$$d_* = \frac{-\sigma e^{kT} + \sqrt{\sigma^2 e^{2kT} + \alpha a_p}}{\alpha} = \frac{a_p}{\sigma e^{kT} + \sqrt{\sigma^2 e^{2kT} + \alpha a_p}}.$$

Then

$$\min_{t \ge \theta_p} h(t) = h(t_*) = h_1(t_*) = \alpha d_*^2 = \frac{\alpha a_p^2}{\left(\sigma e^{kT} + \sqrt{\sigma^2 e^{2kT} + \alpha a_p}\right)^2}. \tag{3.26}$$

Since by (3.4) $a_1 < \frac{1}{2} < e^{2kT}$, and in view of $\alpha < \sigma^2$, we have $\alpha a_p \le \alpha a_1 < \sigma^2 e^{2kT}$, therefore (3.26) implies that

$$|z_p(t) - x_p(t)| \ge \min_{t \ge \theta_p} h(t) > \frac{\alpha a_p^2}{6\sigma^2 e^{2kT}}, \quad \theta_p \le t \le \theta_p'. \tag{3.27}$$

Next, using the fact that $y_2(\theta_p) - x_{p2}(\theta_p) \ge -|y(\theta_p) - z_p(\theta_p)| = -a_p$, and the equality $z_{p2}(\theta_p) = y_2(\theta_p)$ by (3.17) we obtain

$$z_{p2}(\theta_p') - x_{p2}(\theta_p') = y_2(\theta_p) - x_{p2}(\theta_p) + \int_{\theta_p}^{\theta_p'} (U(s) - u_{p2}(s))ds \ge -a_p + \int_{\theta_p}^{\theta_p'} \left( \alpha + \left( \sum_{i=1}^m u_{i2}^2(s) \right)^{1/2} - u_{p2}(s) \right) ds$$

$$\ge -a_p + \alpha \int_{\theta_p}^{\theta_p'} ds = -a_p + \alpha \left(\theta_p' - \theta_p\right) = -a_p + \alpha \left( \theta_p + \frac{2a_p}{\alpha} - \theta_p \right) = a_p. \tag{3.28}$$

Finally, let $t \ge \theta_p'$. By (3.17) $y_2(\theta_p') = z_{p2}(\theta_p')$, and by (3.7), (3.8) and (3.10), $v_2(t) \ge |u_{p2}(t)|$. Then using (3.10), (3.28) we get

$$y_2(t) - x_{p2}(t) = z_{p2}(\theta_p') - x_{p2}(\theta_p') + \int_{\theta_p'}^t (v_2(s) - u_{p2}(s))ds \ge z_{p2}(\theta_p') - x_{p2}(\theta_p') \ge a_p, \quad t \ge \theta_p'.$$

Thus, we have the following inequalities:

$$|z_p(t) - x_p(t)| > \frac{\alpha a_p^2}{6\sigma^2 e^{2kT}}, \quad \theta_p \le t \le \theta_p', \tag{3.29}$$

$$y_2(t) - x_{p2}(t) \ge a_p, \quad t \ge \theta_p'. \tag{3.30}$$

This completes the proof of Lemma 3.7. □

3.8. **Distance between real and fictitious evader.**

**Lemma 3.8.** *The following estimate holds*

$$|y(t) - z_p(t)| \le 6\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}, \quad \theta_p \le t \le \theta_p'. \tag{3.31}$$

*Proof.* Since $z_p(\theta_p) = y(\theta_p)$, we have

$$|y(t) - z_p(t)| = \left| \int_{\theta_p}^t (v(s) - V_p(s))ds \right|, \quad \theta_p \le t \le \theta_p'. \tag{3.32}$$

By (3.8) and (3.17)

$$v(t) = (V_{r1}(t), U(t)), \quad V_p(t) = (V_{p1}(t), U(t)), \quad \theta_p \le t < \theta_p'. \tag{3.33}$$

Consider two cases: (i) $\theta_p' \le \theta_{p+1}$ and (ii) $\theta_{p+1} \le \theta_p'$.

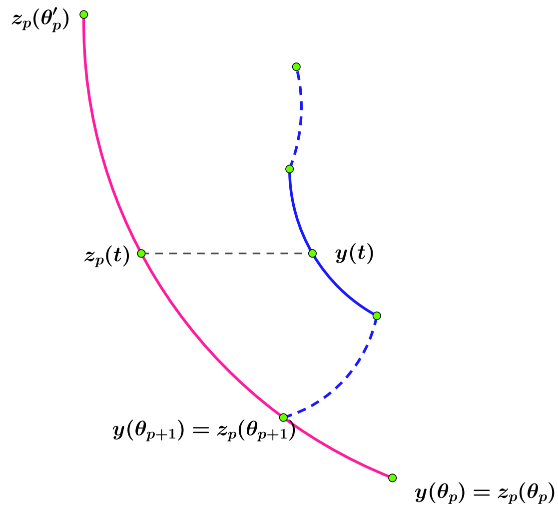


FIGURE 8. Points $y(t)$ and $z_p(t)$ are on one horizontal line.

**Case (i).** Let $\theta'_p \le \theta_{p+1}$. Then by item $(i)$ of Property 3.3 $r = r(t) = p$ for $\theta_p \le t < \theta'_p$. Therefore by (3.11) we have $v(t) = V_p(t)$, $\theta_p \le t < \theta'_p$. Hence, by (3.10)

$$|y(t) - z_p(t)| = 0. \tag{3.34}$$

**Case (ii).** Assume now $\theta_{p+1} \le \theta'_p$. Then by item $(ii)$ of Property 3.3 we have $v(t) = V_p(t)$, $\theta_p \le t < \theta_{p+1}$, therefore, $y(t) = z_p(t)$, for $t \in [\theta_p, \theta_{p+1}]$, and so (3.31) satisfied. This means that the trajectories of $y(t)$ and $z_p(t)$ coincide on $[\theta_p, \theta_{p+1}]$ (see Fig.8). Then, starting from the time $\theta_{p+1}$ the evader applies the maneuver $V_{p+1}(t)$ against the pursuer $x_{p+1}$.

Next, we estimate $|y(t) - z_p(t)|$ for $t \in [\theta_{p+1}, \theta'_p)$, we then obtain

$$\begin{aligned} |y(t) - z_p(t)| &= \left| \int_{\theta_{p+1}}^{t} (v(s) - V_p(s))ds \right| \le \int_{\theta_{p+1}}^{t} |v(s) - V_p(s)|ds \\ &\le \int\limits_{[\theta_{p+1},t)\setminus I_{p+1}} |v(s) - V_p(s)|ds + \int\limits_{[\theta_{p+1},t)\cap I_{p+1}} |v(s) - V_p(s)|ds. \end{aligned} \tag{3.35}$$

Since by definition $r(t) = p$ for $t \in [\theta_p, \theta'_p) \setminus I_{p+1}$, and $[\theta_{p+1}, t) \setminus I_{p+1} \subset [\theta_p, \theta'_p) \setminus I_{p+1}$ for $t \in [\theta_p, \theta'_p)$, therefore we have $r = r(t) = p$, and hence, $v(t) = V_p(t)$ for $t \in [\theta_{p+1}, t) \setminus I_{p+1}$. Consequently, the first integral in (3.35) is 0, and so (3.35) takes the form

$$|y(t) - z_p(t)| \le \int_{[\theta_{p+1},t]\cap I_{p+1}} |v(s) - V_p(s)|ds. \tag{3.36}$$

By (3.9) and (3.11)

$$|v(s) - V_p(s)| = |V_{r1}(s) - V_{p1}(s)| \le 2\alpha + |u_{r1}(s)| + |u_{p1}(s)|,$$

and therefore (3.36) implies that

$$|y(t) - z_p(t)| \le \int_{I_{p+1}} (2\alpha + |u_{r1}(s)| + |u_{p1}(s)|)ds. \tag{3.37}$$

To estimate the integral in (3.37), we need to estimate the integrals

$$\int_{I_{p+1}} 2\alpha ds, \ \int_{I_{p+1}} |u_{r1}(s)|ds, \ \text{and} \ \int_{I_{p+1}} |u_{p1}(s)|ds. \tag{3.38}$$

The first integral can be estimated using the definition of $\theta'_k$ and Property 3.2 as follows

$$\int_{I_{p+1}} 2\alpha ds \le \sum_{i=p+1}^{m} \int_{\theta_i}^{\theta'_i} 2\alpha ds = 2\alpha \sum_{i=p+1}^{m} (\theta'_i - \theta_i) = 2\alpha \sum_{i=p+1}^{m} \frac{2a_i}{\alpha} \le 8a_{p+1}. \quad (3.39)$$

Next, we estimate the second integral in (3.38). Using the Cauchy-Schwartz inequality we have

$$\int_{I_{p+1}} |u_{r1}(s)|ds \le \left(\int_{I_{p+1}} ds\right)^{1/2} \left(\int_{I_{p+1}} |u_{r1}(s)|^2 ds\right)^{1/2}. \quad (3.40)$$

Since $\theta_p \le t \le \theta'_p \le T$, we have

$$\int_{I_{p+1}} |u_{r1}(s)|^2 ds \le \sum_{i=1}^{m} \int_0^t |u_i(s)|^2 ds \le \sigma^2 e^{2kt} \le \sigma^2 e^{2kT}$$

and similar to (3.39) for the first integral in the right hand side of (3.40) we get

$$\int_{I_{p+1}} ds \le \sum_{i=p+1}^{m} \int_{\theta_i}^{\theta'_i} ds \le \frac{4a_{p+1}}{\alpha}.$$

Then it follows from (3.40) that

$$\int_{I_{p+1}} |u_{r1}(s)|ds \le 2\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}. \quad (3.41)$$

Similarly, for the third integral in (3.38), we have

$$\int_{I_{p+1}} |u_{p1}(s)|ds \le 2\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}. \quad (3.42)$$

Combining (3.39), (3.41), and (3.42) we obtain from (3.37) that

$$|y(t) - z_p(t)| \le 8a_{p+1} + 4\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}} \le 6\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}$$

using the inequality

$$16a_{p+1} < 2\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}$$

which follows from the inequalities $a_{p+1} \le a_1 < \frac{\sigma^2}{8\alpha}$ (see (3.4)).

Thus,

$$|y(t) - z_p(t)| \le 6\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}. \quad (3.43)$$

The proof of the lemma is complete. □

3.9. **Distance between evader and pursuer.** Using (3.29) and (3.43) we obtain

$$|y(t) - x_p(t)| \ \ge \ |x_p(t) - z_p(t)| - |z_p(t) - y(t)| \ge \frac{\alpha a_p^2}{6\sigma^2 e^{2kT}} - 6\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}} = \frac{\alpha a_p^2}{12\sigma^2 e^{2kT}}$$

for $t \in [\theta_p, \theta'_p]$ since by (3.5)

$$a_{p+1} = \frac{\alpha^3}{4 \cdot 6^4 \sigma^6 e^{6kT}} a_p^4.$$

Also, it follows from the definition of $\beta$ and the inequality $a_p < 1$ that

$$a_{p+1} \le \frac{\alpha}{16\sigma^2 e^{2kT}} a_p^4 \le \frac{\alpha}{12\sigma^2 e^{2kT}} a_p^2.$$

Therefore, (3.44) implies that $|y(t) - x_p(t)| > a_{p+1},\ \theta_p \le t \le \theta'_p$. Also, by (3.30)

$$y_2(t) - x_{p2}(t) \ge a_p, \quad t \ge \theta'_p.$$

Thus, starting from the time $\theta'_p$ we can ignore the pursuer $x_p$ since $x_p(t) \ne y(t)$ for all $t \ge \theta'_p$ for this pursuer. We now can conclude that

(1) if $y_2^0 \ge x_{i2}^0$ for the pursuer $x_i$, then by item (ii) of Lemma 3.5 $y_2(t) > x_{i2}(t)$ for all $t > 0$ and hence $x_i(t) \ne y(t)$ for all $t > 0$. This means the evader ensures evasion from such a pursuer.

(2) if $x_{i2}^0 > y_2^0$ for all $i \in \{1, 2, ..., m\}$, then the $a_i$-approach of the pursuer $x_i$ may occur at some $\theta_i$. Then, as proved, we have

$$|y(t) - x_p(t)| \ge a_p,\ \text{for } 0 \le t \le \theta_p, \ \ \text{(by definition of } \theta_p) \tag{3.44}$$

$$|y(t) - x_p(t)| \ge \frac{\alpha a_p^2}{12\sigma^2 e^{2kT}} > a_{p+1},\ \text{for } \theta_p \le t \le \theta'_p, \ \ \text{(by (3.44))} \tag{3.45}$$

$$y_2(t) - x_{p2}(t) \ge a_p,\ \text{for } t \ge \theta'_p, \ \ \text{(by (3.30))} \tag{3.46}$$

Based on these relations, we summarize as follows:

If an $a_p$-approach time $\theta_p$ of pursuer $x_p$ to the evader $y$ occurs, then $x_p(t) \ne y(t)$, for all $t \ge 0$ (see (3.44)- (3.46)). Moreover, for any $i \ge p + 1$, there is no $a_i$-approach time $\theta_i$ of the pursuer $x_p$ to the evader $y$. This means that even all the pursuers are in the upper half-plane, and the evader ensures evasion by applying its own maneuver.

The proof of Theorem 3.1 is completed.

## 4. Conclusion

We have analyzed a differential game of evasion from many pursuers. The control functions of the players are subject to exponential integral constraints. We have constructed a strategy for the evader and demonstrated that evasion is possible. The evader uses the control $v(t) = \left(0, \alpha + \left(\sum_{i=1}^m |u_i(t)|^2\right)^{1/2}\right)$ on the set $[0, T] \setminus I_1$ and applies a maneuver on the set $I_1$. The measure of the set $I_1$ can be made by choosing the parameters $a_1$ and $\alpha$ as small as we wish. We have also shown that all the approach times $\theta_i$ for each pursuer can occur only before a specific time $T_0$, and the approach times $\theta'_i$ satisfy $\theta'_i \le T$. The total number of approach times $\theta_i$ associated with all pursuers does not exceed the total number of pursuers, $m$. The evader uses the control $v(t) = \left(0, \left(\sum_{i=1}^m |u_i(t)|^2\right)^{1/2}\right)$ for $t \ge T$, and the approach time no longer occurs.

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