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About one composition of partial mapping of Euclidean space E_5

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Abstract. In the domain $\Omega \subset E_5$ we consider a set of smooth lines such that through each point $X \in \Omega$ there passes exactly one line ω^1 from the given set. The moving frame of the domain Ω is a Frenet frame [1] associated with the line ω^1. The integral lines of the coordinate vector fields form a Frenet net [1]. We define the point F_1^5 on the tangent of the line ω^1 in an invariant manner. As the point X moves within the domain Ω, the point F_1^5 traces out a new domain $\Omega_1^5 \subset E_5$. This defines the partial mapping $f_1^5 : \Omega \to \Omega_1^5$ such that $f_1^5(X) = F_1^5$.

Similarly, we define another partial mapping $f_5^4 : \Omega \to \Omega_5^4$ Next, we consider the composition of these two partial mappings, specifically the inverse mapping $(f_1^5)^{-1}$ and f_5^4 given by:

$f_5^4 \circ (f_1^5)^{-1} : \Omega_1^5 \to \Omega_5^4$ such that $f_5^4 \circ (f_1^5)^{-1}(F_1^5) = F_5^4$, where $(f_1^5)^{-1}$ - is the inverse mapping f_1^5.

Let the line γ, which belongs to the distribution $\Delta_4 = (X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5)$ be a quasi-double line of the pair of distributions (Δ_4, Δ_4') in the partial mapping f_1^5 (where $\Delta_4' = f_1^5(\Delta_4)$).

We establish necessary and sufficient conditions for the line $f_5^4 \circ (f_1^5)^{-1}(\gamma)$ to be a quasi-double line of the pair (Δ_4, Δ_4') of distributions Δ_4, Δ_4' in the partial mapping $f_5^4 \circ (f_1^5)^{-1}$.

Keywords: Euclidean space, Frenet frame, cyclic Frenet net, partial mapping, quasi-double line, distribution.

MSC (2020): 53A04, 53A15

1. Introduction

In the domain $\Omega \subset E_5$ we consider a set of smooth lines such that through each point $X \in \Omega$ exactly one line of the given set passes. The moving frame $\Re = (X, \overrightarrow{e}_i)(i, j, k = \overline{1,5})$ is the Frenet frame for the line ω^1 of the given set of smooth lines. The derivation formula for the frame $\Re$ take the form:

$$d\overrightarrow{X} = \omega^i \overrightarrow{e}_i, d\overrightarrow{e}_i = \omega_i^k \overrightarrow{e}_k. \tag{1.1}$$

The forms ω^i, ω_i^k satisfy the structure equations of Euclidean space:

$$D\omega^i = \omega^k \wedge \omega_k^i, D\omega_i^k = \omega_i^j \wedge \omega_j^k, \omega_i^j + \omega_j^i = 0. \tag{1.2}$$

Integral lines of the vector fields $\overrightarrow{e}_i$ form the Frenet net Σ_5 [1] for the line ω^1 of the given set of lines. Since frame $\Re$ is constructed on the tangent of the lines of the net Σ_5, the forms ω_i^k are principal forms. In other words:

$$\omega_i^k = \Lambda_{ij}^k \omega^j. \tag{1.3}$$

Taking in to account (1.3) from (1.2) we have:

$$\Lambda_{ij}^k = -\Lambda_{kj}^i. \tag{1.4}$$

If we differentiate externally (1.3), we obtain:

$$D\omega_i^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k D\omega^j.$$

Using equation (1.2) we get:

$$\omega_i^j \wedge \omega_j^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k \wedge \omega^\ell \wedge \omega_\ell^j.$$

Taking in to account equation (1.3) the last formula simplifies to:

$$\omega_i^j \wedge \Lambda_{j\ell}^k \omega^\ell = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \wedge \omega_\ell^j \wedge \omega^\ell$$

or equivalently,

$$\Lambda^k_{j\ell}\omega^j_i \wedge \omega^\ell = d\Lambda^k_{ij} \wedge \omega^j - \Lambda^k_{ij} \wedge \omega^j_\ell \wedge \omega^\ell.$$

From here, we obtain:

$$d\Lambda^k_{ij} \wedge \omega^j - \Lambda^k_{i\ell}\omega^\ell_j \wedge \omega^j - \Lambda^k_{j\ell}\omega^j_i \wedge \omega^\ell = 0$$

or

$$(d\Lambda^k_{ij} - \Lambda^k_{i\ell}\omega^\ell_j - \Lambda^k_{\ell j}\omega^\ell_i) \wedge \omega^j = 0.$$

Using Cartan's lemma [2], [6], we conclude:

$$d\Lambda^k_{ij} - \Lambda^k_{i\ell}\omega^\ell_j - \Lambda^k_{\ell j}\omega^\ell_i = \Lambda^k_{ijm}\omega^m.$$

Which simplifies to:

$$d\Lambda^k_{ij} = (\Lambda^k_{ijm} + \Lambda^k_{i\ell}\Lambda^\ell_{jm} + \Lambda^k_{\ell j}\Lambda^\ell_{im})\omega^m. \tag{1.5}$$

The system of variable $\{\Lambda^k_{ij}, \Lambda^k_{ijm}\}$ forms a second - order geometrical object. The Frenet formulas for the line ω^1 of given set take the form:

$$d_1\overrightarrow{e}_1 = \Lambda^2_{11}\overrightarrow{e}_2, \qquad d_1\overrightarrow{e}_2 = \Lambda^1_{21}\overrightarrow{e}_1 + \Lambda^3_{21}\overrightarrow{e}_3, \qquad d_1\overrightarrow{e}_3 = \Lambda^2_{31}\overrightarrow{e}_2 + \Lambda^4_{31}\overrightarrow{e}_4,$$

$$d_1\overrightarrow{e}_4 = \Lambda^3_{41}\overrightarrow{e}_3 + \Lambda^5_{41}\overrightarrow{e}_5, \qquad d_1\overrightarrow{e}_5 = \Lambda^4_{51}\overrightarrow{e}_4,$$

and

$$\Lambda^3_{11} = -\Lambda^3_{31} = 0, \Lambda^4_{11} = -\Lambda^4_{41} = 0, \Lambda^5_{11} = -\Lambda^5_{51} = 0. \tag{1.6}$$

$$\Lambda^5_{21} = -\Lambda^2_{51} = 0, \Lambda^4_{21} = -\Lambda^2_{41} = 0, \Lambda^5_{31} = -\Lambda^3_{51} = 0. \tag{1.7}$$

Here $k^1_1 = \Lambda^2_{11}, k^1_2 = \Lambda^3_{21}, k^1_3 = \Lambda^4_{31}, k^1_4 = \Lambda^5_{41} = -\Lambda^4_{51}$ - represent the first, second, third and fourth curvature of the line ω^1 respectively (where d_1 - denotes differentiation along the line ω^1).

A pseudofocus [3], [5] F^j_i (where $i \neq j$) of the tangent of the line ω^1 in the Frenet net Σ_5 is defined by the radius-vector:

$$\overrightarrow{F}^j_i = \overrightarrow{X} - \frac{1}{\Lambda^j_{ij}}\overrightarrow{e}_i = \overrightarrow{X} + \frac{1}{\Lambda^i_{jj}}\overrightarrow{e}_i. \tag{1.8}$$

There exist four pseudofocuses on each tangent $(X, \overrightarrow{e}_i)$:
on the straight line $(X, \overrightarrow{e}_1)$ - F^2_1, F^3_1, F^4_1, F^5_1;
on $(X, \overrightarrow{e}_2)$ - F^1_2, F^3_2, F^4_2, F^5_2;
on $(X, \overrightarrow{e}_3)$ - F^1_3, F^2_3, F^4_3, F^5_3;
on $(X, \overrightarrow{e}_4)$ - F^1_4, F^2_4, F^3_4, F^5_4;
on $(X, \overrightarrow{e}_5)$ - F^1_5, F^2_5, F^3_5, F^4_5.

The net Σ_5 in E_5 is called a cycle Frenet net [3] if the frames

$$\Re_1 = (X, \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5), \qquad \Re_2 = (X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5, \overrightarrow{e}_1), \qquad \Re_3 = (X, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5, \overrightarrow{e}_1, \overrightarrow{e}_2),$$

$$\Re_4 = (X, \overrightarrow{e}_4, \overrightarrow{e}_5, \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3), \qquad \Re_5 = (X, \overrightarrow{e}_5, \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4),$$

are Frenet frames for the lines $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ respectively, of the net Σ_5 simultaneously.

If the net Σ_5 be a cycle Frenet net, we denote it by $\widetilde{\Sigma}_5$. [3]. The pseudofocus $F^5_1 \in (X, \overrightarrow{e}_1)$ is defined by the radius-vector:

$$\overrightarrow{F}^5_1 = \overrightarrow{X} - \frac{1}{\Lambda^1_{15}}\overrightarrow{e}_1. \tag{1.9}$$

As the point X moves within the domain $\Omega \subset E_5$, the pseudofocus F^5_1 traces out its own domain $\Omega^5_1 \subset E_5$. Thus, we define the partial mapping $f^5_1 : \Omega \to \Omega^5_1$ such that $f^5_1(X) = F^5_1$. We associate the domain $\Omega^5_1 \subset E_5$ with the moving frame $\Re' = (F^5_1, \overrightarrow{b}_i)(i = \overline{1,5})$, where the vectors $\overrightarrow{b}_i$ have the form [4], [7]:

$$\overrightarrow{b}_1 = \overrightarrow{e}_1 + \frac{B^5_{151}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{11}}{\Lambda^5_{15}}\overrightarrow{e}_i;$$

$$\begin{aligned}
\overrightarrow{b}_2 &= \overrightarrow{e}_2 + \frac{B^5_{152}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{12}}{\Lambda^5_{15}}\overrightarrow{e}_i;\\
\overrightarrow{b}_3 &= \overrightarrow{e}_3 + \frac{B^5_{153}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{13}}{\Lambda^5_{15}}\overrightarrow{e}_i;\\
\overrightarrow{b}_4 &= \overrightarrow{e}_4 + \frac{B^5_{154}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{14}}{\Lambda^5_{15}}\overrightarrow{e}_i;\\
\overrightarrow{b}_5 &= \overrightarrow{e}_5 + \frac{B^5_{155}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{15}}{\Lambda^5_{15}}\overrightarrow{e}_i.
\end{aligned} \tag{1.10}$$

2. Setting and solving the problem

It is considered that the line γ, belongs to the distribution $\Delta_4 = (X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5)$.

The tangent vector $\overrightarrow{\gamma}$ of this line γ has the form: $\overrightarrow{\gamma} = \gamma^2\overrightarrow{e}_2 + \gamma^3\overrightarrow{e}_3 + \gamma^4\overrightarrow{e}_4 + \gamma^5\overrightarrow{e}_5$. We will find the tangent vector of the line

$$\overline{\gamma} = f^5_1(\gamma) : \overrightarrow{\overline{\gamma}} = \gamma^2\overrightarrow{b}_2 + \gamma^3\overrightarrow{b}_3 + \gamma^4\overrightarrow{b}_4 + \gamma^5\overrightarrow{b}_5.$$

Taking into account formulas (1.10) from here, we find:

$$\overrightarrow{\overline{\gamma}} = (\gamma^2 b^1_2 + \gamma^3 b^1_3 + \gamma^4 b^1_4 + \gamma^5 b^1_5)\overrightarrow{e}_1 + (\gamma^2 + \gamma^3 b^2_3 + \gamma^4 b^2_4 + \gamma^5 b^2_5)\overrightarrow{e}_2 +$$
$$+\gamma^3\overrightarrow{e}_3 + \gamma^4\overrightarrow{e}_4 + (\gamma^2 b^5_2 + \gamma^3 b^5_3 + \gamma^4 b^5_4)\overrightarrow{e}_5,$$

where b^j_i - j-th coordinate of the vector $\overrightarrow{b}_i$.

Definition 2.1. If line tangent of the line $\gamma \subset \Delta_4$ at the point X and the tangent vector of line $\overline{\gamma} = f^5_1(\gamma)$ at the point $F^5_1 = f^5_1(X)$ belong to the same four-dimensional space $(X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5)$, then the lines γ and $\overline{\gamma}$ are called quasi-double lines of the pair of distributions (Δ_4, Δ'_4) in the partial mapping f^5_1 (where $\Delta'_4 = f^5_1(\Delta_4)$).

From the condition $\overrightarrow{\gamma}, \overrightarrow{\overline{\gamma}} \in (X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5)$ we have:

$$\gamma^2 b^1_2 + \gamma^3 b^1_3 + \gamma^4 b^1_4 + \gamma^5 b^1_5 = 0.$$

Substituting the coordinates b^j_i from the (1.10) we get:

$$\gamma^2 B^5_{152} + \gamma^3 B^5_{153} + \gamma^4 B^5_{154} + \gamma^5 B^5_{155} = 0. \tag{2.1}$$

If the coordinates of the tangent vector $\overrightarrow{\gamma}$ of the line γ satisfy the conditions (2.1), then the lines $\gamma, \overline{\gamma}$ are quasi-double lines of the pair of distributions (Δ_4, Δ'_4) in the partial mapping f^5_1. Thus, the following theorem is proved.

Theorem 2.2. *Lines γ and $\overline{\gamma} = f^5_1(\gamma)$ are quasi-double lines of the pair of distributions (Δ_4, Δ'_4) in the partial mapping f^5_1 if and only if the coordinates of the tangent vector $\overrightarrow{\gamma}$ of the line γ satisfy the conditions (2.1).*

The pseudofocus $F^4_5 \in (X, \overrightarrow{e}_5)$ is defined by the radius-vector:

$$\overrightarrow{F}^4_5 = \overrightarrow{X} - \frac{1}{\Lambda^4_{54}}\overrightarrow{e}_5. \tag{2.2}$$

When the point X is moving in the domain $\Omega \subset E_5$, the pseudofocus F^4_5 describes its domain $\Omega^4_5 \subset E_5$. Thus, the partial mapping $f^4_5 : \Omega \to \Omega^4_5$ is defined such that: $f^4_5(X) = F^4_5$.

We will associate with $\Omega^4_5 \subset E_5$ the moving frame $\Re'' = (F^4_5, \overrightarrow{d}_i)$, $(i = \overline{1,5})$ [4], where:

$$\overrightarrow{d}_1 = \overrightarrow{e}_1 - \frac{\Lambda^4_{51}}{\Lambda^4_{54}}\overrightarrow{e}_4 - \frac{D^4_{541}}{(\Lambda^4_{54})^2}\overrightarrow{e}_5;$$

$$\overrightarrow{d}_2 = -\frac{\Lambda^1_{52}}{\Lambda^4_{54}}\overrightarrow{e}_1 + \overrightarrow{e}_2 - \frac{\Lambda^4_{52}}{\Lambda^4_{54}}\overrightarrow{e}_4 + \frac{D^4_{542}}{(\Lambda^4_{54})^2}\overrightarrow{e}_5;$$
$$\overrightarrow{d}_3 = -\frac{\Lambda^1_{53}}{\Lambda^4_{54}}\overrightarrow{e}_1 + \overrightarrow{e}_3 - \frac{\Lambda^4_{53}}{\Lambda^4_{54}}\overrightarrow{e}_4 + \frac{D^4_{543}}{(\Lambda^4_{54})^2}\overrightarrow{e}_5; \quad (2.3)$$
$$\overrightarrow{d}_4 = -\frac{\Lambda^1_{54}}{\Lambda^4_{54}}\overrightarrow{e}_1 + \frac{D^4_{544}}{(\Lambda^4_{54})^2}\overrightarrow{e}_5;$$
$$\overrightarrow{d}_5 = -\frac{\Lambda^1_{55}}{\Lambda^4_{54}}\overrightarrow{e}_1 + [1 + \frac{D^4_{545}}{(\Lambda^4_{54})^2}]\overrightarrow{e}_5.$$

In the work [4], necessary and sufficient conditions were found for the lines γ and $\overline{\overline{\gamma}} = f^4_5(\gamma)$ to be a quasi-double lines of the partial mapping f^4_5 (in which case these lines will automatically be quasi-double lines of the pair (Δ_4, Δ''_4) where $\Delta''_4 = f^4_5(\Delta)$):

$$\gamma^2\Lambda^1_{52} + \gamma^3\Lambda^1_{53} + \gamma^4\Lambda^1_{54} + \gamma^5\Lambda^1_{55} = 0. \quad (2.4)$$

From conditions (2.1) and (2.4) it follows that the line $\overline{\gamma}$ in the domain Ω^5_1 transforms into the line $\overline{\overline{\gamma}}$ in the domain Ω^4_5.

It is considered the composition $f^4_5 \circ (f^5_1)^{-1} : \Omega^5_1 \to \Omega^4_5$ of the partial mapping $(f^5_1)^{-1}$ and f^4_5, where $(f^5_1)^{-1}$ - is a reverse mapping such that $(f^5_1)^{-1}(F^5_1) = X \in \Omega$. Let $\overline{\gamma}$ is a line which tangent vectors coordinates satisfy the conditions (2.1). Then $(f^5_1)^{-1}(\overline{\gamma}) = \gamma$ and $f^4_5(\gamma) = \overline{\overline{\gamma}} \subset \Omega^4_5$, where $\overline{\overline{\gamma}}$ is a line which tangent vectors coordinates satisfy the conditions (2.4).

Thus, the following theorem is proven:

Theorem 2.3. *If the coordinates of the tangent vector of the line $\overline{\gamma} \subset \Omega^5_1$ satisty the condition (2.1) than $f^4_5 \circ (f^5_1)^{-1}(\overline{\gamma}) = \overline{\overline{\gamma}}$, where $\overline{\overline{\gamma}} \subset \Omega^4_5$ is line, which tangent vector's coordinates satisfy the condition (2.4).*

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