Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.152-164
DOI: 10.29229/uzmj.2025-4-15

An initial-boundary value problem with local and nonlocal conditions for a degenerate partial differential equation of high even order

Karimov K., Oripov D.

Abstract. In this paper, an initial-boundary value problem with local and nonlocal conditions for a degenerate partial differential equation of high even order in a rectangle is formulated and investigated. The method of energy integrals is used to prove the uniqueness of the solution to the problem, and using spectral analysis methods and the Green function, the solution to the considered problem is constructed as the sum of a Fourier series.

Keywords: degenerate equation, initial-boundary value problem, method of separation of variables, spectral problem, Green's function method, integral equation, Fourier series.
MSC (2020): 35G16

1. Introduction. Formulation of the problem

In a rectangle $\Omega = \{(x,t):\ 0<x<1;\ 0<t<T\}$, we consider the following degenerate equation of high even order

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^{2n}}{\partial x^{2n}}\left(x^{\alpha}\frac{\partial^{2n}u}{\partial x^{2n}}\right)=f\left(x,t\right),\ \ n\in\mathbb{N}, \tag{1.1}$$

where $u=u\left(x,t\right)$ is an unknown function, $f\left(x,t\right)$ is a given function, and $\alpha\in\mathbb{R}$ such that $0<\alpha<1$.

Many problems on vibrations of beams and plates, which are of great importance in structural mechanics, lead to differential equations of even order [1, 2, 3].

In 1956, F.I. Frankl in his work [4], considering the flow around a finite symmetrical profile by a subsonic flow with a supersonic zone ending in a normal shock wave, formulated a problem for the Chaplygin equation in a mixed domain with a non-local condition of type $u\left(0,y\right)=u\left(0,-y\right)$. In this case, local condition $u_x\left(0,y\right)=0$ was additionally specified. In the work [5] N.I. Ionkin proved the existence of a solution to a non-local problem with conditions $u_x\left(0,y\right)=u_x\left(1,y\right),\ u\left(0,y\right)=0,\ 0\le y\le T$ and $u\left(x,0\right)=\tau\left(x\right),\ 0\le x\le 1$ for the heat conduction equation using the spectral analysis method. In his work [6] he substantiated the uniqueness of the solution to this problem. Such conditions are encountered, for example, when solving problems describing the process of particle diffusion in turbulent plasma, as well as in the processes of heat propagation in a thin heated rod, if the law of change in the total amount of heat of the rod is given. Due to its wide range of applications, scientific research in these areas continues to advance.

This paper aims to study a nonlocal boundary value problem for an even-order higher-order differential equation. We first provide a brief overview of closely related results to situate our work within existing research.

In the work [7], in a rectangular domain of the plane xOt for the following equation

$$B^{t}_{\gamma-1/2}u+\left(-1\right)^{k}\frac{\partial^{k}}{\partial x^{k}}\left(x^{\alpha}\frac{\partial^{k}u}{\partial x^{k}}\right)=f\left(x,t\right),\ \ k\in\mathbb{N},$$

an initial-boundary value problem with local conditions was investigated, where $B^{y}_{q}\equiv\frac{\partial^2}{\partial y^2}+\frac{2q+1}{y}\frac{\partial}{\partial y}$ is the Bessel operator [8], $\alpha,\ 2\gamma\in(0,1)$. They proved the existence and uniqueness of the solution to the considered problem by applying the method of separation of variables.

For equation (1.1) with $\alpha=0$, where the derivative with respect to x is of order $2n$, various initial-boundary value problems involving non-local conditions have been studied in [9, 10].

Several studies, particularly [11, 12, 13], have examined both local and nonlocal problems for partial differential equations containing higher-order derivatives in the spatial variable x and time variable t.

For partial differential equations with multiple variables, [14] analyzed initial-boundary value problems involving both local and nonlocal conditions, while [15] focused on initial-boundary value problems with local conditions for a $4n$th-order partial differential equation in the spatial variable x.

We also note that for the case $n = 1$ in equation (1.1), the authors of [16] investigated an initial-boundary value problem for a degenerate equation with three lines of degeneracy.

The main goal and novelty of our work consists in the formulation and analysis of a new boundary value problem for a degenerate higher-order even partial differential equation, incorporating both local and nonlocal conditions. As this problem was first introduced by Professor A. Urinov, we hereafter refer to it as the Urinov problem (Problem U).

Problem U. Find a function $u(x,t)$ that with the following properties:
1) u_t, $(\partial^j/\partial x^j)\,u$, $(\partial^j/\partial x^j)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)u\right] \in C\left(\overline{\Omega}\right)$, $j = \overline{0, 2n-1}$,
$\left(\partial^{2n}/\partial x^{2n}\right)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)u\right]$, $u_{tt} \in C(\Omega)$;
2) in the domain Ω it satisfies equation (1.1);
3) it satisfies the following initial and boundary conditions:

$$u(x,0) = \varphi_1(x), \quad x \in [0,1], \quad u_t(x,0) = \varphi_2(x), \quad x \in [0,1], \tag{1.2}$$

$$\frac{\partial^{2j}}{\partial x^{2j}}u(0,t) + \frac{\partial^{2j}}{\partial x^{2j}}u(1,t) = 0, \quad \frac{\partial^{2j}}{\partial x^{2j}}\left(x^\alpha \frac{\partial^{2n}}{\partial x^{2n}}u(x,t)\right)\Big|_{x=0} = 0, \quad j = \overline{0, n-1}, \quad t \in [0,T], \tag{1.3}$$

$$\begin{gathered}\frac{\partial^{2j+1}}{\partial x^{2j+1}}u(1,t) = 0, \quad \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha \frac{\partial^{2n}}{\partial x^{2n}}u(x,t)\right)\Big|_{x=0} + \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha \frac{\partial^{2n}}{\partial x^{2n}}u(x,t)\right)\Big|_{x=1} = 0, \\ j = \overline{0, n-1}, \quad t \in [0,T],\end{gathered} \tag{1.4}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are given sufficiently smooth functions.

2. Study of the spectral problem

When formally applying the Fourier method to the problem U, we obtain the following spectral problem:

$$Mv(x) = \lambda v(x), \quad 0 < x < 1, \tag{2.1}$$

$$\left.\begin{aligned} & v^{(j)}(x), \quad \left(x^\alpha v^{(2n)}(x)\right)^{(j)} \in C[0,1], \quad j = \overline{0, 2n-1}, \\ & v^{(2j)}(0) + v^{(2j)}(1) = 0, \quad \left[x^\alpha v^{(2n)}(x)\right]^{(2j)}\Big|_{x=0} = 0, \\ & v^{(2j+1)}(1) = 0, \quad \left[x^\alpha v^{(2n)}(x)\right]^{(2j+1)}\Big|_{x=0} + \left[x^\alpha v^{(2n)}(x)\right]^{(2j+1)}\Big|_{x=1} = 0, \quad j = \overline{0, n-1}, \end{aligned}\right\} \tag{2.2}$$

where $M \equiv \partial^{2n}/\partial x^{2n}\ \left[x^\alpha \partial^{2n}/\partial x^{2n}\right]$.

Assume that there exists a non-trivial solution $v(x)$ to the problem $\{(2.1), (2.2)\}$. Under this assumption, multiplying both parts of equation (2.1) by the function $v(x)$ and integrating the resulting equality on the segment $[0,1]$, and then applying the rule of integration by parts $2n$ times and considering (2.2), we obtain

$$\lambda \int_0^1 v^2(x)\,dx = \int_0^1 x^\alpha \left[v^{(2n)}(x)\right]^2 dx. \tag{2.3}$$

From (2.3) it follows that $\lambda \geq 0$. Let $\lambda = 0$. Then from (2.3), we have $v^{(2n)}(x) = 0$, $0 < x < 1$. Hence, by virtue of conditions $v^{(2j)}(0) + v^{(2j)}(1) = 0$, $v^{(2j+1)}(1) = 0$, $j = \overline{0, n-1}$, we obtain $v(x) \equiv 0$, $0 \leq x \leq 1$. Consequently, problem $\{(2.1), (2.2)\}$ can have non-trivial solutions only for $\lambda > 0$.

Let us assume that $\lambda > 0$. We will prove the existence of eigenvalues of the problem $\{(2.1),(2.2)\}$ using the Green's function method. The Green's function $G(x,s)$ of this problem has the following properties:

1) the functions $(\partial^j/\partial x^j)\,G(x,s)$, $j = \overline{0, 2n-1}$ and $(\partial^j/\partial x^j)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]$,
$j = \overline{0, 2n-2}$ are continuous for all x, $s \in [0,1]$;

2) in each of the intervals $[0,s)$ and $(s,1]$ there exists a continuous derivative $(\partial^{2n-1}/\partial x^{2n-1})\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]$, and at $x=s$ it has a jump:

$$\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]\Big|_{x=s-0}^{x=s+0}=1; \tag{2.4}$$

3) in intervals $(0,s)$ and $(s,1)$ with respect to the argument x there is a continuous derivative $MG(x,s)$ and equality $MG(x,s)=0$ is satisfied;
4) at $s\in(0,1)$ by argument x satisfies the conditions

$$\frac{\partial^{2j}G(0,s)}{\partial x^{2j}}+\frac{\partial^{2j}G(1,s)}{\partial x^{2j}}=0,\ \frac{\partial^{2j}}{\partial x^{2j}}\left(x^\alpha\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\bigg|_{x=0}=0,\ j=\overline{0,n-1}, \tag{2.5}$$

$$\frac{\partial^{2j+1}G(1,s)}{\partial x^{2j+1}}=0,\ \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\bigg|_{x=0}+\frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\bigg|_{x=1}=0,\ j=\overline{0,n-1}. \tag{2.6}$$

Considering the form of the general solution of the equation $MG(x,s)=0$ in intervals $(0,s)$ and $(s,1)$, we will seek the function $G(x,s)$ in the following form

$$G(x,s)=\begin{cases}\displaystyle\sum_{j=1}^{2n}\frac{a_jx^{4n-\alpha-j}}{(2n-j)!(2n-\alpha-j+1)_{2n}}+\sum_{j=1}^{2n}\frac{a_{2n+j}x^{2n-j}}{(2n-j)!},\ 0\le x\le s,\\ \displaystyle\sum_{j=1}^{2n}\frac{b_jx^{4n-\alpha-j}}{(2n-j)!(2n-\alpha-j+1)_{2n}}+\sum_{j=1}^{2n}\frac{b_{2n+j}x^{2n-j}}{(2n-j)!},\ s\le x\le 1,\end{cases} \tag{2.7}$$

where a_j and b_j, $j=\overline{1,4n}$ are unknown functions of the variable s, and $(z)_n=z(z+1)(z+2)...(z+n-1)$ is the Pochhammer's symbol [17].

Satisfying the function (2.7) with properties 1) and 2) of the Green's function, we obtain the following system of equations with respect to b_j-a_j, $j=\overline{1,4n}$:

$$\begin{cases}b_1-a_1=1,\\ \displaystyle\sum_{j=1}^{m_1}\frac{s^{m_1-j}}{(m_1-j)!}(b_j-a_j)=0,\quad m_1=\overline{2,2n},\\ \displaystyle\sum_{j=1}^{2n}\frac{s^{2n-\alpha+m_2-j}(b_j-a_j)}{(2n-j)!(2n-\alpha-j+1)_{m_2}}+\sum_{j=1}^{m_2}\frac{s^{m_2-j}(b_{2n+j}-a_{2n+j})}{(m_2-j)!}=0,\ \ m_2=\overline{1,2n}.\end{cases}$$

This system has a unique solution [18]:

$$b_j-a_j=\frac{(-1)^{j-1}s^{j-1}}{(j-1)!},\ b_{2n+j}-a_{2n+j}=\frac{(-1)^{j-1}s^{2n+j-1-\alpha}}{(j-1)!(j-\alpha)_{2n}},\quad j=\overline{1,2n}. \tag{2.8}$$

Obeying (2.7) into the second condition of (2.5), we obtain

$$a_{2j+2}=0,\quad j=\overline{0,n-1}, \tag{2.9}$$

and we substituting this expression into the first equality of (2.8), we have

$$b_{2j+2}=-\frac{s^{2j+1}}{(2j+1)!},\quad j=\overline{0,n-1}. \tag{2.10}$$

Hence, t he second of the conditions (2.6) and (2.5), we obtain

$$a_{2j+1}+\sum_{i=1}^{2j+1}\frac{b_i}{(2j+1-i)!}=0,\quad j=\overline{0,n-1} \tag{2.11}$$

and
$$\sum_{i=1}^{2j+2}\frac{b_i}{(2j+2-i)!}=0,\quad j=\overline{0,n-1},\tag{2.12}$$
respectively.

Changing j to $2j+1$ in the first relation (2.8), we get
$$a_{2j+1}=b_{2j+1}-\frac{s^{2j}}{(2j)!},\quad j=\overline{0,n-1},\tag{2.13}$$
and substituting this expression into the system of equations (2.11), we obtain
$$b_1=\frac{1}{2},\quad b_{2j+1}=\frac{s^{2j}}{2\,(2j)!}-\sum_{i=1}^{2j}\frac{b_i}{2\,(2j+1-i)!},\quad j=\overline{1,n-1}.\tag{2.14}$$

From the system of equations (2.12), it is easy to determine the following equalities
$$b_2=-\frac{1}{2},\quad b_{2j+2}=-\sum_{i=1}^{2j+1}\frac{b_i}{(2j+2-i)!},\quad j=\overline{1,n-1}.\tag{2.15}$$

Solving the system of equations $\{(2.14),\,(2.15)\}$, we find $b_j,\ \ j=\overline{1,2n}$ and substituting their values into (2.13), we find $a_{2j+1},\ j=\overline{0,n-1}$, and the value of $a_{2j+2},\ j=\overline{0,n-1}$, by virtue of (2.8), is found from relations
$$a_{2j+2}=b_{2j+2}+\frac{s^{2j+1}}{(2j+1)!},\quad j=\overline{0,n-1}.\tag{2.16}$$

Now, substituting (2.7) into the first part of the condition (2.5), and as a result, we compose the following system of equations
$$a_{2n+2j}+\sum_{i=1}^{2n}\frac{b_i}{(2n-i)!(2n+1-i-\alpha)_{2j}}+\sum_{i=1}^{2j}\frac{b_{2n+i}}{(2j-i)!}=0,\quad j=\overline{1,n}.\tag{2.17}$$

In this case, the ratio
$$a_{2n+2j}=b_{2n+2j}+\frac{s^{2n+2j-1-\alpha}}{(2j-1)!(2j-\alpha)_{2n}},\quad j=\overline{1,n}\tag{2.18}$$
is determined by substituting $j\sim 2j$ into the second part of ratio (2.8).

We substitute the found relations (2.18) into the system of equations (2.17) and, taking into account (2.10), we obtain
$$\begin{gathered}b_{2n+2j}=-\frac{s^{2n+2j-1-\alpha}}{2\,(2j-1)!(2j-\alpha)_{2n}}-\\-\sum_{i=1}^{2n}\frac{b_i}{2\,(2n-i)!(2n+1-i-\alpha)_{2j}}-\sum_{i=1}^{2j-1}\frac{b_{2n+i}}{2\,(2j-i)!}=0,\quad j=\overline{1,n}.\end{gathered}\tag{2.19}$$

Substituting the results obtained from (2.19) into relation (2.18), we determine $a_{2n+2j},\ \ j=\overline{1,n}$.

Consequently, the Green function satisfying conditions 1)-4) exists, is unique and has the form (2.7), and the coefficients a_j and b_j, $j=\overline{1,4n}$ are determined by the equalities (2.9), (2.10), (2.13), (2.14), (2.15), (2.16), (2.18) and (2.19).

Now, let us prove that it is symmetric for its arguments. It is not easy to prove this fact using formulas (2.9), (2.10), (2.13), (2.14), (2.15), (2.16), (2.18) and (2.19). Therefore, we will use properties 1)-4) of the Green's function.

Let $v(x),\ h(x) \in C^{2n-1}[0,1];\ x^{\alpha} v^{(2n)}(x),\ x^{\alpha} h^{(2n)}(x) \in C^{2n-1}[0,1] \cap C^{2n}(0,1),\ n \in \mathbb{N}$. Then, the following identity holds true:

$$hM[v] - vM[h] = h(x)\left(x^{\alpha} v^{(2n)}(x)\right)^{(2n)} - v(x)\left(x^{\alpha} h^{(2n)}(x)\right)^{(2n)} =$$

$$\sum_{j=0}^{2n-1} \frac{d}{dx}\left\{(-1)^j\left[h^{(j)}(x)\left(x^{\alpha} v^{(2n)}(x)\right)^{(2n-1-j)} - v^{(j)}(x)\left(x^{\alpha} h^{(2n)}(x)\right)^{(2n-1-j)}\right]\right\}, 0 < x < 1. \tag{2.20}$$

If we assume that $v(x) = G(x,s)$ and $h(x) = G(x,\xi)$, then at all points of segment $(0,1)$, except points $x \neq \xi$, $x \neq s$, the equalities $M[v] = 0$ and $M[h] = 0$ hold. Then equality (2.20) takes the form

$$\sum_{j=0}^{2n-1} \frac{\partial}{\partial x}\left\{(-1)^j\left[\frac{\partial^j}{\partial x^j} G(x,\xi) \frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right) - \right.\right.$$
$$\left.\left. - \frac{\partial^j}{\partial x^j} G(x,s) \frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,\xi)\right)\right]\right\} = 0,\ x \in (0,1)/\{s,\xi\}. \tag{2.21}$$

Without loss of generality, assume that $s < \xi$. Then segment $[0,1]$ is divided into three segments, i.e. $[0,s]$, $[s,\xi]$ and $[\xi,1]$. Integrating the equality (2.21) over these segments, we obtain

$$\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{\partial^j}{\partial x^j} G(x,\xi) \frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right) - \right.\right.$$
$$\left.\left. - \frac{\partial^j}{\partial x^j} G(x,s) \frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,\xi)\right)\right]\right\}_{x=0}^{x=s-0} +$$
$$+ \sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{\partial^j}{\partial x^j} G(x,\xi) \frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right) - \right.\right.$$
$$\left.\left. - \frac{\partial^j}{\partial x^j} G(x,s) \frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,\xi)\right)\right]\right\}_{x=s+0}^{x=\xi-0} +$$
$$+ \sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{\partial^j}{\partial x^j} G(x,\xi) \frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right) - \right.\right.$$
$$\left.\left. - \frac{\partial^j}{\partial x^j} G(x,s) \frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,\xi)\right)\right]\right\}_{x=\xi+0}^{x=1} = 0.$$

Taking into account properties 1) and 4) of the Green's function $G(x,s)$, the last equality takes the form

$$-\left[G(x,\xi) \frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right)\right]\Bigg|_{x=s-0}^{x=s+0} + \left[G(x,s) \frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,\xi)\right)\right]\Bigg|_{x=s-0}^{x=s+0} -$$
$$-\left[G(x,\xi) \frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right)\right]\Bigg|_{x=\xi-0}^{x=\xi+0} + \left[G(x,s) \frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,\xi)\right)\right]\Bigg|_{x=\xi-0}^{x=\xi+0} = 0.$$

According to 2) property of the Green's function $G(x,s)$, the function $\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right) G(x,s)\right]$ is continuous at $x \neq s$. Considering this, from the last, we have

$$\left[G(x,\xi) \frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right)\Bigg|_{x=s-0} - G(x,\xi) \frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,s)\right)\Bigg|_{x=s+0}\right] +$$
$$+\left[G(x,s) \frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,\xi)\right)\Bigg|_{x=\xi+0} - G(x,s) \frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha} \frac{\partial^{2n}}{\partial x^{2n}} G(x,\xi)\right)\Bigg|_{x=\xi-0}\right] = 0.$$

Hence, by virtue of equality (2.4), we have $-G(s,\xi)+G(\xi,s)=0$, which was required to be proved.

In a particular case, for $n=1$, Green's function $G(x,s)$ has the form

$$G(x,s)=\begin{cases} -\dfrac{x^{3-\alpha}}{2(2-\alpha)_2}+\dfrac{x^{2-\alpha}s}{(1-\alpha)_2}-\dfrac{x^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{s^{3-\alpha}}{2(2-\alpha)_2}-\dfrac{s^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{1}{2(1-\alpha)_3}, & 0\le x\le s,\\ -\dfrac{s^{3-\alpha}}{2(2-\alpha)_2}+\dfrac{s^{2-\alpha}x}{(1-\alpha)_2}-\dfrac{s^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{x^{3-\alpha}}{2(2-\alpha)_2}-\dfrac{x^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{1}{2(1-\alpha)_3}, & s\le x\le 1,\end{cases}$$

and for $n=2$, it has the following form

$$G(x,s)=\begin{cases} -\dfrac{x^{7-\alpha}}{2\cdot 3!(4-\alpha)_4}+\left(s-\dfrac{1}{2}\right)\dfrac{x^{6-\alpha}}{2!(3-\alpha)_4}+\left(\dfrac{1}{2}-s^2\right)\dfrac{x^{5-\alpha}}{4(2-\alpha)_4}+\\ +\left(\dfrac{s^3}{3!}-\dfrac{s^2}{4}+\dfrac{1}{24}\right)\dfrac{x^{4-\alpha}}{(1-\alpha)_4}+\dfrac{s^{7-\alpha}}{2\cdot 3!(4-\alpha)_4}-\dfrac{s^{6-\alpha}}{2\cdot 2!(3-\alpha)_4}+\\ +\left(x^2+\dfrac{1}{2}\right)\dfrac{s^{5-\alpha}}{4(2-\alpha)_4}+\left(\dfrac{1}{6}-x^2\right)\dfrac{s^{4-\alpha}}{4(1-\alpha)_4}+\dfrac{x^2s^2}{8(1-\alpha)_3}-\dfrac{\alpha^2-9\alpha+12}{16(1-\alpha)_5}x^2-\\ -\dfrac{\alpha^2-9\alpha+12}{16(1-\alpha)_5}s^2+\dfrac{\alpha^4-22\alpha^3+163\alpha^2-430\alpha+328}{32(1-\alpha)_7}, \quad 0\le x\le s,\\ -\dfrac{s^{7-\alpha}}{2\cdot 3!(4-\alpha)_4}+\left(x-\dfrac{1}{2}\right)\dfrac{s^{6-\alpha}}{2!(3-\alpha)_4}+\left(\dfrac{1}{2}-x^2\right)\dfrac{s^{5-\alpha}}{4(2-\alpha)_4}+\\ +\left(\dfrac{x^3}{3!}-\dfrac{x^2}{4}+\dfrac{1}{24}\right)\dfrac{s^{4-\alpha}}{(1-\alpha)_4}+\dfrac{x^{7-\alpha}}{2\cdot 3!(4-\alpha)_4}-\dfrac{x^{6-\alpha}}{2\cdot 2!(3-\alpha)_4}+\\ +\left(s^2+\dfrac{1}{2}\right)\dfrac{x^{5-\alpha}}{4(2-\alpha)_4}+\left(\dfrac{1}{6}-s^2\right)\dfrac{x^{4-\alpha}}{4(1-\alpha)_4}+\dfrac{s^2x^2}{8(1-\alpha)_3}-\dfrac{\alpha^2-9\alpha+12}{16(1-\alpha)_5}s^2-\\ -\dfrac{\alpha^2-9\alpha+12}{16(1-\alpha)_5}x^2+\dfrac{\alpha^4-22\alpha^3+163\alpha^2-430\alpha+328}{32(1-\alpha)_7}, \quad s\le x\le 1.\end{cases}$$

The last two examples demonstrate that the Green's function is symmetric in its arguments.

Now, using the method applied in [19], problem $\{(2.1),(2.2)\}$ can be equivalently reduced to the following integral equation

$$v(x)=\lambda\int_0^1 G(x,s)\,v(s)\,ds. \tag{2.22}$$

Since the kernel $G(x,s)$ is continuous, symmetric and positive ($\lambda>0$), then the integral equation (2.22), and therefore the problem $\{(2.1),(2.2)\}$ has a countable set of eigenvalues $0<\lambda_1\le\lambda_2\le\lambda_3\le \ldots\le\lambda_k<\ldots,\quad \lambda_k\to+\infty$, and the corresponding eigenfunctions $v_1(x), v_2(x), v_3(x), \ldots, v_k(x)\ldots$ form an orthonormal system in the space $L_2(0,1)$ [20].

Moreover, it is easy to verify by direct verification that the system of functions $s^{\alpha/2}v_k^{(2n)}(s)/\sqrt{\lambda_k}$, $k=1,2,\ldots$ also constitutes an orthonormal system in $L_2(0,1)$.

Lemma 2.1. *Let the function $g(x)$ satisfy conditions (2.2) and $Mg(x)\in C(0,1)\cap L_2(0,1)$. Then, it can be expanded on the segment $[0,1]$ into an absolutely and uniformly convergent series in the system of eigenfunctions of the problem $\{(2.1),(2.2)\}$.*

Proof. Using the rule of integration by parts, the properties of Green's function $G(x,s)$ and the conditions imposed on the function $g(x)$, it is easy to verify that the following equality holds:

$$\int_0^1 G(x,s)\,Mg(s)\,ds=\int_0^1 G(x,s)\left[s^\alpha g^{(2n)}(s)\right]^{(2n)}ds=g(x).$$

Since $Mg(x)\in L_2(0,1)$, it follows from the last equality that $g(x)$ is a function that can be representable through the kernel of $G(x,s)$. In addition, the function $G(x,s)$, i.e. the kernel of equation

(2.22), is continuous in $\overline{\Omega}$. Then, based on Theorem 2, p. 153, of the book [20], the statement of Lemma 2.1 is true. □

Lemma 2.2. *The following series converge uniformly on the segment* $[0,1]$*:*

$$\sum_{k=1}^{+\infty}\frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k},\quad \sum_{k=1}^{+\infty}\frac{\left(\left[x^\alpha v_k^{(2n)}(x)\right]^{(j)}\right)^2}{\lambda_k^2},\quad j=\overline{0,2n-1}. \tag{2.23}$$

Proof. Taking into account equality (2.1) and the properties of function $G(x,s)$, from (2.22) for $v(x)\equiv v_k(x)$ we obtain

$$v_k^{(j)}(x)=\lambda_k\int_0^1\frac{\partial^j}{\partial x^j}G(x,s)\,v_k(s)\,ds=\int_0^1\left[s^\alpha v_k^{(2n)}(s)\right]^{(2n)}\frac{\partial^j}{\partial x^j}G(x,s)\,ds,\quad j=\overline{0,2n-1}.$$

Hence, applying the rule of integration by parts $2n$ times, and then taking into account conditions (2.2), we have

$$v_k^{(j)}(x)=\int_0^1 s^\alpha v_k^{(2n)}(s)\,\frac{\partial^{2n+j}}{\partial x^j\partial s^{2n}}G(x,s)\,ds,\ j=\overline{0,2n-1},$$

from which, by virtue of $\lambda_k>0$, the equality

$$\frac{v_k^{(j)}(x)}{\sqrt{\lambda_k}}=\int_0^1\left(s^{\alpha/2}\frac{\partial^{2n+j}}{\partial x^j\partial s^{2n}}G(x,s)\right)\left(\frac{s^{\alpha/2}v_k^{(2n)}(s)}{\sqrt{\lambda_k}}\right)ds,\quad j=\overline{0,2n-1} \tag{2.24}$$

follows.

From (2.24) it follows that $v_k^{(j)}(x)/\sqrt{\lambda_k}$ is the Fourier coefficient of the function by the orthonormal system of functions $\left\{s^{\alpha/2}v_k^{(2n)}(s)/\sqrt{\lambda_k}\right\}_{k=1}^{+\infty}$. Therefore, according to Bessel's inequality [20], we have

$$\sum_{k=1}^{+\infty}\frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k}\le\int_0^1 s^\alpha\left[\frac{\partial^{2n+j}}{\partial x^j\partial s^{2n}}G(x,s)\right]^2 ds,\quad j=\overline{0,2n-1}. \tag{2.25}$$

By $0<\alpha<1$, the integral on the right-hand side of inequality (2.25) is uniformly bounded for $j=\overline{0,2n-1}$, which implies that the first series in (2.23) converges uniformly.

The convergence of the remaining series is proved similarly.

Lemma 2.2 has been proved. □

From here and the rest of the paper g_k denotes the Fourier coefficient of the function $g(x)$ by the system of eigenfunctions $\{v_k(x)\}_{k=1}^{+\infty}$, i.e.

$$g_k=\int_0^1 g(x)\,v_k(x)\,dx,\quad k\in\mathbb{N}. \tag{2.26}$$

Lemma 2.3. *Let* $g^{(j)}(x)\in C[0,1],\ j=\overline{0,2n-1},\ x^{\alpha/2}g^{(2n)}(x)\in C(0,1)\cap L_2(0,1),\ g^{(2j)}(0)+g^{(2j)}(1)=0,\ g^{(2j+1)}(1)=0,\ j=\overline{0,n-1}$. *Then, the following inequality is valid*

$$\sum_{k=1}^{+\infty}\lambda_k g_k^2\le\int_0^1 x^\alpha\left[g^{(2n)}(x)\right]^2dx. \tag{2.27}$$

In particular, the series on the left-hand side converges.

Proof. Using the equation (2.1), we can rewrite (2.26) as follows

$$\lambda_k^{1/2} g_k = \lambda_k^{1/2} \int_0^1 g(x)\, v_k(x)\, dx = \lambda_k^{-1/2} \int_0^1 g(x) \left[x^\alpha v_k^{(2n)}(x)\right]^{(2n)} dx.$$

Hence, applying the rule of integration by parts $2n$ times and considering the properties of the functions $g(x)$ and $v_k(x)$, we obtain

$$\lambda_k^{1/2} g_k = \int_0^1 \left\{x^{\alpha/2} g^{(2n)}(x)\right\} \left\{\lambda_k^{-1/2} x^{\alpha/2} v_k^{(2n)}(x)\right\} dx.$$

From the last, it follows that the number $\lambda_k^{1/2} g_k$ is the Fourier coefficient of the function $x^{\alpha/2} g^{(2n)}(x)$ in the orthonormal system $\left\{x^{\alpha/2} v^{(2n)}(x)/\sqrt{\lambda_k}\,\right\}_{k=1}^{+\infty}$. Then, according to Bessel's inequality [20], we have the validity of (2.27). Lemma 2.3 has been proved. □

Lemma 2.4. *If the function $g(x)$ satisfies the conditions* (2.2) *and $Mg(x) \in C(0,1) \cap L_2(0,1)$, then the inequality holds true*

$$\sum_{k=1}^{+\infty} \lambda_k^2 g_k^2 \leq \int_0^1 \left[Mg(x)\right]^2 dx. \tag{2.28}$$

In particular, the series on the left-hand side converges.

Proof. Considering (2.1), we rewrite (2.26), in the following form

$$\lambda_k g_k = \lambda_k \int_0^1 g(x)\, v_k(x)\, dx = \int_0^1 g(x) \left[x^\alpha v_k^{(2n)}(x)\right]^{(2n)} dx.$$

Hence, applying the rule of integration by parts $4n$ times to the integral on the right-hand side of the last equality and considering the properties of the functions $g(x)$ and $v_k(x)$, we obtain

$$\lambda_k g_k = \int_0^1 \left[x^\alpha g^{(2n)}(x)\right]^{(2n)} v_k(x)\, dx = \int_0^1 [Mg(x)] v_k(x)\, dx.$$

From the last, it follows that the number $\lambda_k g_k$ is the Fourier coefficient of the function $Mg(x)$ with respect to the orthonormal system of functions $\{v_k(x)\}_{k=1}^{+\infty}$. Then, according to Bessel's inequality, inequality (2.28) is valid. Lemma 2.4 has been proved. □

Similarly, the following lemma can be proved:

Lemma 2.5. *If the function $g(x)$ satisfies conditions* (2.2) *and $[Mg(x)]^{(j)} \in C[0,1]$, $j = \overline{0, 2n-1}$, $x^{\alpha/2}[Mg(x)]^{(2n)} \in C(0,1) \cap L_2(0,1)$, $[Mg(x)]^{(2j)}\Big|_{x=0} + [Mg(x)]^{(2j)}\Big|_{x=1} = 0$, $[Mg(x)]^{(2j+1)}\Big|_{x=1} = 0$, $j = \overline{0, n-1}$, then inequality*

$$\sum_{k=1}^{+\infty} \lambda_k^3 g_k^2 \leq \int_0^1 x^\alpha \left\{[Mg(x)]^{(2n)}\right\}^2 dx \tag{2.29}$$

is true, in particular, the series on the left-hand side of (2.29) *converges.*

3. Existence, uniqueness and stability of the solution of problem U

We will seek a solution to the problem U in the form

$$u(x,t)=\sum_{k=1}^{+\infty}u_k(t)\,v_k(x), \tag{3.1}$$

where $v_k(x)$, $k\in\mathbb{N}$ are the eigenfunctions of the problem $\{(2.1),(2.2)\}$, and $u_k(t)$, $k\in\mathbb{N}$ are the unknown functions to be determined.

Substituting (3.1) into equation (1.1) and into the initial conditions (1.2), with respect to $u_k(t), k\in\mathbb{N}$, we obtain the following problem

$$u_k''(t)+\lambda_k u_k(t)=f_k(t),\ \ t\in(0,T),\ \ k\in\mathbb{N},$$

$$u_k(0)=\varphi_{1k},\quad u_k'(0)=\varphi_{2k},$$

where

$$\varphi_{jk}=\int_0^1\varphi_j(x)\,v_k(x)\,dx,\ \ j=\overline{1,2};\ \ f_k(t)=\int_0^1 f(x,t)\,v_k(x)\,dx,\ \ k\in\mathbb{N}. \tag{3.2}$$

It is known that the solution to the last problem exists, is unique, and is determined by the following formula:

$$u_k(t)=\varphi_{1k}\cos\left(t\sqrt{\lambda_k}\right)+\varphi_{2k}\lambda_k^{-1/2}\sin\left(t\sqrt{\lambda_k}\right)+\lambda_k^{-1/2}\int_0^t f_k(\tau)\sin\left[(t-\tau)\sqrt{\lambda_k}\right]d\tau, 0\le t\le T. \tag{3.3}$$

From this, the estimate

$$|u_k(t)|\le|\varphi_{1k}|+\frac{1}{\sqrt{\lambda_k}}|\varphi_{2k}|+\frac{1}{\sqrt{\lambda_k}}\sqrt{\int_0^T f_k^2(\tau)d\tau},\ \ 0\le t\le T \tag{3.4}$$

easily follows.

Theorem 3.1. *Let the function $\varphi_1(x)$ satisfy the conditions of Lemma 2.5, the function $\varphi_2(x)$ satisfy the conditions of Lemma 2.4, and the function $f(x,t)$ satisfy the conditions of Lemma 2.4 with respect to the argument x uniformly with respect to t. Then the series* (3.1)*, whose coefficients are defined by equalities* (3.3)*, determines the solution of the problem U.*

Proof. To do this, we need to prove the uniform convergence in $\overline{\Omega}$ of the series (3.1) and the following series, formally obtained from (3.1):

$$u_t(x,t)=\sum_{k=1}^{+\infty}u_k'(t)\,v_k(x),\ \ \frac{\partial^j u(x,t)}{\partial x^j}=\sum_{k=1}^{+\infty}u_k(t)\,v_k^{(j)}(x),\ \ j=\overline{1,2n-1},$$

$$\frac{\partial^j}{\partial x^j}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial^{2n}}\right)=\sum_{k=1}^{+\infty}u_k(t)\left(x^\alpha v_k^{(2n)}(x)\right)^{(j)},\ \ j=\overline{0,2n-1},$$

and the uniform convergence in any compact set $\Omega_0\subset\Omega$ of the series

$$\frac{\partial^{2n}}{\partial x^{2n}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial^{2n}}\right)=\sum_{k=1}^{+\infty}u_k(t)\left(x^\alpha v_k^{(2n)}(x)\right)^{(2n)}, \tag{3.5}$$

$$u_{tt}(x,t)=\sum_{k=1}^{+\infty}u_k''(t)\,v_k(x). \tag{3.6}$$

Let us consider the series (3.1). By virtue of (3.4) from (3.1), for any $(x,t)\in\overline{\Omega}$, we have

$$|u(x,t)|\le\sum_{k=1}^{+\infty}|u_k(t)|\,|v_k(x)|\le\sum_{k=1}^{+\infty}\frac{|v_k(x)|}{\sqrt{\lambda_k}}\left(\sqrt{\lambda_k}\,|\varphi_{1k}|+|\varphi_{2k}|+\sqrt{\int_0^T f_k^2(\tau)d\tau}\right).$$

Hence, applying the Cauchy-Schwarz inequality, we obtain

$$|u(x,t)|\le\sqrt{\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}}\left(\sqrt{\sum_{k=1}^{+\infty}\lambda_k\varphi_{1k}^2}+\sqrt{\sum_{k=1}^{+\infty}\varphi_{2k}^2}+\sqrt{\int_0^T\sum_{k=1}^{+\infty}[f_k(\tau)]^2d\tau}\right). \tag{3.7}$$

The series on the right-hand side of this inequality, by the condition of Theorem 3.1, according to Lemmas 2.2 and 2.3, converge uniformly. Therefore, the series on the left-hand side, i.e., series (3.1) converges uniformly in $\overline{\Omega}$.

Now, let us consider the series (3.5). By virtue of equation (2.1), in any compact Ω_0 the series in (3.5) can be written as

$$\sum_{k=1}^{+\infty}\lambda_k u_k(t)\,v_k(x). \tag{3.8}$$

To prove the uniform convergence of series (3.8), according to (3.4), it is sufficient to prove the absolute and uniform convergence of the series

$$\sum_{k=1}^{+\infty}\lambda_k\varphi_{1k}v_k(x),\quad\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\varphi_{2k}v_k(x),\quad\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\sqrt{\int_0^T[f_k(\tau)]^2d\tau}\,v_k(x). \tag{3.9}$$

Similarly, applying Cauchy-Schwarz inequality to each of these series, we have

$$\left|\sum_{k=1}^{+\infty}\lambda_k\varphi_{1k}v_k(x)\right|\le\sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k^3}\varphi_{1k}\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le\left[\sum_{k=1}^{+\infty}\lambda_k^3\varphi_{1k}^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\left|\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\varphi_{2k}v_k(x)\right|\le\sum_{k=1}^{+\infty}\left|\lambda_k\varphi_{2k}\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le\left[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_{2k}^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\left|\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\sqrt{\int_0^T[f_k(\tau)]^2d\tau}\,v_k(x)\right|\le\sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k^2\int_0^T[f_k(\tau)]^2d\tau}\,\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le$$

$$\le\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^2[f_k(\tau)]^2d\tau\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

The series on the right-hand sides of these inequalities, by the conditions of Theorem 3.1, according to Lemmas 2.2, 2.4, and 2.5, converge uniformly. Then the series on the left-hand sides, i.e., series (3.9), converge absolutely and uniformly in Ω_0. Therefore, the series (3.8), and consequently the series in (3.5) converge uniformly in the any compact set Ω_0. Uniform convergence in Ω_0 of series (3.6) follows from the convergence of series (3.5) and the validity of equation (1.1).

The uniform convergence of the remaining series can be proved similarly. Theorem 3.1 has been proved. □

Theorem 3.2. *Problem U cannot have more than one solution.*

Proof. Suppose that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of the problem U. We denote their difference by $u(x,t)$. Then the function $u(x,t)$ satisfies equation (1.1) for $f(x,t)\equiv 0$, and conditions (1.2) and (1.3) for $\varphi_1(x)\equiv\varphi_2(x)\equiv 0$.

Let $\forall T_0\in(0,T]$, $\Omega_0=\{(x,t):\ 0<x<1,\ 0<t<T_0\}$. Obviously $\overline{\Omega}_0\subset\overline{\Omega}$. Let us introduce the following function:

$$\omega(x,t)=-\int\limits_t^{T_0}u(x,\xi)d\xi,\quad (x,t)\in\overline{\Omega}_0\,.$$

This function has the following properties:

1) $\omega_t,\ \omega_{tt},\ \frac{\partial^j\omega}{\partial x^j},\ \frac{\partial^j}{\partial x^j}\left(x^\alpha\frac{\partial^{2n}\omega}{\partial x^{2n}}\right)\in C\left(\overline{\Omega}_0\right),\ \ j=\overline{0,2n-1};$

2) it satisfies conditions (1.3) for $t\in[0,T_0]$.

Let us consider equation (1.1) for $f(x,t)\equiv 0$, multiplying it by the function $\omega(x,t)$ and integrating the resulting equality over the domain Ω_0, we have

$$\int\limits_{\Omega_0}\omega(x,t)\left\{u_{tt}(x,t)+\frac{\partial^{2n}}{\partial x^{2n}}\left[x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right]\right\}dtdx=0.$$

Let us rewrite this equality as

$$\int\limits_0^{T_0}dt\int\limits_0^1\omega(x,t)\frac{\partial^{2n}}{\partial x^{2n}}\left[x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right]dx+\int\limits_0^1dx\int\limits_0^{T_0}\omega(x,t)u_{tt}(x,t)dt=0.$$

Now, applying the rule of integration by parts, we obtain the equality

$$\int\limits_0^{T_0}\left[\omega(x,t)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)-\frac{\partial\omega(x,t)}{\partial x}\frac{\partial^{2n-2}}{\partial x^{2n-2}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)+...\right.$$

$$\left....-\frac{\partial^{2n-1}\omega(x,t)}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)\right]_{x=0}^{x=1}dt+\int\limits_0^{T_0}dt\int\limits_0^1x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx+$$

$$+\int\limits_0^1\left[\omega(x,t)\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}^{t=T_0}-\int\limits_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}\right]dx=0,$$

from which, due to the properties of the functions $\omega(x,t)$ and $u(x,t)$, it follows that

$$\int\limits_0^{T_0}dt\int\limits_0^1x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx-\int\limits_0^1dx\int\limits_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}dt=0.$$

Hence, taking into account the equalities $u=\frac{\partial\omega}{\partial t}$ and $\frac{\partial^{2n}u}{\partial x^{2n}}=\frac{\partial^{2n+1}\omega}{\partial x^{2n}\partial t}$, we have

$$\int\limits_0^1x^\alpha dx\int\limits_0^{T_0}\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}dt-\int\limits_0^1dx\int\limits_0^{T_0}u(x,t)\frac{\partial u(x,t)}{\partial t}dt=0.$$

Next, considering the equalities

$$u(x,t)\frac{\partial u(x,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}[u(x,t)]^2,\quad \frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\right]^2,$$

and applying the rule of integration by parts to the integrals over t, taking into account $\omega(x,T_0)=0$, $u\,(x,0)=0$, we obtain

$$\int\limits_0^1 u^2(x,T_0)\,dx+\int\limits_0^1 x^\alpha\left[\frac{\partial^{2n}\omega\,(x,t)}{\partial x^{2n}}\right]^2_{t=0}dx=0.$$

Hence, it follows that $u\,(x,T_0)\equiv 0,\ x\in[0,1]$. Since $\forall T_0\in(0,T]$, then $u\,(x,t)\equiv 0,\ (x,t)\in\overline{\Omega}$. Then $u_1\,(x,t)\equiv u_2\,(x,t)\,,\ (x,t)\in\overline{\Omega}$. Theorem 3.2 is proven. □

Theorem 3.3. *Let the functions $\varphi_1\,(x)\,,\ \varphi_2\,(x)$ and $f\,(x,t)$ satisfy the conditions of Theorem (3.1). Then the following estimates are valid for the solution of problem U:*

$$\|u\,(x,t)\|^2_{L_2(0,1)}\le K_0\left[\|\varphi_1\,(x)\|^2_{L_2(0,1)}+\|\varphi_2\,(x)\|^2_{L_2(0,1)}+\|f\,(x,t)\|^2_{L_2(\Omega)}\right],\tag{3.10}$$

$$\|u\,(x,t)\|_{C\left(\overline{\Omega}\right)}\le K_1\left[\left\|\varphi_1^{(2n)}\,(x)\right\|_{L_{2,r}(0,1)}+\|\varphi_2\,(x)\|_{L_2(0,1)}+\|f\,(x,t)\|_{L_2(\Omega)}\right],\tag{3.11}$$

where $\|\varphi_1\,(x)\|_{L_{2,r}(0,1)}=\left[\int\limits_0^1 x^\alpha[\varphi_1\,(x)]^2dx\right]^{1/2}$ *and* $r=r\,(x)=x^\alpha$, *and* K_0 *and* K_1 *are some real positive numbers.*

Proof. Here, taking into account the orthonormality of the system $\{v_k\,(x)\}_{k=1}^{+\infty}$ and inequality (3.4), from (3.1), we have

$$\|u\,(x,t)\|^2_{L_2(0,1)}=\sum_{k=1}^{+\infty}u_k^2\,(t)\le$$

$$\le\sum_{k=1}^{+\infty}\left[|\varphi_{1k}|+\frac{1}{\sqrt{\lambda_k}}\,|\varphi_{2k}|+\frac{1}{\sqrt{\lambda_k}}\|f_k\,(t)\|_{L_2(0,T)}\right]^2\le 3\sum_{k=1}^{+\infty}\left[\varphi_{1k}^2+\frac{1}{\lambda_k}\varphi_{2k}^2+\frac{1}{\lambda_k}\,\|f_k(t)\|^2_{L_2(0,T)}\right].$$

Hence, applying Bessel's inequality, we get

$$\|u\,(x,t)\|^2_{L_2(0,1)}\le K_0\left(\|\varphi_1\,(x)\|^2_{L_2(0,1)}+\|\varphi_2\,(x)\|^2_{L_2(0,1)}+\sum_{k=1}^{+\infty}\|f_k\,(t)\|^2_{L_2(0,T)}\right),\tag{3.12}$$

where $K_0=3C,\ C=\max\left(1,\ 1/\lambda_1\ \right)$.

Taking into account the easily verified equality

$$\|f\,(x,t)\|^2_{L_2(\Omega)}=\sum_{n=1}^{+\infty}\|f_k\,(t)\|^2_{L_2(0,T)},$$

from (3.12) we obtain inequality (3.10).

Taking into account the statements of Lemmas 2.2 and 2.3, from (3.7), we obtain the inequality

$$\|u\,(x,t)\|_{C\left(\overline{\Omega}\right)}=\sup_{\overline{\Omega}}\,|u\,(x,t)|\le K_1\left\{\sqrt{\int\limits_0^1 x^\alpha\left[\varphi_1^{(2n)}\,(x)\right]^2dx+\|\varphi_2\,(x)\|^2_{L_2(0,1)}+\sum_{k=1}^{+\infty}\|f_k\,(t)\|^2_{L_2(0,T)}}\right\},$$

where $K_1=\sup\limits_{[0,1]}\sqrt{\sum\limits_{k=1}^{+\infty}v_k^2\,(x)/\lambda_k}$.

From here, by the introduced notations, inequality (3.11) follows.

Theorem 3.3 has been proved. □

References

[1] Tikhonov A. N., Samarskii A. A.; Equations of Mathematical Physics. Moscow, Nauka, –1966.

[2] Strutt J.; Theory of sound. Vol. 1, Gostekhizdat, Moscow, USSR. –1955.

[3] Timoshenko S.P., Young D.Kh., Weaver U.; Oscillations in engineering. Moscow, Mechanical Engineering,–1985.

[4] Frankl F.I.; Flowing profiles by a stream of subsonic velocity with a supersonic zone ending with a direct surge of seal. PMM,–1956.–47(2).–P.196-202.

[5] Ionkin N.I.; The Solution of a boundary value problem of thermal conduction theory with a nonlocal boundary condition. Dier. equations.,–1977.–13(2).–P.294-304.

[6] Ionkin N.I.; On the sustainability of one problem theory of thermal conductivity with a nonclassical edge condition. Differ. equations,–1979.–15(7).–P.1279-1283.

[7] Urinov A.K., Azizov M.S.; On the solvability of an initialboundary value problem for a high even order partial differential equation degenerating on the domain boundary. Journal of Applied and Industrial Mathematics,–2023.–17(2).–P.414-426.

[8] Kipriyanov I.A.; Singular elliptic boundary value problems. Moscow, Nauka,–1997.

[9] Urinov A.K., Azizov M.S.; On the solvability of nonlocal initial-boundary value problems for a partial differential equation of high even order. Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompyuternye Nauki,–2022.–32(2).–P.240-255.

[10] Azizov M.S.; About an initial-boundary value problem for a partial differential equation of higher even order with the Bessel operator. Bull. Inst. Math.,–2022.–5(1).–P.14-24.

[11] Sabitov K.B.; The Dirichlet problem for higher-order partial dierential equations. Math. Notes,–2015.–97(2).–P.255-267.

[12] Irgashev B.Yu.; On the conditions for the solvability of a problem of Dirichlet type for the equation of high order with discontinuous coecients. Uzbek Mathematical Journal. –2021.–65(4).–P.13-26.

[13] Irgashev B.Yu.; Boundary value problem for high order equation with discontinuous coecients . Uzbek Mathematical Journal. –2020.–3.–P.85-97.

[14] Kasimov Sh.G., Ataev Sh.K.; On solvability of the mixed problem for a partial equation of a fractional order with Laplace operators and nonlocal boundary conditions in the Sobolev classes. Uzbek Mathematical Journal,–2018.–1.–P.73-89.

[15] Kasimov Sh.G., Madrakhimov U.S; On the unique solvability for initial - boundary problems of vibrations of a beam, one end of which is xed and the other is free, in the Sobolev classes in the multidimensional case. Uzbek Mathematical Journal,–2019.–4.–P.89-103.

[16] Urinov A.K., Usmonov D.A.; Initial boundary value problems for the fourth order equation with three lines of degeneracy. Uzbek Mathematical Journal. –2023.–67(1).–P.129-136.

[17] Beytmen G., Erdeyi A.; Higher transcendental functions. Vol. 1. Moskva, Nauka,–1965.

[18] Urinov A.K., Oripov D.D.; On the solvability of an initial boundary problem for a high even order degenerate equation. J. Samara State Tech. Univ., Ser. Phys. Math. Sci.,–2023.–27(4).–P.621-644.

[19] Naimark M.A.; Linear Differential Operators. Moscow, Nauka,–1969.

[20] Mikhlin S.G.; Lectures on Linear Integral Equations. Moscow, Fizmatgiz,–1959.

Karimov K.T.,
Romanovskii Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan,
Fergana State University, Fergana, Uzbekistan,
email: karimovk80@mail.ru

Oripov D.D.,
Fergana State University, Fergana, Uzbekistan,
email: dastonbekoripov94@gmail.com