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On a Boundary Value Problem for a Loaded Parabolic-Hyperbolic Equation of the Second Kind Degenerating Inside the Domain

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Abstract. This paper studies an analog of the Tricomi problem for a loaded parabolic-hyperbolic equation of the second kind, which degenerates inside the domain. In proving the existence and uniqueness theorem for a classical solution to the Tricomi-type problem, a general representation of the solution to the loaded parabolic-hyperbolic equation degenerating within the domain is derived. The uniqueness of the solution is established using the extremum principle and the energy integral method. The existence of the solution is equivalently reduced to integral equations of the second kind, specifically Volterra and Fredholm equations, which remain relatively unexplored. Furthermore, a class of prescribed functions is determined to ensure the solvability of the obtained integral equations.

Keywords: Second-kind equation, loaded equation, extremum principle, method of energy integrals, Fredholm integral equation of the second kind.

MSC (2020): 35M10, 35M12, 35L80, 35K65, 35K20.

1. Introduction

Many important problems in mathematical physics and biology, particularly long-term groundwater prediction and regulation problems [1], modeling of particle transport processes [2], heat and mass transfer problems with finite velocity, liquid filtration modeling in porous media [3], inverse problem studies [4], and various optimal control problems in agroecosystems [5] lead to boundary value problems for loaded partial differential equations.

The term "loaded equation" first appeared in [6]. The general definition of loaded equations, which is now widely accepted in scientific literature, was introduced by A.M. Nakhushev in 1976. In [7], a more comprehensive definition and detailed classification of various types of loaded equations were provided, including loaded differential, integral, integro-differential, and functional equations, along with their numerous applications.

Boundary value problems for non-degenerate loaded equations of mixed type of the second and third orders, where the loaded part contains a trace or derivative of the unknown function, have been studied in [8–20]. For degenerate-loaded equations of hyperbolic, parabolic, and mixed types, investigations have been carried out in [21–26]. A three-dimensional analog of the Tricomi problem for a loaded parabolic-hyperbolic equation was examined in [27]. The theory of boundary value problems for second-order loaded equations with integro-differential operators has been explored in [28–33].

In the study of degenerate loaded equations of mixed type of the second kind, difficulties arise associated with the lack of a general representation of the solution, as well as the impossibility of direct application of classical methods. This problem is solved in this paper. A new method for constructing a representation of the general solution of a loaded parabolic-hyperbolic equation of the second kind in a form convenient for further studies of various boundary value problems is developed, and a new type of extremum principle for a degenerate loaded parabolic-hyperbolic equation of the second kind is proved. The analysis of the state of affairs in this direction shows that boundary value problems for degenerate loaded equations leading to less studied integral Volterra and Fredholm equations with shifts.

In this paper, we study problems with the Tricomi condition for a loaded parabolic-hyperbolic equation of the second kind, degenerating inside the domain. We prove the existence and uniqueness theorems of the classical solution of the problems posed. The proofs of the theorem are based on energy identities and the extremum principle, as well as on the theory of Volterra and Fredholm integral equations.

2. Formulation of Problem A_T

Let Ω be a finite simply connected domain in the plane of variables x, y bounded by curves:

$$S_j: \ \ (-1)^{j-1}x = 1, \ 0 < y < 1, \ \ S_3: \ \ 0 < x < 1, \ y = 1, \ \ S_4: \ \ -1 < x < 0, \ y = 1,$$

$$\Gamma_j: (-1)^{j-1}x - \frac{2}{2-m}(-y)^{2/(2-m)} = 0, \ \Gamma_{j+2}: (-1)^{j-1}x + \frac{2}{2-m}(-y)^2/(2-m) = 1, \ y < 0, (j = 1, 2).$$

We introduce denotations

$$\Omega_j^+ = \Omega \cap \left\{(x, y) : (-1)^{j-1}x > 0, \ y > 0\right\}, \ \Omega_j^- = \Omega \cap \left\{(x, y) : \ (-1)^{j-1}x > 0, \ y < 0\right\},$$

$$I_j = \left\{(x, y) : \ 0 < (-1)^{j-1}x < 1, \ y = 0\right\}, \ \Omega_j = \Omega_j^+ \cup \Omega_j^- \cup I_j, \ I_3 = \left\{(x, y) : \ x = 0, \ 0 < y < 1\right\},$$

$$\Omega_3 = \Omega_1^+ \cup \Omega_2^+ \cup I_3, \quad A_j\left((-1)^{j-1}, 0\right) = \bar{I}_j \cap \bar{S}_j, \ O\left(0, 0\right) = \bar{I}_1 \cap \bar{I}_2, \ B_1\left(1, 1\right) = \bar{S}_1 \cap \bar{S}_3,$$

$$B_2\left(-1, 1\right) = \bar{S}_2 \cap \bar{S}_4, \ B_0\left(0, 1\right) = \bar{S}_3 \cap \bar{S}_4, \ C_j\left[\tfrac{(-1)^{j-1}}{2}; -\left(\left(-1\right)^{j-1}\tfrac{2-m}{4}\right)^{2/(2-m)}\right] = \bar{\Gamma}_j \cap \bar{\Gamma}_{j+2}, (j = 1, 2).$$

Next, we assume that the domains Ω_1^+ and Ω_1^- should be symmetric domains of Ω_2^+ and Ω_2^- with respect to the axes Oy.

In the domain Ω we consider the following equation

$$0 = \begin{cases} u_{xx} - ((-1)^{j-1}x)^p u_y - \rho_j u\left(x, 0\right), & (x, y) \in \Omega_j^+, \\ u_{xx} - \left(-y\right)^m u_{yy} + \mu_j u\left(x, 0\right), & (x, y) \in \Omega_j^-, \end{cases} \tag{2.1}$$

where $m, \quad p, \quad \rho_j, \quad \mu_j \ (j = 1, 2)$ are any real numbers, and

$$0 < m < 1, \quad p > 0, \quad \rho_j > 0, \quad \mu_j > 0, \ \ (j = 1, 2). \tag{2.2}$$

In the domain Ω for equation (2.1) we investigate the following problem.

Problem A_T. Find a function $u(x, y)$ that satisfies the following properties:

1) $u(x, y) \in (\overline{\Omega}) \cap C^1(\Omega)$; 2) $u(x, y) \in C_{x,y}^{2,1}(\Omega_1^+ \cup \Omega_2^+)$ and it is a regular solution of equation (2.1) in the domain Ω_j^+; 3) $u(x, y)$ is a generalized solution of equation (2.1) from the class R_2 [34] in the domain Ω_j^- $(j = 1, 2)$; 4) $u(x, y)$ satisfies the boundary conditions:

$$u|_{S_j} = \varphi_j(y), \quad 0 \leq y \leq 1, \tag{2.3}$$

$$u|_{\Gamma_j} = \psi_j(x), \qquad 0 \leq (-1)^{j+1}x \leq \frac{1}{2}, \tag{2.4}$$

5) $u(x, y)$ satisfies the matching conditions on the degeneration line I_i $\left(i = \overline{1, 3}\right)$:

$$\lim_{y \to +0} u(x, y) = \lim_{y \to -0} u(x, y), \quad \lim_{y \to +0} u_y(x, y) = \lim_{y \to -0} u_y(x, y), \quad (x, 0) \in I_j \ (j = 1, 2), \tag{2.5}$$

$$\lim_{x \to +0} u(x, y) = \lim_{x \to -0} u(x, y), \quad \lim_{x \to +0} u_x(x, y) = \lim_{x \to -0} u_x(x, y), \quad (0, y) \in I_3, \tag{2.6}$$

where $\varphi_j(y)$, $\psi_j(x) (j = 1, 2)$ are given function, and $\psi_1\left(0\right) = \psi_2\left(0\right)$,

$$\varphi_j(y) \in C(\bar{I}_3) \cap C^1(I_3), \tag{2.7}$$

$$\psi_1(x) \in C^2\left[0, \frac{1}{2}\right], \ \ \psi_2(x) \in C^2\left[-\frac{1}{2}, 0\right]. \tag{2.8}$$

3. Investigation of Problem A_T for equation (2.1)

If the conditions 1) and 2) of Problem A_T, any regular solution of equation (2.1) can be represented as in [16], ([35],p.3-6):

$$u(x,y)=\upsilon(x,y)+\omega(x) \tag{3.1}$$

where

$$\upsilon(x,y)=\begin{cases}\upsilon_j(x,y), & (x,y)\in\Omega_j^+,\\ w_j(x,y) & (x,y)\in\Omega_j^-,\end{cases} \tag{3.2}$$

$$\omega(x)=\begin{cases}\omega_j^+(x), & (x,+0)\in\bar{I}_j,\\ \omega_j^-(x), & (x,-0)\in\bar{I}_j,\end{cases} \tag{3.3}$$

here $\upsilon_j(x,y)$ and $w_j(x,y)(j=1,2)$ are regular solutions of the equation

$$L\upsilon_j\equiv\upsilon_{j\,xx}-((-1)^{j-1}x)^p\upsilon_{jy}=0,\qquad (x,y)\in\Omega_j^+, \tag{3.4}$$

$$Lw_j\equiv w_{j\,xx}-(-y)^m w_{j\,yy}=0,\qquad (x,y)\in\Omega_j^-\quad (j=1,2) \tag{3.5}$$

and $\omega_j^+(x)$ and ω_j^- $(j=1,2)$ are arbitrary twice continuously differentiable solutions of the equations

$$\omega_j^{+''}(x)-\rho_j\omega_j^+(x)=\rho_j\upsilon_j(x,0),\qquad (x,0)\in I_j, \tag{3.6}$$

$$\omega_j^{-''}(x)+\mu_j\omega_j^-(x)=-\mu_j w_j(x,0),\qquad (x,0)\in I_j. \tag{3.7}$$

Since the function $ax+b$ is a solution to equations (3.4) and (3.6), arbitrary functions $\omega_j^+(x)$ and $\omega_j^-(x)$ can be chosen to satisfy the conditions

$$\omega_j^+\left((-1)^{j+1}\right)=\omega_j^{+'}\left((-1)^{j+1}\right)=0, \tag{3.8}$$

$$\omega_j^-(0)\;=\;\omega_j^{-'}(0)=0,\quad (j=1,2)\,. \tag{3.9}$$

The solution to the Cauchy problem (3.7), (3.9) and (3.8), (3.10) is given by:

$$\omega_j^+(x)=\sqrt{\rho_j}\int_{(-1)^{j-1}}^{x}\tau_j(t)\;sh\sqrt{\rho_j}(x-t)dt,\qquad (x,0)\in\bar{I}_j, \tag{3.10}$$

and

$$\omega_j^-(x)=-\sqrt{\mu_j}\int_0^x\tau_j(t)sin\sqrt{\mu_j}\,(x-t)\,dt,\qquad (x,0)\in\bar{I}_j, \tag{3.11}$$

where

$$\tau_j(x)\equiv\upsilon_j(x,+0)=w_j(x,-0),\qquad (x,0)\in\bar{I}_j(j=1,2). \tag{3.12}$$

In view of (2.1),(2.3), (2.4), (3.2), (3.3), (3.9), (3.10) Problem A_T reduces to Problem A_T^* for the equation

$$0=\begin{cases}L\,\upsilon_j, & (x,y)\in\Omega_j^+,\\ L\,w_j, & (x,y)\in\Omega_j^-\end{cases} \tag{3.13}$$

with boundary conditions

$$\upsilon_j|_{S_j}=\varphi_j(y),\;\; 0\le y\le 1, \tag{3.14}$$

$$w_j|_{\Gamma_j}=\psi_j(x)-\omega_j^-(x),\qquad 0\le(-1)^{j+1}x\le\frac{1}{2}, \tag{3.15}$$

where $\omega_j^-(x)$ is determined from (3.12), $(j=1,2)$.

4. Uniqueness of solution of the Problem A_T

To prove the uniqueness of the solution of Problem A_T, we first prove the uniqueness of the solution of Problem A_T^* for equation (3.14). The following lemmas play an important role in proving the uniqueness of the solution to Problem A_T^* for equation (3.14).

Lemma 1. *If the conditions (2.2),* $-1 < 2\beta < 0$, $p + 2\beta > 0$, $\varphi_1(y) \equiv \varphi_2(y) \equiv 0$, $\forall y \in [0,1]$ *and* $\psi_1(x) \equiv 0$, $\forall x \in \left[0; \frac{1}{2}\right]$, $\psi_2(x) \equiv 0$, $\forall x \in \left[-\frac{1}{2}; 0\right]$ *are satisfied, then*

$$\tau_j(x) \equiv 0, \quad \forall (x,0) \in \bar{I}_j, \tag{4.1}$$

where $\tau_j(x)$ *is defined from (3.13)* $(j = 1, 2)$, $2\beta = m/(m-2)$.

Lemma 1 is proven in exactly the same way as in ([25], p. 39-41, Lemma 3.1).

From (4.1), using (3.11) and (3.12) , we obtain

$$\omega_j^+(x) = \omega_j^-(x) = 0, \quad \forall x \in \bar{I}_j.$$

Thus, from (3.3), we have

$$\omega(x) \equiv 0, \ \ \forall\, x \in \bar{I}_1 \ \cup \bar{I}_2\,. \tag{4.2}$$

Lemma 2. *The solution* $\upsilon(x,y) \in C\left(\bar{\Omega}_3\right) \cap C^1(\Omega_3) \cap C^{2,1}_{x,y}(\Omega_1^+ \cup \Omega_2^+)$ *of equation (3.4) in the closed domain* $\bar{\Omega}_3$ *attains its positive maximum and negative minimum only on* $\overline{A_1B_1} \cup \overline{A_2B_2} \cup I_1 \cup I_2$.

Proof. By the extremum principle for parabolic equations [36, 37, 38], the solution of equation (3.4) inside the domain Ω_1^+ and Ω_2^+ cannot attain its positive maximum and negative minimum. We will show that the solution $\upsilon(x,y)$ of equation (3.4) in the domain Ω_3 does not attain its positive maximum (or negative minimum) on $\bar{I}_3$.

Assume the contrary. Suppose that $\upsilon(x,y)$ attains its positive maximum (or negative minimum) at some point $(0, y_0)$ on the interval I_3. Then, based on the extremum principle [36, 39] from the domain Ω_1^+ , we have

$$\upsilon_x(+0, y_0) < 0 \ (> 0). \tag{4.3}$$

On the other hand, from the domain Ω_2^+ , we obtain

$$\upsilon_x(-0, y_0) > 0 \ (< 0).$$

This inequality, due to the matching condition $\upsilon_x(+0,y) = \upsilon_x(-0,y)$, $(0,y) \in I_3$, contradicts the relation (4.3). Therefore, $\upsilon(x,y)$ does not attain its positive maximum (or negative minimum) on the interval I_3.

By condition (3.10) from (3.16), and considering (3.2) and $\psi_1(0) = \psi_2(0) = 0$, it follows that $\upsilon(0,0) = w_j(0,0) = \upsilon_j(0,0) = 0$. Thus, $\upsilon(x,y)$ does not attain its extremum at the point $O(0,0)$.

Using Lemmas 1.1 and 1.2 ([36], ch. 2, § 2.3. p. 93-94), it can be proven that at the point $B_0(0,1)$, there is no positive maximum (or negative minimum).

Therefore, $\upsilon(x,y)$ does not attain its positive maximum (or negative minimum) on the interval $\bar{I}_3$. □

Theorem 1. *If the conditions of Lemmas 1-2 and (4.2) are satisfied, then the solution of Problem* A_T^* *for equation (3.14) is unique in the domain* Ω.

Proof. According to the maximum principle for parabolic equations [37, 38, 40], and considering Lemma 2, the Problem with conditions(3.13) and (3.15) for equation (3.4) in the domain $\bar{\Omega}_3$ with $\tau_j(x) \equiv \varphi_j(y) \equiv 0 (j = 1,2)$, has no non-zero solution, i.e.

$$\upsilon_j(x,y) \equiv 0 \ \ in \ \ \bar{\Omega}_j^+ \ (j = 1,2). \tag{4.4}$$

Due to the uniqueness of a solution of the Cauchy problem with homogeneous conditions $w_j(x,y)\big|_{y=0} = 0$, $(x,0) \in \bar{I}_j$, $w_{j\,y}(x,y)\big|_{y=0} = 0$, $(x,0) \in I_j$ for equation (3.6) in the domain Ω_j, it follows that

$$w_j(x,y) \equiv 0, (x,y) \in \bar{\Omega}_j. \tag{4.5}$$

Due to (4.4) and (4.5), from (3.2), we have

$$\upsilon(x,y) \equiv 0, \ (x,y) \in \bar{\Omega}\,. \tag{4.6}$$

From (4.6) the uniqueness of the solution of Problem A_T^* for equation (3.14) follows. □

Theorem 2. *If the conditions of Theorem 1 are satisfied, then the solution of Problem A_T for equation (2.1) is unique in the domain Ω.*

Proof. From (4.2), (4.6) and (3.1), it follows that $u(x,y)\equiv 0,\ \ (x,y)\in\bar{\Omega}$. Thus, the uniqueness of the solution of Problem A_T for equation (2.1) follows. □

5. Existence of a Solution to Problem A_T

The existence of a solution to Problem A_T is proven using the method of integral equations. To prove the existence of a solution to Problem A_T, we first prove the existence of a solution to Problem A_T^* for equation (3.14) with the conditions (3.15) and (3.16).

Theorem 3. *If the conditions (2.2), (2.8), (2.9) and*

$$-1<2\beta<0,\ \ p+2\beta>0,\ \ \psi_1(0)=\psi_2(0)=0 \tag{5.1}$$

are satisfied, then the solution to Problem A_T^* exists in the domain Ω .

Proof. The proof of Theorem 3 relies on the following problems, which have independent interest.

Problem $B_j (j=1,2)$. Find a solution $\upsilon(x,y)\in C(\bar{\Omega}_j)\cap C^1(\Omega_j)\cap C^{2,1}(\Omega_j^+)\cap C^2(\Omega_j^-)$ to equation (3.14), satisfying the conditions (3.15), (3.16) and

$$\upsilon(0,y)=\tilde{\tau}_3(y)-\omega_j^+(0),\quad (0,y)\in\bar{I}_3,(j=1,2), \tag{5.2}$$

where $\tilde{\tau}_3(y)=u(0,y),\ \ (0,y)\in\bar{I}_3$ is a given function, and $\tilde{\tau}_3(0)=\psi_1(0)=\psi_2(0)=0$,

$$\tilde{\tau}_3(y)\in C(\bar{I}_3)\cap C^1(I_3), \tag{5.3}$$

and $\omega_j^+(0)$ is determined from (3.11) .

Problem B_3. Find a solution $\upsilon(x,y)\in C(\bar{\Omega}_3)\cap C^1(\Omega_3)\cap C^{2,1}_{x,y}(\Omega_1^+\cup\Omega_2^+)$ to equation (3.4), satisfying the conditions (3.15) and

$$\upsilon(x,y)|_{y=0}=\tau_j(x)-\omega_j^+(x),\qquad (x,0)\in\bar{I}_j, \tag{5.4}$$

where $\tau_j(x)$ and $\omega_j^+(x)(j=1,2)$ are determined respectively from (3.11), and

$$\tau_j(x)=w_j(x,-0)=\upsilon_j(x,+0),\quad (x,0)\in\bar{I}_j.$$

5.1. Investigation of Problem $B_j,\ \ (j=1,2)$.

Theorem $4_j,(j=1,2)$. *If the conditions (2.2), (2.8), (2.9), (5.1), and (5.3) are satisfied, then a solution to Problem B_j, exists and is unique in the domain Ω_j.*

Proof. By Lemma 1 and from the extremum principle for degenerate parabolo-hyperbolic equations [36], it follows that the solution $\upsilon(x,y)$ to Problem B_j with $\psi_j(x)\equiv 0$ has a positive maximum (PM) and a negative minimum (NM) in the closed domain $\bar{\Omega}_j^+$, which is attained only at $\overline{S}_j\cup\bar{I}_3,\ \ (j=1,2)$.

According to the extremum principle, the homogeneous Problem B_j, taking into account conditions (3.3) and (4.2), i.e., the problem with zero boundary conditions, has no solution other than zero. This implies the uniqueness of the solution to Problem B_j.

Now we proceed to prove the existence of a solution to the problem with conditions (3.15), (3.16) and (5.2).

The generalized solution of the class R_2 [25, 34] of the Cauchy problem with initial conditions

$$\lim_{y\to 0-}w_j(x,y)=\tau_j(x),(x,0)\in\bar{I}_j,\quad \lim_{y\to 0-}w_{jy}(x,y)=\nu_j(x),\quad (x,0)\in I_j,$$

for equation (3.6) in the domain Ω_j^- $(j=1,2)$ is given by the formula

$$w_j(\xi,\eta)=(-1)^{j-1}\left[\int_0^\xi(\eta-t)^{-\beta}(\xi-t)^{-\beta}T_j(t)\,dt+\int_\xi^\eta(\eta-t)^{-\beta}(t-\xi)^{-\beta}N_j(t)\,dt\right], \tag{5.5}$$

where $\xi=(-1)^{j-1}x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}}, \qquad \eta=(-1)^{j-1}x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}},$

$$N_j\left(x\right)=T_j\left(x\right)/2\cos\pi\beta-\gamma_2\nu_j\left(x\right), \quad \gamma_2=\left[2\left(1-2\beta\right)\right]^{2\beta-1}\frac{\Gamma\left(2-2\beta\right)}{\Gamma^2\left(1-\beta\right)}. \tag{5.6}$$

$$\tau_j(x)=(-1)^{j-1}\int_0^x\frac{T_j(t)dt}{((-1)^{j-1}(x-t))^{2\beta}}=\Gamma(1-2\beta)(-1)^{j-1}D_{0x}^{2\beta-1}T_j(x),(x,0)\in\bar I_j, \tag{5.7}$$

where the function $T_j(x)$ is continuous in I_j and integrable on $\bar I_j$, and $\tau_j\left(x\right)$ and becomes zero of order no less than at $1-2\beta$ when $x\to 0$, and $D_{0x}^{2\beta-1}\left[\bullet\right]$ is the integral-differential operator of fractional order ([41], p. 42-43).

By setting $\xi=0$, $\eta=x$, in (5.5) considering (3.16), (5.6), and the properties of integral-differential operators of fractional order ([41], p. 42-43), ([42], p. 16-21), we obtain

$$T_j(x)=\gamma_3\nu_j(x)+\frac{2(-1)^{j-1}\cos\pi\beta}{\Gamma(1-\beta)}((-1)^{j-1}x)^{\beta}D_{0x}^{1-\beta}\left[\psi_j(x)-\omega_j^-(x)\right], \quad (x,0)\in I_j, \tag{5.8}$$

where $\gamma_3=2\gamma_2\cos\pi\beta$, and $\omega_j^-(x)$ is determined from (3.12) $(j=1,2)$,

Substituting (5.8) into (5.7), we find the first functional relationship between $\tau_j(x)$ and $\nu_j(x)$, transferred from the domain Ω_j^- to domain I_j $(j=1,2)$:

$$\tau_j(x)=\Gamma(1-2\beta)\gamma_3(-1)^{j-1}D_{0x}^{2\beta-1}\nu_j(x)+\Psi_j(x), \quad (x,0)\in\bar I_j, \tag{5.9}$$

where

$$\Psi_j(x)=\frac{2\Gamma(1-2\beta)\cos\pi\beta}{\Gamma(1-\beta)}D_{0x}^{-(1-2\beta)}(\pm x)^{\beta}D_{0x}^{1-\beta}\left[\psi_j(x)-\omega_j^-(x)\right].$$

Therefore, as in the work ([43], p.39-48), according to conditions 1) - 2) of Problem A_T, by taking the limit as $y\to +0$ in equation (3.4) and condition (3.15), (5.4) considering (5.1), we obtain

$$\upsilon_j(x,+0)=\tau_j(x), \ \ (x,0)\in\bar I_j, \quad \upsilon_{j\,y}(x,+0)=\nu_j(x), \quad (x,0)\in I_j$$

it follows that

$$\tau_j''(x)=\left((-1)^{j-1}x\right)^p\nu_j(x), \tag{5.10}$$

$$\tau_j(0)=-\omega_j^+(0),\tau_j\left((-1)^{j-1}\right)=\varphi_j(0), \tag{5.11}$$

where $\omega_j^+(x)\,(j=1,2)$ is determined from (3.11).

Solving the problem (5.10) and (5.11), we obtain the functional relationship between $\tau_j(x)$ and $\nu_j(x)$, transferred from the domain Ω_j^+ to domain $I_j(j=1,2)$:

$$\tau_j(x)=(-1)^{j-1}\int_0^{(-1)^{j-1}}G_j\left(x,t\right)\left((-1)^{j-1}t\right)^p\nu_j(t)dt+f_j(x), \quad x\in\bar I_j, \tag{5.12}$$

where

$$G_1\left(x,t\right)=\begin{cases}(x-1)\,t, & 0\le t\le x,\\ (t-1)\,x, & x\le t\le 1,\end{cases} \qquad G_2\left(x,t\right)=\begin{cases}(t+1)\,x, & -1\le t\le x,\\ (x+1)\,t, & x\le t\le 0,\end{cases}$$

$$f_j(x)=-\omega_j^+(0)+(-1)^{j-1}x\left[\varphi_j\left(0\right)+\omega_j^+(0)\right]. \tag{5.13}$$

By virtue of (3.11), from (5.12) and (5.13), we obtain the Fredholm integral equation of the second kind with respect to: $\tau_j(x)$ $(j=1,2)$:

$$\tau_j(x)+\int_0^{(-1)^{j-1}}K_j\left(x,t\right)\tau_j(t)dt=\Phi_j(x), \quad (x,0)\in\bar I_j, \tag{5.14}$$

where

$$K_j\left(x,t\right)=\sqrt{\rho_j}\left(1-(-1)^{j-1}x\right)sh\sqrt{\rho_j}\,t\,,$$

$$\Phi_j(x) = (-1)^{j-1}\int_0^{(-1)^{j-1}} G_j\left(x,t\right)\left((-1)^{j-1}t\right)^p \nu_j(t)dt + (-1)^{j-1}x\varphi_j(0). \tag{5.15}$$

According to the theory of Fredholm integral equations ([44], ch.1, § 15) and from the uniqueness of the solution to Problem B_j, we conclude that the integral equation (5.19) is uniquely solvable in the class $C^1(\bar{I}_j)\cap C^2(I_j)$, $(j=1,2)$ and its solution is given by the formula:

$$\tau_j(x) = \Phi_j(x) - \int_0^{(-1)^{j-1}} K_j^*(x,t)\Phi_j(t)dt, \quad (x,0)\in \bar{I}_j, \tag{5.16}$$

where $K_j^*(x,t)$ is the resolvent of the kernel $K_j(x,t)$ $(j=1,2)$ (see [44], p. 81-88).

Eliminating $\tau_j\left(x\right)$ from (5.9) and (5.16), considering the matching condition and (3.12), we obtain the integral equation with respect to: $\nu_j(x)\,(j=1,2)$:

$$\nu_j(x) - \int_0^{(-1)^{j-1}} M_j\left(x,t\right)\,\nu_j(t)dt = F_j(x), \quad (x,0)\in I_j, \tag{5.17}$$

where

$$M_j\left(x,t\right) = \frac{\left((-1)^{j-1}t\right)^p}{\gamma_3\Gamma(1-2\beta)}D_{0x}^{1-2\beta}G_j(x,t) - \frac{\left((-1)^{j-1}t\right)^p}{\gamma_3\Gamma(1-2\beta)}\int_0^{(-1)^{j-1}} G_j\left(z,t\right)D_{0x}^{1-2\beta}K_j^*(x,z)dz -$$

$$-\frac{2(-1)^{j+1}\sqrt{\mu_j}\cos\pi\beta\cdot\left((-1)^{j-1}x\right)^\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j+1}t\right)^p D_{0x}^{1-\beta}\int_0^x sin\sqrt{\mu_j}\left(x-z\right)G_j\left(z,t\right)dz +$$

$$+\frac{2(-1)^{j-1}\sqrt{\mu_j}\cos\pi\beta\cdot\left((-1)^{j-1}x\right)^\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j-1}t\right)^p \times$$

$$\times\; D_{0x}^{1-\beta}\int_0^x sin\sqrt{\mu_j}\left(x-z\right)dz\int_0^{(-1)^{j-1}} K_j^*(t,s)G_j\left(s,z\right)ds, \tag{5.18}$$

$$F_j(x) = \frac{\varphi_j(0)}{\gamma_3\Gamma(1-2\beta)}D_{0x}^{1-2\beta}x - \frac{\varphi_j(0)}{\gamma_3\Gamma(1-2\beta)}\int_0^{(-1)^{j-1}} t\,D_{0x}^{1-2\beta}K_j^*(x,t)dt -$$

$$-\frac{2\cos\pi\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j-1}x\right)^\beta D_{0x}^{1-\beta}\psi_j(x) - \frac{2(-1)^{j-1}\sqrt{\mu_j}\varphi_j(0)\cos\pi\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j-1}x\right)^\beta \times$$

$$\times D_{0x}^{1-\beta}\int_0^x t\,sin\sqrt{\mu_j}\left(x-t\right)dt + \frac{2(-1)^{j-1}\sqrt{\mu_j}\varphi_j(0)\cos\pi\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j-1}x\right)^\beta \times$$

$$\times\; D_{0x}^{1-\beta}\int_0^x sin\sqrt{\mu_j}(x-t)dt\int_0^{(-1)^{j-1}} zK_j^*(t,z)dz. \tag{5.19}$$

Based on (2.2), (2.8), (2.9) and (5.1), considering the properties of the integral-differentiation operator, Beta, the hypergeometric function ([42], ch. 1, § 1, 2 and 4, p.4-32) and function $G_j\left(x,t\right)$ $(j=1,2)$ from (5.18) and (5.19), it follows that the kernel and the right-hand side of equation (5.17) admit estimates

$$|M_j\left(x,t\right)| \le C_j((-1)^{j-1}x)^{2\beta-1}((-1)^{j-1}t)^p, \tag{5.20}$$

$$|F_j\left(x\right)| \le C_{j+2}((-1)^{j-1}x)^{2\beta}, \quad C_k = const > 0, \ \ (k=\overline{1,4}). \tag{5.21}$$

Based on (2.2), (2.8), (2.9) and considering (5.21), we conclude that $F_j(x)\in C^1(I_j)$, where the function $F_j(x)\,(j=1,2)$ may have a singularity of order less than -2β as $(-1)^{j-1}x\to 0$, and it is bounded as $x\to(-1)^{j-1}$.

By virtue of (2.2),(5.20), (5.21), equation (5.17) is a Fredholm integral equation of the second kind. According to the theory of Fredholm integral equations ([44], ch.1, § 15) and from the uniqueness of the solution to the problem$j$, we conclude that the integral equation (5.17) is uniquely solvable in the class $C^1(I_j)$, and the function $\nu_j(x)$ may have a singularity of order less than -2β as $(-1)^{j-1}x \to 0$, and it is bounded as $x \to (-1)^{j-1}$, and its solution is given by the formula:

$$\nu_j(x) = F_j(x) + \int_0^{(-1)^{j-1}} M_j^*(x,t)\ F_j(t)dt, \quad (x,0) \in I_j, \tag{5.22}$$

where $M_j^*(x,t)\,(j=1,2)$ is the resolvent of the kernel $M_j(x,t)$ (see [44], p. 81-88).

Substituting (5.22) into (5.15) and (5.16), we find the function $\tau_j(x)$:

$$\tau_j(x) = (-1)^{j-1}\int_0^{(-1)^{j-1}} G_j(x,t)\left((-1)^{j-1}t\right)^p F_j(t)dt + (-1)^{j-1}\int_0^{(-1)^{j-1}} G_j(x,t)\left((-1)^{j-1}t\right)^p dt\times$$

$$\times\int_0^{(-1)^{j-1}} M_j^*(t,z)\ F_j(z)dz - (-1)^{j-1}\int_0^{(-1)^{j-1}} K_j^*(x,t)dt\int_0^{(-1)^{j-1}} G_j(t,s)\left((-1)^{j-1}s\right)^p F_j(s)ds-$$

$$(-1)^{j-1}\int_0^{(-1)^{j-1}} K_j^*(x,t)dt\int_0^{(-1)^{j-1}} G_j(t,z)\left((-1)^{j-1}z\right)^p dz\int_0^{(-1)^{j-1}} M_j^*(z,s)\ F_j(s)ds-$$

$$(-1)^{j-1}\varphi_j(0)\int_0^{(-1)^{j-1}} t\,K_j^*(x,t)dt + (-1)^{j-1}x\varphi_j(0),$$

and it belongs to the class

$$\tau_j(x) \in C^1(\bar{I}_j)\cap C^2(I_j),\ \ (j=1,2)\,. \tag{5.23}$$

Therefore, Problem B_j is uniquely solvable due to the equivalence of its Fredholm integral equation of the second kind (5.17).

Thus, the solution to Problem B_j can be recovered in the domain Ω_j^+ as the solution of the first boundary value problem for equation (3.4) [45], and in the domain Ω_j^- as the generalized solution of the Cauchy problem for equation (3.6) (see (5.5)).

From this, it follows that in the domain Ω_j, the solution to Problem B_j for equation (3.14) exists and is unique. *Theorem* $4_j,(j=1,2)$ *is proven.* □

5.2. **Investigation of Problem B_3 for Equation (3.4).**

Theorem 4_3. *Let the conditions (2.2), (2.8), (5.23) and* $-1<2\beta<0,\quad p+2\beta>0$ *be satisfied, then in the domain* Ω_3*, the solution to Problem* B_3 *exists and is unique.*

Proof of Theorem 4_3. The solution to the first boundary value problem with conditions (3.15), (5.4) for equation (3.4) in the domain Ω_j^+ has the form:

$$\upsilon_j(x,y) = (-1)^{j+1}\left\{\int_0^{(-1)^{j+1}} R_j(x,t,y;\delta)\ \left((-1)^{j+1}t\right)^p\tau_j(t)dt+\right.$$

$$\left.+\frac{\partial}{\partial y}\int_0^y R_j^{(1)}(x,y-t;\delta)\left(\tilde{\tau}_3(t)-\omega_j^+(0)\right)dt - \frac{\partial}{\partial y}\int_0^y R_j^{(2)}(x,y-t;\delta)\ \varphi_j(t)dt\right\},\ \ (j=1,2) \tag{5.24}$$

and belongs to the class $\upsilon_j(x,y)\in C(\bar{\Omega}_j^+)\cap C^{2,1}_{x,y}(\Omega_j^+)$, if the conditions (2.8), (5.3), (5.23) are satisfied, where $R_j(x,t,y;\alpha)$ is the Green's function for the first boundary value problem for equation (3.4) in the domain $\Omega_j^+$is:

$$R_j(x,\xi,y;\delta) = \sum_{k=0}^{\infty} exp\left\{-\frac{\lambda_k^2 y}{4}\right\}\frac{(1-\delta)\sqrt{x\xi}}{J_{2-\delta}^2(\lambda_k)}\times$$

$$\times J_{1-\delta}\left(\lambda_k(1-\delta)\left((-1)^{j+1}x\right)^{1/2(1-\delta)}\right)J_{1-\delta}\left(\lambda_k(1-\delta)\left((-1)^{j+1}\xi\right)^{1/2(1-\delta)}\right), \tag{5.25}$$

$$R_j^{(1)}(x,y;\delta) = 1 + (-1)^j(1-\delta)^{2(1-\delta)}x-$$

$$-\int_0^{(-1)^{j+1}} R_j(x,t,y;\delta)\left[1+(-1)^j(1-\delta)^{2(1-\delta)}\xi\right]\left((-1)^{j+1}\xi\right)^p d\xi, \tag{5.26}$$

$$R_j^{(2)}(x,y;\delta)=(-1)^{j+1}(1-\delta)^{2(1-\delta)}x-$$

$$-\int_0^{(-1)^{j+1}} R_j(x,t,y;\delta)\left[(-1)^{j+1}(1-\delta)^{2(1-\delta)}\xi\right]\left((-1)^{j+1}\xi\right)^p d\xi, \tag{5.27}$$

$J_\theta(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\theta+2k}}{k!\Gamma(k+\theta+1)}$ is the Bessel function of the first kind [46], λ_k are the positive roots of the equation $J_{1-\delta}(\lambda_k)=0,\ \ k\in N\cup\{0\},\quad \delta=\frac{p+1}{p+2}$, where

$$\frac{1}{2}<\delta<1. \tag{5.28}$$

Note that in [46] the convergence of Bessel series and (5.25) was shown, and in [36], [45] the existence of integrals (5.26) and (5.27) was proven.

Differentiating (5.24) with respect to x and taking the limit as $(-1)^{j-1}x\to\ 0$, we obtain

$$\nu_3(y)=\frac{\partial}{\partial y}\int_0^y \mathrm{N}_j(y-t;\delta)\ \tilde{\tau}_3(t)dt+\mathrm{H}_j(y),\qquad (0,y)\in I_3, \tag{5.29}$$

where $\nu_3(y)=\upsilon_{j\,x}(0,y),\ (0,y)\in I_3$,

$$\mathrm{H}_j(y)=\lim_{x\to 0}(-1)^{j+1}\frac{\partial}{\partial x}\left\{\int_0^1 R_j(x,t,y;\delta)\ ((-1)^{j+1}t)^p\tau_j(t)dt-\right.$$

$$\left.-\ \omega_j^+(0)\frac{\partial}{\partial y}\int_0^y R_j^{(1)}(x,y-t;\delta)\,dt-\frac{\partial}{\partial y}\int_0^y R_j^{(2)}(x,y-t;\delta)\ \varphi_j(t)\,dt\right\}, \tag{5.30}$$

$$\mathrm{N}_j(y-t;\delta)\equiv(1-\delta)^{2\delta-1}(-1)^{j+1}\lim_{x\to 0}\frac{\partial}{\partial x}\left[R_j^{(1)}(x,y-t;\delta)\right]=$$

$$=(-1)^j\left\{(1-\delta)+\sum_{k=0}^{\infty}exp\left\{-\frac{\lambda_k^2(y-t)}{4}\right\}\frac{2^{2\delta}\lambda_k^{-2\delta}}{\Gamma^2(1-\delta)J_{2-\delta}^2(\lambda_k)}\right\},$$

where $\omega_j^+(0)\,(j=1,2)$ are determined from (3.11).

Based on the properties of the function $J_\theta(z)$ and $\mathrm{N}_j(y-t;\alpha)$ can be represented as ([46], p. 12):

$$\mathrm{N}_j(y-t;\delta)=\frac{(-1)^j}{\Gamma(1-\alpha)}(y-t)^{\delta-1}+B_j(y-t), \tag{5.31}$$

where the function $B_j(y-t)$ is continuously differentiable at $y\ge t$.

Substituting (5.31) into (5.29), we obtain the functional relationship between $\tau_3(y)$ and $\nu_3(y)$, transferred from the domain Ω_j^+ to domain I_3:

$$\nu_3(y)=\frac{(-1)^j\Gamma(\delta)}{(1-\delta)}D_{0y}^{1-\delta}\tilde{\tau}_3(y)+B_j(0)\,\tilde{\tau}_3(y)+\int_0^y B_j'(y-t)\,\tilde{\tau}_3(t)dt+\mathrm{H}_j(y). \tag{5.32}$$

Eliminating $\nu_3(y)$ from the relations (5.32) as $j=1$ and $j=2$, and then applying the integral operator $D_{0y}^{\delta-1}[\bullet]$ considering $\tilde{\tau}_3(0)=0$ and $D_{0y}^{\delta-1}D_{0y}^{1-\delta}\tilde{\tau}_3(y)=\tilde{\tau}_3(y)$, we obtain

$$\tilde{\tau}_3(y)-\int_0^y M_3(y,t)\,\tilde{\tau}_3(t)dt=\mathrm{H}_3(y),\quad (0,y)\in\bar{I}_3, \tag{5.33}$$

where

$$M_3(y,t)=\frac{1}{2\Gamma(\alpha)}\left\{\frac{B_2(0)-B_1(0)}{(y-t)^{\delta}}-\int_t^y[B'_2(z-t)-B'_1(z-t)](y-z)^{-\delta}dz\right\}, \tag{5.34}$$

$$\mathrm{H}_3(y) = \frac{\Gamma(1-\delta)}{2(\delta)} D_{0y}^{\delta-1}\left[\mathrm{H}_2(y) - \mathrm{H}_1(y)\right], \tag{5.35}$$

here $\mathrm{H}_j(y)$ $(j = 1, 2)$ are determined from (5.30).

By virtue of (2.2), (2.8), (5.28), (5.32) and the properties of the function $B_j\,(y-t)$ from (5.34) and (5.35), it follows that:

1) The kernel $M_3\,(y,t)$ is continuous in $\{(y,t):\ 0 \le t < y \le 1\}$ and, for $y \to t$ admit the estimate

$$|M_3\,(y,t)| \le C_5\,(y-t)^{-\delta}; \tag{5.36}$$

2) The function $H_3(y)$ belongs to the class $C\left(\bar{I}_3\right) \bigcap C^1\left(I_3\right)$ and admits the estimate

$$|\mathrm{H}_3(y)| \le C_6 y^{1-\delta} \tag{5.37}$$

where C_5 and C_6 are arbitrary positive constants.

From (5.36) and (5.37), it follows that the integral equation (5.33) is a Volterra integral equation of the second kind with a weak singularity. According to the theory of Volterra integral equations of the second kind [44], we conclude that the integral equation (5.33) is uniquely solvable in the class $C\left(\overline{I_3}\right) \bigcap C^1\left(I_3\right)$, and its solution is given by the formula:

$$\tilde{\tau}_3(y) = \int_0^y M_3{}^*\,(y,t)\,\mathrm{H}_3(t)dt + \mathrm{H}_3(y), \quad (0,y) \in \bar{I}_3, \tag{5.38}$$

where $M_3^*\,(y,t)$ is the resolvent kernel of $M_3\,(y,t)$.

Substituting (5.38) into (5.32), considering (5.36) and (5.37), we determine the function $\nu_3(y)$ from the class

$$\nu_3(y) \in C^1\left(I_3\right),$$

where the function $\nu_3(y)$ may have a singularity of order less than $1-\delta$ at $y \to 0$ and is bounded at $y \to 1$. Therefore, Problem B_3 is uniquely solvable.

Thus, the solution to Problem B_3 can be recovered in the domain Ω_j^+ $(j = 1, 2)$ as the solution to the first boundary value problem for equation (3.4) [45].

This completes the investigation of the existence of the solution to Problem B_3 for equation (3.4) in the domain Ω_3. *Theorem 4_3 is proven.* □

From Theorems 4_j and 4_3, it follows the existence of the solution to Problem A_T^* for equation (3.14) in the domain Ω. *Theorem 3 is proven.* □

We proceed to the proof of the existence of the solution to Problem A_T .

Theorem 5. *If the conditions (2.2), (2.8), (2.9) and (5.1) are satisfied, then the solution to Problem A_T exists in the domain Ω .*

*Proof.*Let the solution $u\,(x,y)$ to Problem A_T in the domain Ω with conditions (2.3), (2.4), (2.5), (2.7) exist, then, using the results of Theorems 4_1 and 4_2 (see Section 5.1), we recover the solution to the problem A_T . By virtue of (5.12), (5.22) from (3.11), (3.12), considering (3.1), (3.2), (3.3), we determine the functions $\omega_j^+(x)$ and $\omega_j^-(x)$. Then, in the domain Ω_j^+, the solution to Problem A_T is expressed as

$$u\,(x,y) = \upsilon_j\,(x,y) + \omega_j^+\,(x),$$

where $\upsilon_j\,(x,y)$ is the solution to the first boundary value problem with conditions (3.15) and (5.2) for equation (3.4) [36, 45], here $\tilde{\tau}_3(y)$ is determined from formula (5.38), and in the domain Ω_j^- , it is expressed as:

$$u(x,y) = w_j(x,y) + \omega_j^-(x), (j = 1, 2),$$

where $w_j(x,y)$ is the generalized solution to the Cauchy problem for equation (3.6) in the domain Ω_j^- $(j = 1, 2)$ (see (5.5)).

Thus, in the domain Ω the solution to Problem A_T for equation (2.1) exists.

Theorem 5 is proven. □

This concludes the study of the problem A_T for equation (2.1).

6. Conclusion

In the study of degenerate loaded equations of mixed type of the second kind, difficulties arise associated with the absence of a general representation of the solution, as well as the impossibility of direct application of classical methods. This problem is solved in this paper. A new method for constructing a representation of the general solution of a loaded parabolic-hyperbolic equation of the second kind in a form convenient for further studies of various boundary value problems is developed, and a new type of extremum principle for a degenerate loaded parabolic-hyperbolic equation of the second kind is proved. The analysis of the state of affairs in this direction shows that boundary value problems for degenerate loaded equations leading to less studied integral equations of Volterra and Fredholm with shifts. Moreover, boundary value problems for loaded parabolic-hyperbolic equations of the second kind, degenerating inside the domain, have not yet been studied.

In this paper, we study problems with the Tricomi condition for a loaded parabolic-hyperbolic equation of the second kind, degenerating inside the domain. Theorems of existence and uniqueness of the classical solution of the problems are proved. The proofs of the theorem are based on energy identities and the extremum principle, as well as on the theory of Volterra and Fredholm integral equations. A class of given functions is determined that ensures the solvability of the obtained integral equations. The studied boundary value problems for such equations are effectively used in modeling processes that are associated with the dynamics of soil moisture, groundwater and biology.

References

[1] Nakhushev A.M. ; Equations of Mathematical Biology. Higher School, Moscow,–1995. - 301 p.

[2] Wiener J., Debnath L.; A survey of partial differential equations with piecewise continuous arguments // *Internet Journal of Mathematics and Mathematical Sciences.* – 1995. -**18**(2). -P. 209–228.

[3] Shkhanukov M.Kh.; On some boundary value problems for third-order equations arising in the modeling of liquid filtration in porous media. // *Differential Equations.* –1982. -**18**(4). -P. 689–699.

[4] Kozhanov A.I.; On a nonlinear loaded parabolic equation and its related inverse problem. // *Mathematical Notes.* –2004. -**76**(6). -P. 840–853.

[5] Jenaliev M.T., Ramazanov M.I.; *Loaded Equations as Perturbations of Differential Equations.* Almaty: "FYLIM", –2010. – 334 p.

[6] Gunter N.M., Gabib-zade A.Sh.; Investigation of a class of linear and nonlinear loaded integral equations with different parameters. // *Scientific Notes of AGU. Ser. Phys. Math.and Chem. Sciences.* Baku. –1958. –No. 1. –P. 41–59.

[7] Nakhushev A.M.; Loaded Equations and Their Applications. // *Differential Equations.* –1983. -**19**(1). -P. 86–94.

[8] Nakhushev A.M.; Boundary value problems for loaded integro-differential equations of hyperbolic type and some applications to soil moisture forecasting. // *Differential Equations.* –1979. -**15**(1). -P. 96–105.

[9] Jenaliev M.T.; On the theory of linear boundary value problems for loaded differential equations. Almaty, –1995. –270 p.

[10] Jenaliev M.T., Ramazanov M.I, Kosmakova M.T, Tuleutaeva Zh.M.; *On the solution to a two-dimensional heat conduction problem in a degenerate domain.* Eurasian Math. J. –**11**(2020). –No. 3. –P.334 .

[11] Sabitov K.B., Melisheva E.P.; Dirichlet's problem for a loaded mixed-type equation in a rectangular domain. // *Izvestiya VUZ. Mathematics.* –2013. –No. 7. –P. 62–76.

[12] Zikirov O.S., Kholikov D.K.; Solvability of some nonlocal problems for a third-order loaded equation. // *Siberian Electronic Mathematical Journal*, –2020. –**17**(2020). –P. 77–88.

[13] Pulkina L.S.; Nonlocal problem for a loaded hyperbolic equation. // *Transactions of the Mathematical Institute of the Academy of Sciences*, –2002. –Vol. 236. –P. 298–303.

[14] Khubiev K.U.; Analogue of the Tricomi problem for a loaded hyperbolic-parabolic type equation with variable coefficients. // *Bulletin of the Samara State Technical University. Series: Physical and Mathematical Sciences.* –2007. –**2**(15).– P. 155–157.

[15] Lanina I.N.; Boundary value problem for a loaded hyperbolic-parabolic third-order equation. // *Differential Equations*. –1981. –**17**(1). –P. 97–106.

[16] Islomov B., Kuryazov D.M.; On a boundary value problem for a loaded second-order equation. // *Proceedings of the Academy of Sciences of Uzbekistan*. –1996. –No. 1-2. –P. 3–6.

[17] Islomov B., Boltaeva U.I.; Boundary value problem for a loaded third-order equation with a parabolic-hyperbolic operator. // *Uzbek Mathematical Journal*. –2007. –No. 2. –P. 45–55.

[18] Islomov B.I., Kholbekov Zh.A.; On a nonlocal boundary value problem for a loaded parabolic-hyperbolic equation with three lines of type change. // *Bulletin of Samara State Technical University. Series: Physical and Mathematical Sciences*. –2021. -**25**(3). –P. 407–422.

[19] Abdullaev O.Kh.; Nonlocal problem with an integral gluing condition for a loaded parabolic-hyperbolic equation with a fractional Caputo derivative. // *Differential Equations*. –2023. –**59**(3). –P. 350–357.

[20] Islomov B., Kuryazov D.M.; Boundary value problems for a mixed loaded third-order equation of parabolic-hyperbolic type. // *Uzbek Mathematical Journal*. –2000. -No. 2. –P. 29–35.

[21] Nakhushev A.M.; On the Darboux problem for a degenerate loaded integrodifferential equation of the second order. // *Differential Equations*. –1976. -**12**(1). –P. 103–108.

[22] Attaev A.Kh.; Gurs' problem for a loaded degenerate hyperbolic equation of the second order with a Gellerstedt operator in the main part. // *Bulletin of Samara State Technical University. Series: Physical and Mathematical Sciences*. –2016. –**20**(1). –P. 7–21.

[23] Dzhamalov S.Z., Ashurov R.R.; On a nonlocal boundary-value problem for a second-kind second-order mixed-type loaded equation in a rectangle. // *Uzbek Mathematical Journal*.– 2018. –No. 3. –P. 63–72.

[24] Islomov B., Juraev F.; Local boundary value problems for a loaded parabolic-hyperbolic type equation, degenerate inside the domain. // *Ufa Mathematical Journal*. –2022. –**14**(1). –P. 41–56.

[25] Islomov B.I., Nasirova D.A.; A problem with Gellerstedt conditions on different characteristics for a mixed loaded equation of the second kind. // *Eurasian Mathematical Journal*. –2024. -**15**(1). –P. 34–48.

[26] Islomov B.I., Yuldashev T.K., Yunusov O.M.; Nonlocal boundary problem for a loaded equation of mixed type in a special domain. // *Lobachevskii Journal of Mathematics*. –2024. -**45**(7). –P. 3304–3313.

[27] Islomov B.I., Alikulov Y.K.; Boundary value problem for a loaded parabolic-hyperbolic equation of the third order in an infinite three-dimensional domain. // *International Journal of Applied Mathematics*. –2021. -**34**(2). –P. 158–170.

[28] Abdullaev O.Kh., Agarwal P.; A nonlocal problem with an integral gluing condition for a third-order loaded equation with parabolic-hyperbolic operator involving fractional derivatives. // *Mathematical Methods in the Applied Sciences*. –2020. -**43**(6). -P. 3716-3726.

[29] Islomov B., Baltaeva U.I.; Boundary value problems for the classical and mixed integro-differential equations with Riemann-Liouville operators. // *International Journal of Partial Differential Equations*. –2013. -Article ID 157947. –P. 11–17.

[30] Gekkieva S.Kh. Second boundary value problem for a loaded equation with a fractional derivative. // *Reports of the Academy of Mathematical Sciences*. –2008. -**10**(2). –P. 17–19.

[31] Ubaidullaev U.Sh.; Inverse problem for a mixed loaded equation with the Riemann-Liouville operator in a rectangular domain. // *Bulletin of the KRAUNC. Physical and Mathematical Sciences*. –2020. –**31**(2). –P. 18–31.

[32] Islomov B.I., Baltaeva U.I.; Boundary value problems for loaded differential equations of hyperbolic and mixed types of third order. // *Ufa Mathematical Journal*. –2011. -**3**(3). –P. 15–25.

[33] Islomov B.I., Baltaeva U.; Boundary-value problems for a third-order loaded parabolic-hyperbolic equation with variable coefficients. // *Electronic Journal of Differential Equations*. –2015. –No. 221. –P. 1–10.

[34] Karol I.L.; On a boundary value problem for a mixed elliptic-hyperbolic type equation. // *Reports of the USSR Academy of Sciences*, –1953. –**88**(2). –P. 197–200.

[35] Salahitdinov M.S.; Mixed-type equations. Fan, Tashkent: –1974. –156 p.

[36] Akbarova S.Kh.; Boundary value problems for equations of mixed parabolic-hyperbolic and elliptic-parabolic types with two lines and different orders of degeneration: Ph.D. Dissertation, Tashkent: Institute of Mathematics, Academy of Sciences of Uzbekistan, –1995. –120 p .

[37] Ilyin A.M., Kalashnikov A.S., Oleinik O.A.; Linear second-order equations of parabolic type. // *Journal of Mathematical Physics and Mechanics*. –1965. –**4**(6). –P. 1006–1024.

[38] Salahitdinov M.S., Akbarova S.Kh.; Boundary value problem for a mixed elliptic-parabolic equation with different orders of degeneration inside the domain. // *Uzbek Mathematical Journal*. –1994. –No. 4. –P. 51–60.

[39] Tersenov S.A.; Introduction to the theory of equations degenerating at the boundary. Novosibirsk: NSU, –1973. –144 p.

[40] Djuraev T.D., Sopuev A., Mamajonov M.; Boundary value problems for equations of parabolic-hyperbolic type. Tashkent: Fan, –1986. –220 p.

[41] Samko S.G., Kilbas A.A., Marichev O.I.; Fractional integrals and derivatives and some of their applications. Minsk: Science and Technology. –1987. -P.688.

[42] Smirnov M.M.; Equations of mixed type. Moscow: Higher School. 1985. –301 p .

[43] Djuraev T.D.; Boundary value problems for equations of mixed and mixed-composite type. Tashkent: Fan, –1979. –240 p.

[44] Mikhlin S.G.; Lectures on linear integral equations. Fizmatgiz, Moscow, 1959. –232 p.

[45] Tersenov S.A.; First boundary value problem for a parabolic type equation with a changing time direction. Novosibirsk: NSU, –1978.

[46] Bateman G., Erdelyi A.; Higher Transcendental Functions. Science, Moscow, –1966. -Vol. 2. –295 p.

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