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The Samarskii-Ionkin type problem for the fourth-order ordinary differential equation

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Abstract. In this paper the spectral properties of the nonlocal SamarskiiIonkin type problems for a fourth-order ordinary differential equation are investigated. The eigenvalues and corresponding eigenfunctions are found and their completeness are studied. The spectral properties of the adjoint problem are also studied. Further, the Riesz basis property of the systems of root functions of these problems is proved.

Keywords: Ionkin-Samarskii type problems; non-self-adjoint problem; eigenvalues and eigenfunctions; completeness; Riesz basis.
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1. Introduction and statement of the problem

As is known, the application of the Fourier method in solving boundary value problems for partial differential equations leads to an eigenvalue problem, and the main issue here is the expansion of an arbitrary function into a series in terms of the system of eigenfunctions (and associated functions) of this problem. In the case of self-adjoint operators, in general, the system of root functions forms an orthonormal basis, and, in this case, the indicated problem is theoretically solved [1].

As for non-self-adjoint operators, the solution to the above problems is ambiguous [2]. In this case, the system of eigenfunctions may be incomplete, and the problem arises in supplementing them with so-called associated functions. It should be noted that the system of eigenfunctions and associated functions of non-self-adjoint operators is defined ambiguously; that is, there are different approaches to constructing systems of eigenfunctions and related functions of such operators.

Let us note the well-known works [3], [4], where the theory of associated functions was constructed and the completeness of the system of eigenfunctions and associated functions of a wide class of non-self-adjoint differential equations was proven. We also note the work [5], where another method for constructing associated functions of non-self-adjoint differential operators is proposed. In the works [6]-[7], a constructive method for constructing the so-called reduced system of eigen- and associated functions such operators is proposed, and necessary and sufficient conditions of its basis property are also proved.

We note the works [2], [8]-[10], where new formulas for constructing chains of associated functions of non-self-adjoint differential operators are proposed and substantiated. More detailed information on the spectral properties of non-self-adjoint differential operators can be found in the monograph [11].

As noted above, the spectral problems of self-adjoint problems for ordinary differential equations of the fourth order, in the case of a model equation of the form

$$y^{IV}(x) - \lambda y(x) = 0 \tag{*}$$

have been theoretically solved [1], [12]. In this direction, we note the work [13], where the spectral properties of a differential operator defined by differential equations of even (in particular, fourth) order with piecewise smooth weight functions, as well as with separated boundary conditions, are studied.

As for non-classical problems, here we can only note the works of [14],[15], where for equation (*) investigates the spectral properties of a Samarskii-Ionkin type problem. Using the spectral method, eigenvalues and the corresponding root functions are found, and their completeness and basis properties are proven. The associated adjoint problem is also studied. We also note the works [16], where spectral issues of multidimensional spectral problem is studied. It is shown that the system of eigenfunctions is complete and forms a Riesz basis in Sobolev spaces.

As far as we know, the spectral properties of Bitsadze-Samarskii type problems, or interior boundary value problems for fourth-order equations, have not been studied.

In the proposed work, spectral issues of two nonlocal problems of the Samarskii-Ionkin type for a non-self-adjoint fourth-order differential operator are investigated. Problems of this type for the heat equation were first formulated and investigated by N.I. Ionkin [5].

Problem 1. It is required to find such values λ for which problem

$$X^{IV}(x) - \lambda X(x) = 0,\ 0 < x < 1 \tag{1.1}$$

$$X(1) = 0, X''(0) = 0, \tag{1.2}$$

$$X'(0) = X'(1), X'''(0) = X'''(1). \tag{1.3}$$

has a non-trivial solution.

Problem 2. It is required to find such values λ for which problem (1.1), (1.2) and

$$X'(0) + X'(1) = 0, X'''(0) + X'''(1) = 0, \tag{1.4}$$

has a non-trivial solution. Here λ is a spectral parameter.

The necessity of studying such problems arises when studying boundary value problems for partial differential equations by the spectral method, when conditions of the form (1.2), (1.3) are given with respect to one of the spatial variables.

2. Some auxiliary information about the Riesz Basis

Let $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ are two complete system of functions in $L_2(a,b)$. Let $(\varphi,\psi)_0$ denote the scalar product of functions $\varphi(x)$ and $\psi(x)$ in $L_2(a,b)$, that is

$$(\varphi,\psi)_0 = (\varphi,\psi)_{L_2(a,b)} = \int\limits_a^b \varphi(x)\psi(x)dx.$$

Definition 2.1. (see [17]) Two system of functions $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ form a biorthonormal system on some interval $[a,b]$, if

$$(\varphi_n,\psi_k)_0 = \int_a^b \varphi_n\psi_k dx = \delta_{nk} = \begin{cases} 0,\ n \neq k, \\ 1,\ n = k. \end{cases}$$

Thus, the system $\{\psi_n(x)\}$ is called biorthogonally adjoint to the system $\{\varphi_n(x)\}$.

Definition 2.2. (see [17]) System is called minimal if none of the functions of this system is included in the linear envelope of other functions of this system.

The minimality of the system ensures the existence of a biorthogonally adjoint system.

Definition 2.3. (see [17]) The biorthogonal expansion of the function $f \in L_2(a,b)$ in the system $\{\varphi_n(x)\}$ is the series

$$f(x) \sim \sum_{n=1}^{\infty} c_n\varphi_n(x),$$

where $c_n = (f,\psi_n)_0$.

Definition 2.4. (see [17]) We call complete and minimal system of functions $\{\varphi_n(x)\}$ the Bessel system, if for any $f \in L_2(a,b)$ the series of squared coefficients of its biorthogonal expansion in $\{\varphi_n(x)\}$ converges, i.e. if $f \in L_2(a,b)$ implies that

$$\sum_{n=1}^{\infty} |(f,\psi_n)_0|^2 < \infty,$$

where $\{\psi_n\}$ is biorthogonal conjugate system to $\{\varphi_n(x)\}$.

Definition 2.5. (see [17]) We call complete and minimal system of functions $\{\varphi_n(x)\}$ the Hilbert system, if for any sequences of numbers c_n, such that $\sum\limits_{k=1}^{\infty} c_n^2 < \infty$, there is one and only one $f \in L_2(a,b)$ for which these are the coefficients of its biorthogonal expansion in $\{\varphi_n(x)\}$, i.e.

$$c_n = (f, \psi_n)_0, \; n = 1, 2, ...$$

Definition 2.6. We call complete and minimal system a Riesz basis, if it is both Bessel and Hilbert system.

Theorem 2.7. *(see* [18]*) Following statements are equivalent:*

1) Sequence $\{\psi_j\}_1^\infty$ forms a basis in the space R, which is equivalent to an orthonormal one.

2) Sequence $\{\psi_j\}_1^\infty$ will be an orthonormal basis in the space R at the corresponding replacement of the scalar product (f,g) with the new $(f,g)_1$, which topologically equivalent to the previous.

3) Sequence $\{\psi_j\}_1^\infty$ is complete in R and there exist constants $a_1, a_2\,(>0)$, such that, for any natural n and for any complex numbers $\gamma_1, \gamma_2, ..., \gamma_n$

$$a_2 \sum_{j=1}^{n} |\gamma_j|^2 \le \sum_{j=1}^{n} |\gamma_j \psi_j|^2 \le a_1 \sum_{j=1}^{n} |\gamma_j|^2.$$

4) Sequence $\{\psi_j\}_1^\infty$ complete in R and its matrices of Gramm $(\psi_j, \psi_k)_1^\infty$ generate a bounded invertible operator in the space l_2.

5) Sequence $\{\psi_j\}_1^\infty$ complete in R, it corresponds to a complete bi-orthogonal sequence $\{\chi_j\}_1^\infty$ and for any $f \in R$

$$\sum_{j=1}^{n} |(f,\psi_j)|^2 < \infty, \sum_{j=1}^{n} |(f,\chi_j)|^2 < \infty.$$

Lemma 2.8. *(see* [19]*) Let $f(x) \in L_2(0,1)$, $a_n = \int\limits_0^1 f(x) e^{-\lambda n x} dx$, $b_n = \int\limits_0^1 f(x) e^{\lambda n (x-1)} dx$, where λ is any complex number,* $\operatorname{Re}\lambda > 0$. *Then series $\sum\limits_{n=1}^{\infty} |a_n|^2$, $\sum\limits_{n=1}^{\infty} |b_n|^2$ converge.*

3. The solution of Problem 1

Let us find the eigenvalues and eigenfunctions of Problem 1. For this aim, we write the characteristic equation

$$k^4 - \lambda = 0 \Leftrightarrow k^4 = \lambda. \tag{3.1}$$

We have the consider three cases: $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, from which depends the general expression of the solution of the equation (1.1). Let $\lambda = 0$. Then equation (3.1) has multiple roots $k_{1,2,3,4} = 0$, therefore general solution has a form

$$X(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4,$$

where $C_i, i = \overline{1,4}$ are some numbers. Substituting this solution into conditions (1.2),(1.3), obtain $C_1 = C_2 = 0, C_3 = -C_4$. As consequence, all the solutions of Problem 1 with $\lambda = 0$ are given by the expression $X(x) = C_4(1-x)$, with any real number C_4. Thus, we can call $X_0(x) = 1 - x$ the eigenfunction associated to the single eigenvalue $\lambda_0 = 0$.

Let $\lambda < 0$. Assume $\lambda = -4\mu^4$, $(\mu > 0)$ and we write characteristic equation as follows $k^4 = -4\mu^4$, roots of which are

$$k_1 = (1+i)\mu,\; k_2 = (-1+i)\mu,\; k_3 = (1-i)\mu,\; k_4 = (-1-i)\mu.$$

Obviously, the general solution of equation (1.1) has the form

$$X(x) = C_1 \cosh \mu x \cos \mu x + C_2 \cosh \mu x \sin \mu x + C_3 \sinh \mu x \cos \mu x + C_4 \sinh \mu x \sin \mu x.$$

Substituting this expression into conditions (1.2) and (1.3), to find $C_i, i = \overline{1,4}$, we obtain a system of equations

$$\begin{cases} C_1 \cosh\mu\cos\mu + C_2\cosh\mu\sin\mu + C_3\sinh\mu\cos\mu = 0, \\ C_1\sinh\mu\cos\mu + C_2\sinh\mu\sin\mu + C_3(\cosh\mu\cos\mu - 1) = 0, \\ C_1\cosh\mu\sin\mu + C_2(1-\cosh\mu\cos\mu) + C_3\sinh\mu\sin\mu = 0, \\ C_4 = 0. \end{cases}$$

which has only a trivial solution $C_i = 0, i = \overline{1,4}$, so that Problem 1 also has only a trivial solution $X(x) \equiv 0$.

Consider the case $\lambda > 0$. Assume, $\lambda = \mu^4$, $(\mu > 0)$ and we write the characteristic equation with its roots $k^4 = \mu^4$, $k_{1,2} = \pm\mu$, $k_{3,4} = \pm\mu i$. The following functions correspond to these roots

$$X_1(x) = e^{\mu x},\ X_2(x) = e^{-\mu x},\ X_3(x) = \cos\mu x,\ X_4(x) = \sin\mu x,$$

and the general solution is form

$$X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} + C_3\cos\mu x + C_4\sin\mu x.$$

Substituting this solution into conditions (1.2) and (1.3), obtain following system

$$\begin{cases} C_1 + C_2 - C_3 = 0, \\ C_1 e^{\mu} + C_2 e^{-\mu} + C_3\cos\mu + C_4\sin\mu = 0, \\ C_1(e^{\mu}-1) - C_2(e^{-\mu}-1) - C_3\sin\mu + C_4(\cos\mu - 1) = 0, \\ C_1(e^{\mu}-1) - C_2(e^{-\mu}-1) + C_3\sin\mu - C_4(\cos\mu - 1) = 0. \end{cases} \tag{3.2}$$

The resulting system of equations has a nontrivial solution only for those values of μ at which its determinant goes to zero. The determinant of this system is $\Delta(\mu) = 4(2 - e^{\mu} - e^{-\mu})(1-\cos\mu)$. Equating this determinant to zero, we find that the numbers $\lambda_n = \mu_n^4 = (2\pi n)^4, n = 1, 2, ...$ are the eigenvalues of Problem 1.

Let us study the multiplicity of the found eigenvalues. It is easy to see that for $\lambda_n = (2\pi n)^4$ the rank of the main matrix of system (3.2) is equal to 2. It follows that the geometric multiplicity of the eigenvalues is equal to 2, therefore, each eigenvalue corresponds to a pair of eigenfunctions. Since $\Delta'(\mu_k) = 0, \Delta''(\mu_k) \neq 0$, we find that the algebraic multiplicity of the eigenvalues is equal to 2. Consequently, all eigenvalues of the problem under consideration have multiplicity equal to two (geometrically and algebraically), and the eigenfunctions are the functions

$$X_{1n}(x) = -\sin 2\pi n x,\ \ X_{2n}(x) = \frac{e^{2\pi n x} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \cos 2\pi n x.$$

Thus, the eigenvalues of Problem 1 are given by

$$\lambda_0 = 0,\ \lambda_n = (2\pi n)^4,\ n \in N, \tag{3.3}$$

and corresponding eigenfunctions has the form

$$X_0(x) = 2(1-x), X_{1n}(x) = -2\sin 2\pi n x, X_{2n}(x) = \frac{e^{2\pi n x} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \cos 2\pi n x. \tag{3.4}$$

Problem 1 is non-self-adjoint and it is easy to see that the following problem will be adjoint to it

$$Y^{IV}(x) - \lambda Y(x) = 0,\ 0 < x < 1, \tag{3.5}$$

$$Y(0) = Y(1), Y'(1) = 0, Y''(0) = Y''(1), Y'''(0) = 0. \tag{3.6}$$

It is not difficult to show that problem (3.5), (3.6) has eigenvalues (3.3), and the corresponding eigenfunctions have the form

$$Y_0(x) = 1,\ Y_{1n}(x) = \frac{e^{2\pi n x} + e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \sin 2\pi n x,\ Y_{2n}(x) = -2\cos 2\pi n x. \tag{3.7}$$

It should be noted that (3.4) and (3.7) are a non-orthogonal system of functions. Indeed, let us consider, for example, system (3.4) and calculate

$$\left(X_0(x)\,,\,X_n^{(1)}(x)\right)_0 = -4\int\limits_0^1 (1-x)\sin 2\pi nx dx = -\frac{2}{\pi n} \neq 0.$$

We proceed to study the questions of the basis property of systems (3.4) and (3.7) in $L_2(0,1)$.

Lemma 3.1. *System of functions (3.4) and (3.7) are biorthogonal system in $L_2(0,1)$, that is*

$$(X_0, Y_0)_{L_2(a,b)} = 1, (X_{ik}, Y_{jn}) = \begin{cases} 1, & k = n, i = j \\ 0, & k \neq n, i \neq j \end{cases}, i, j = 1, 2; k, n = 1, 2, ...$$

Proof. We present the proof of Lemma 3.1 for the functions $X_{1n}(x)$ and $Y_{1n}(x)$. According to Definition 2.1, we calculate the integral

$$(X_{1k}, Y_{1n}) = -2\int\limits_0^1 \sin 2\pi kx \left(\frac{e^{2\pi nx} + e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \sin 2\pi nx\right) dx =$$

$$-\frac{2}{e^{2\pi n}-1}\int\limits_0^1 (e^{2\pi nx} + e^{2\pi n(1-x)}) \sin 2\pi kx dx + 2\int\limits_0^1 \sin 2\pi nx \sin 2\pi kx dx = I_{kn} + J_{kn}.$$

Simple calculations show that

$$I_{kn} = -\frac{2}{e^{2\pi n}-1}\int\limits_0^1 (e^{2\pi nx} + e^{2\pi n(1-x)}) \sin 2\pi kx dx = 0, k, n \in N,$$

$$J_{kn} = 2\int\limits_0^1 \sin 2\pi kx \cdot \sin 2\pi nx dx = \begin{cases} 1, k = n \\ 0, k \neq n \end{cases}, k, n \in N$$

and so $(X_{1n}, Y_{1n}) = 1$ at $k = n$ and $(X_{1n}, Y_{1n}) = 0$ at $k \neq n$, which is needed what to proven. □

Lemma 3.2. *The system of functions (3.4) and (3.7) are minimal in $L_2(0,1)$.*

The proof of Lemma 3.2 follows from the existence of a biorthonormal system, which was established in Lemma 3.1.

Theorem 3.3. *The system of functions (3.4) and (3.7) are complete in $L_2(0,1)$.*

Proof. First, we prove the completeness of (3.4). Assume, on the contrary, that the system of functions (3.4) is not complete in $L_2(0,1)$. Then there exists a nontrivial function $\varphi(x)$ in $L_2(0,1)$, that is orthogonal to all functions of system (3.4). Let us expand the function $\varphi(x)$ into a Fourier series

$$\varphi(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx),$$

which converge in $L_2(0,1)$. Since $\varphi(x)$ is orthogonal to the system $\{-2\sin 2\pi nx\}_{n=1}^{\infty}$, then the last expansion can be written as

$$\varphi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nx. \tag{3.8}$$

By assumption, $\varphi(x)$ is orthogonal to all functions of the form $X_0(x)$, $X_{2k}(x)$. Then, multiplying series (3.8) sequentially by these functions and integrating along $[0,1]$, we have

$$0 = 2\int_0^1 \varphi(x)(1-x)dx = 2a_0 \int_0^1 (1-x)dx + 2\sum_{n=1}^{\infty} a_n \int_0^1 (1-x)\cos 2\pi nx dx = a_0,$$

$$0=\int_0^1 \varphi(x)\cdot\left(\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\cos 2\pi kx\right)dx=$$
$$=\sum_{n=1}^{\infty} a_n \int_0^1 \left(\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\cos 2\pi kx\right)\cos 2\pi nx dx=-\frac{1}{2}a_k, k=1,2,3,....$$

From here, it follows that $a_k=0,\ k=0,1,2,....$ Therefore, from (3.8) we conclude that $\varphi(x)=0$ in $[0,1]$, which opposing conditions $\varphi(x)\neq 0$. Thus, system (3.4) is complete in the space $L_2(0,1)$.

We prove the completeness of the system (3.7). Let there exists a nontrivial function $\varphi(x)$ in $L_2(0,1)$, that is orthogonal to all functions of system (3.7). Since the function $\varphi(x)$ is orthogonal to the system $\{-2cos2\pi nx\}_{n=0}^{\infty}$, it can be represented in $L_2(0,1)$ as a series of sinus, i.e.

$$\varphi(x)=\sum_{n=1}^{\infty} b_n sin2\pi nx. \tag{3.9}$$

Then multiplying last series to $Y_{1k}(x)$ and integrating along $[0,1]$, taking into account the orthogonality of the functions $\varphi(x)$ and $Y_{1k}(x)$, we obtain

$$0=\int_0^1 \varphi(x)\cdot\left(\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\sin 2\pi kx\right)dx=$$
$$=\sum_{n=1}^{\infty} b_n \int_0^1 \left(\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\sin 2\pi kx\right)\sin 2\pi nx dx=-\frac{1}{2}b_k, k=1,2,...,$$

that is $b_k=0,\ n=1,2,....$ Then from (3.9) it follows that $\varphi(x)=0$ in $[0,1]$, i.e. system (3.7) is complete in $L_2(0,1)$. Theorem 3.3 is proven. □

Theorem 3.4. *The system of functions (3.4) and (3.7) are two bases of Riesz in* $L_2(0,1)$.

Proof. In order to prove this statement, it is sufficient to prove the completeness of systems (3.4) and (3.7), and the convergence of the following series for $\varphi(x)\in L_2(0,1)$ according to Theorem 2.7:

$$(\varphi(x),2(1-x))_0^2+\sum_{n=1}^{\infty}(\varphi(x),-2\sin 2\pi nx)_0^2+\sum_{n=1}^{\infty}\left(\varphi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right)_0^2, \tag{3.10}$$

$$(\varphi(x),1)_0^2+\sum_{n=1}^{\infty}(\varphi(x),-2\cos 2\pi nx)_0^2+\sum_{n=1}^{\infty}\left(\varphi(x),\frac{e^{2\pi nx}+e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\sin 2\pi nx\right)_0^2. \tag{3.11}$$

Since the completeness of systems (3.4) and (3.7) has been proven in Lemma 3.1 we only must verify the convergence of the previous series. To this end, we consider (3.10) and introduce the following notations

$$I_1=4\left(\varphi(x),(1-x)\right)_0^2, I_2=4\sum_{n=1}^{\infty}(\varphi(x),\sin 2\pi nx)_0^2,$$
$$I_3=\sum_{n=1}^{\infty}\left(\varphi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right)_0^2.$$

From I_1, applying the Cauchy-Bunyakovsky inequality we obtain

$$I_1=4\left(\int_0^1(1-x)\varphi(x)dx\right)^2\leq 4\int_0^1(1-x)^2dx\cdot\int_0^1\varphi^2(x)dx=\frac{4}{3}\|\varphi(x)\|^2{}_{L_2(0,1)},$$

i.e. I_1 is finite.

I_2 will be represented in the form

$$I_2=4\sum_{n=1}^{\infty}(\varphi(x),\sin 2\pi nx)_0^2=2\sum_{n=1}^{\infty}\left(\varphi(x),\sqrt{2}\sin 2\pi nx\right)^2=2\sum_{n=1}^{\infty}c_n^2,$$

where $c_n = \left(\varphi(x), \sqrt{2}\sin 2\pi nx\right)$ are the coefficients of Fourier of the function $\varphi(x)$ on the orthonormal system $\{\sqrt{2}\sin 2\pi nx\}$. From here, applying Bessels inequality, we obtain that $I_2 = 2\sum\limits_{n=1}^{\infty} c_n^2 \le 2\left\|\varphi(x)\right\|^2_{L_2(0,1)}$, i.e. I_2 is finite.

Consider I_3. Let

$$A = \left(\varphi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \cos 2\pi nx\right)^2_0.$$

Since

$$A = \left(\left(\varphi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right) - \left(\varphi(x), \cos 2\pi nx\right)\right)^2_0,$$

from here, applying inequality

$$(a+b)^2 \le 2(a^2 + b^2)$$

obtain that

$$A \le 2\left(\varphi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)^2_0 + 2\left(\varphi(x), \cos 2\pi nx\right)^2_0$$
$$= 2\left(\left(\varphi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right) - \left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)\right)^2_0 + 2\left(\varphi(x), \cos 2\pi nx\right)^2_0.$$

Applying the previous inequality again, we get that

$$A \le 4\left(\varphi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right)^2_0 + 4\left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)^2_0 + 2\left(\varphi(x), \cos 2\pi nx\right)^2_0.$$

Thus

$$I_3 \le 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right)^2_0 + 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)^2_0 + 2\sum_{n=1}^{\infty}\left(\varphi(x), \cos 2\pi nx\right)^2_0 =$$
$$= J_1 + J_2 + J_3.$$

Consider J_3.

$$J_3 = 2\sum_{n=1}^{\infty}\left(\varphi(x), \cos 2\pi nx\right)^2 = \sum_{n=1}^{\infty} a_n^2,$$

where $a_n = \left(\varphi(x), \sqrt{2}\cos 2\pi nx\right)$ are the coefficients of Fourier of the function on the orthonormal system $\{\sqrt{2}\cos 2\pi nx\}$. Then applying Bessel's inequality, we get

$$J_3 = \sum_{n=1}^{\infty} a_n^2 \le \left\|\varphi(x)\right\|^2_{L_2(0,1)}.$$

Consider J_1. Since

$$\left(\varphi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right)^2_0 = \left(\int_0^1 \varphi(x)\cdot\frac{e^{2\pi n}}{e^{2\pi n} - 1}e^{2\pi n(x-1)}dx\right)^2_0 =$$

$$= \left(\int_0^1 \varphi(x)\left[1 + \frac{1}{e^{2\pi n} - 1}\right]e^{2\pi n(x-1)}dx\right)^2 \le 4\left(\int_0^1 \varphi(x)e^{2\pi n(x-1)}dx\right)^2,$$

hence

$$J_1 \leq 16 \sum_{n=1}^{\infty} \left(\int_0^1 \varphi(x) e^{2\pi n(x-1)} dx \right)^2 = 16 \sum_{n=1}^{\infty} b_n^2, b_n = \left(\int_0^1 \varphi(x) e^{2\pi n(x-1)} dx \right).$$

Here, from Lemma 3.2, it follows that J_1 is finite. Similarly, we can prove that J_2 is finite too. Thus, the series I_1 and I_2 are converge, and therefore series (3.10) also converges. The convergence of series (3.11) is proved similarly. Theorem 3.4 is proved. □

4. The solution of Problem 2

Let us find the eigenvalues and eigenfunctions of Problem 2. After similar calculations, as in the case of Problem 1, it is easy to see that the eigenvalues of Problem 2 are given by

$$\lambda_n = (\pi(2n-1))^4, \; n \in N \tag{4.1}$$

and corresponding eigenfunctions has the form

$$X_{1n}(x) = 2\sin(\pi(2n-1)x), \; X_{2n}(x) = \frac{e^{\pi(2n-1)x} + e^{\pi(2n-1)(1-x)}}{e^{\pi(2n-1)} + 1} + \cos(\pi(2n-1)x). \tag{4.2}$$

Note that Problem 2 is a non self-adjoint problem. On the contrary, it is not difficult to verify that the following problem is self-adjoint.

$$Y^{IV}(x) - \lambda Y(x) = 0, \; 0 < x < 1, \tag{4.3}$$

$$Y(0) + Y(1) = 0, Y'(1) = 0, Y''(0) + Y''(1) = 0, Y'''(0) = 0. \tag{4.4}$$

Problem (4.3), (4.4) have eigenvalues (4.1), and the corresponding eigenfunctions have the form

$$Y_{1n}(x) = \frac{e^{\pi(2n-1)x} - e^{\pi(2n-1)(1-x)}}{e^{\pi(2n-1)} + 1} + \sin(\pi(2n-1)x), Y_{2n}(x) = 2\cos(\pi(2n-1)x). \tag{4.5}$$

It is not difficult to show that systems (4.2) and (4.5) are bi-orthonormalized in $L_2(0,1)$. This also implies the minimality of these systems in $L_2(0,1)$.

Theorem 4.1. *The system of functions (4.2) and (4.5) are complete in the space* $L_2(0,1)$.

Proof. We prove the completeness of system (4.2). Assume, on the contrary, that the system of functions (4.2) is not complete in $L_2(0,1)$. Then there exists a nontrivial function $\varphi(x)$ in $L_2(0,1)$, that is orthogonal to all functions of system (4.2). Let us expand the function $\varphi(x)$ into a Fourier series

$$\varphi(x) = \sum_{n=1}^{\infty} (a_n \cos(2n-1)\pi x + b_n \sin(2n-1)\pi x),$$

according to the complete orthogonal system $\{\cos(2n-1)\pi x, \sin(2n-1)\pi x\}_{n=1}^{\infty}$, which converges in $L_2(0,1)$. Since $\varphi(x)$ is orthogonal to the system $\{\sin(2n-1)\pi x\}_{n=1}^{\infty}$, then the last expansion can be written as

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \cos(2n-1)\pi x. \tag{4.6}$$

Further, multiplying last series by the function $X_{2k}(x)$, and integrating along $[0,1]$, by using the orthogonality of this last function and $\varphi(x)$, we obtain the following equality:

$$0 = \int_0^1 \varphi(x) \cdot \left(\frac{e^{(2k-1)\pi x} + e^{\pi(2k-1)(1-x)}}{e^{(2k-1)\pi} + 1} + \cos(2k-1)\pi x \right) dx =$$

$$= \sum_{n=1}^{\infty} a_n \int_0^1 \left(\frac{e^{(2k-1)\pi x} + e^{\pi(2k-1)(1-x)}}{e^{(2k-1)\pi} + 1} + \cos(2k-1)\pi x \right) \cos(2n-1)\pi x dx = \frac{1}{2} a_k, k = 1, 2, 3, ...$$

From here, it follows that $a_k = 0$, $k = 1, 2, ...$ Therefore, from (4.6) we conclude that $\varphi(x) = 0$ in $[0,1]$, which opposing conditions $\varphi(x) \neq 0$. Thus, the system (4.2) is complete in the space $L_2(0,1)$. The completeness of the system (4.5) is proved similarly. Theorem 4.1 is proven. □

Theorem 4.2. *The system of functions (4.2) and (4.5) are two bases of Riesz in $L_2(0,1)$.*

Proof. In order to prove this statement, it is sufficient to prove the completeness of systems (4.2) and (4.5), and the convergence of the following series for $\varphi(x) \in L_2(0,1)$ according to Theorem 2.7:

$$\sum_{n=1}^{\infty} (\varphi(x), 2\sin(2n-1)\pi x)_0^2 + \sum_{n=1}^{\infty} \left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \cos(2n-1)\pi x\right)_0^2, \tag{4.7}$$

$$\sum_{n=1}^{\infty} (\varphi(x), 2\cos(2n-1)\pi x)_0^2 + \sum_{n=1}^{\infty} \left(\frac{e^{(2n-1)\pi x} - e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \sin(2n-1)\pi x, \varphi(x)\right)_0^2.$$

Since the completeness of systems (4.2) and (4.5) has been proven in Theorem 4.1, we only must verify the convergence of the previous series. Let us prove the convergence of series (4.7). To this end, we consider (4.7) and introduce the following notations

$$I_1 = 4\sum_{n=1}^{\infty} (\varphi(x), \sin(2n-1)\pi x)_0^2,$$

$$I_2 = \sum_{n=1}^{\infty} \left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \cos(2n-1)\pi x\right)_0^2.$$

I_1 represent in the form

$$I_1 = 4\sum_{n=1}^{\infty} (\varphi(x), \sin(2n-1)\pi x)_0^2 = 2\sum_{n=1}^{\infty} \left(\varphi(x), \sqrt{2}\sin(2n-1)\pi x\right)_0^2 = 2\sum_{n=1}^{\infty} c_n^2,$$

where $c_n = (\varphi(x), \sqrt{2}\sin(2n-1)\pi x)_0$ are the Fourier coefficients of the function $\varphi(x)$ on the orthonormal system $\{\sqrt{2}\sin(2n-1)\pi x\}$. From here, applying Bessels inequality, we obtain that $I_1 = 2\sum\limits_{n=1}^{\infty} c_n^2 \leq 2\|\varphi(x)\|_{L_2(0,1)}^2$, i.e. I_1 is finite.

Consider I_2. Let

$$A = \left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \cos(2n-1)\pi x\right)_0^2.$$

Since

$$A = \left(\left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0 + (\varphi(x), \cos(2n-1)\pi x)_0\right)^2,$$

from here, applying inequality $(a+b)^2 \leq 2(a^2+b^2)$ we obtain that

$$A \leq 2\left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\,(\varphi(x), \cos(2n-1)\pi x)_0^2 =$$

$$= 2\left(\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0 + \left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{(2n-1)\pi}+1}\right)_0\right)^2 + 2\,(\varphi(x), \cos(2n-1)\pi x)_0^2.$$

Applying the previous inequality again, we get that

$$A \leq 4\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0^2 + 4\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\,(\varphi(x), \cos(2n-1)\pi x)_0^2.$$

Thus

$$I_2 \leq 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0^2 + 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\sum_{n=1}^{\infty}(\varphi(x), \cos(2n-1)\pi x)_0^2 = J_1 + J_2 + J_3.$$

Consider $J_3 = 2\sum\limits_{n=1}^{\infty}(\varphi(x), \cos(2n-1)\pi x)_0^2 = \sum\limits_{n=1}^{\infty} a_n^2$, where $a_n = (\varphi(x), \sqrt{2}\cos(2n-1)\pi x)_0$ are the Fourier coefficients of the function $\varphi(x)$ on the orthonormal system $\{\sqrt{2}\cos(2n-1)\pi x\}$. Then applying Bessel's inequality, we get $J_3 = \sum\limits_{n=1}^{\infty} a_n^2 \le \|\varphi(x)\|^2_{L_2(0,1)}$. Consider J_2. Since

$$\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{\pi(2n-1)}+1}\right)_0 = \frac{e^{\pi(2n-1)}}{e^{\pi(2n-1)}+1}\left(\varphi(x)e^{\pi x}, e^{-2\pi n x}\right)_0,$$

hence

$$J_2 = 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 \le 4\sum_{n=1}^{\infty}\left(\varphi(x)e^{\pi x}, e^{-2\pi n x}\right)_0^2 = 4\sum_{n=1}^{\infty} c_n^2,$$

where $c_n = (\varphi(x)e^{\pi x}, e^{-2n\pi x})_0$. Then from Lemma 3.2, it follows that J_2 is finite. Similarly, we can prove that J_1 is finite too. Thus, the series I_1 and I_2 are converge, and therefore series (4.7) also converges. The convergence of the second series is proved similarly. Theorem 4.2 is proven. □

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