Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.169-180
DOI: 10.29229/uzmj.2025-4-17

Impact of almost η-Ricci-Bourguignon solitons on anti-invariant submanifolds of trans-Sasakian manifolds coupled with generalized symmetric non-metric connection of type (α, β)

Mishra R.K., Yadav S.K.

Abstract. We classify almost η-Ricci-Bourguignon solitons on anti-invariant submanifolds of trans-Sasakian manifolds admitting a generalized symmetric non-metric connection of type (α, β). Certain results of such solitons on submanifolds of trans-Sasakian manifolds with respect to a generalized symmetric non-metric connection (GSNM) of type (α, β) are obtained.

Keywords: trans-Sasakian manifold, anti-invariant submanifold, almost η-Ricci Bourguignon soliton, generalized symmetric non-metric connection of type (α, β).

MSC (2020): 53C05, 53C025, 53C40.

1. Introduction

The Ricci-Bourguignon soliton leads to an equivalent soliton to the Ricci-Bourguignon flow, which is explained by [1, 3, 8].

$$\frac{\partial g}{\partial t} = -2(S - \rho R g),\ g(0) = g_0, \tag{1.1}$$

where S is the Ricci curvature tensor, R is the scalar curvature w.r.t. g, and ρ is a real non-zero constant. It should be noticed that for special values of the constant ρ in equation (1.1), we obtain the following situations for the tensor $S - \rho R g$ appearing in the equation. The PDE system (1.1) defines the evolution equation of special interest if
(i) $\rho = \frac{1}{2}$, the Einstein tensor $S - \frac{R}{2} g$ (Einstein soliton),
(ii) $\rho = \frac{1}{2}$, the traceless Ricci tensor $S - \frac{R}{n} g$,
(iii) $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $S - \frac{R}{2(n-1)} g$ (Schouten soliton),
(iv) $\rho = 0$, the Ricci tensor S (Ricci soliton).
In fact, for small t, equation (1.1) has a unique solution for $\rho < \frac{1}{2(n-1)}$.
A pseudo-Riemannian manifold of dimension $n \geq 3$ is called the Ricci-Bourguignon soliton if

$$L_V g(X, Y) + 2S(X, Y) + 2(\mu + \rho r) g(X, Y) = 0, \tag{1.2}$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, the scalar curvature is denoted by r, S is the Ricci curvature tensor, and ρ, μ are scalars. A Ricci-Bourguignon soliton is called expanding if $\mu > 0$, steady if $\mu = 0$, and shrinking if $\mu < 0$.
From equation (1.2), we have defined a more general view, namely η-Ricci -Bourguignon soliton.

$$L_V g(X, Y) + 2S(X, Y) + 2(\mu + \rho r) g(X, Y) + 2C\eta(X)\eta(Y) = 0. \tag{1.3}$$

For more details, see ([19]-[21]).
Friedman and Schouten first introduced the concept of a semi-symmetric linear connection on a differentiable manifold in 1924 [5]. In 1930, the geometric significance of a semi-symmetric linear connection was given by Bartolotti [2]. In 1932, Hayden [7] was first introduced and investigated a metric connection known as a semi-symmetric metric connection with a non-zero torsion on a Riemannian manifold. Yano has conducted a detailed investigation of the semi-symmetric metric connection on a Riemannian manifold [25]. In 1975, Golab [6] was first introduced, a quarter-symmetric linear connection on a differentiable manifold. Rastogi [16] carried out a subsequent systematic investigation into the quarter-symmetric metric connection on a Riemannian manifold. The study of these connections was further studied by various authors ([12, 14, 18, 22]). If the torsion tensor T of a linear connection on a semi-Riemannian manifold M is said to be a generalized symmetric connection, then T is defined as

$$T(u_1, u_2) = \alpha\{(u_2)u_1 - \pi(u_1)u_2\} + \beta\{\pi(u_2)\varphi u_1 - \pi(u_1)\varphi u_2\}, \tag{1.4}$$

where u_1, u_2 are vector fields on M, α and β are smooth functions on M. Here, φ denotes a tensor of type (1,1) and π is a 1-form and satisfies $\pi(u_1) = g(u_1, v)$ for a vector field v in M. If $\nabla^G g = 0$, then the connection is called a generalized symmetric metric connection (shortly, GSM-connection) of type (α, β). The connection in equation (1.4) is referred to as a β-quarter-symmetric connection and α-semi-symmetric connection, respectively, if $\alpha = 0$ and $\beta = 0$. If $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, then the GSM-connection of type (α, β) is called a semi-symmetric connection and a symmetric connection, respectively. The concircular curvature tensor C of type (1,3) on a Riemannian manifold [23] is defined by the equation

$$C(X,Y)Z = R(X,Y)Z + \frac{r}{n(n-1)}\left[g(X,Z)Y - g(Y,Z)X\right], \tag{1.5}$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type (1,3) and r is the scalar curvature. Let us consider C^G be the concircular curvature tensor with respect to the generalized symmetric non-metric connection, then we have

$$C^G(X,Y)Z = R^G(X,Y)Z + \frac{r^G}{2n(2n+1)}[g(X,Z)Y - g(Y,Z)X], \tag{1.6}$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where R^G is called the concircular curvature tensor and r^G is said to be the scalar curvature with respect to the GSNM-connection. According to [10] and [13], the M-projective curvature tensor $\overline{M}$ of rank 3 on an n-dimensional manifold M is given by the following equation

$$\overline{M}(X,Y)Z = R(X,Y)Z + \frac{1}{2(n-1)}S(X,Z)Y - S(Y,Z)X + \frac{1}{2(n-1)}[g(X,Z)QY - g(Y,Z)QX],$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where Q denotes the Ricci operator. Hence, the M-projective curvature tensor with respect to the GSNM connection is given by

$$\overline{M}^G(X,Y)Z = R^G(X,Y)Z + \frac{1}{4n}[S^G(X,Z)Y - S^G(Y,Z)X] + \frac{1}{4n}[g(X,Z)Q^GY - g(Y,Z)Q^GX], \tag{1.7}$$

where Q^G is the Ricci operator with respect to the GSNM connection.
The pseudo-projective curvature tensor on a Riemannian manifold is given by [15].

$$P(X,Y)Z = AR(X,Y)Z + B[S(Y,Z)X - S(X,Z)Y] + cr[g(Y,Z)X - g(X,Z)Y]. \tag{1.8}$$

For all smooth vector fields $X, Y, Z \in \chi(M)$, where A, B, and c are non-zero constants related as

$$c = -\frac{1}{n}\left(\frac{A}{n-1} + B\right).$$

Now we consider P^G as the pseudo-projective curvature tensor with respect to the GSNM connection as

$$P^G(X,Y)Z = AR^G(X,Y)Z + B[S^G(Y,Z)X - S^G(X,Z)Y] + cr^G[g(Y,Z)X - g(X,Z)Y], \tag{1.9}$$

where A, B, c are non-zero constants related as

$$c = -\frac{1}{2n+1}\left(\frac{A}{2n} + B\right). \tag{1.10}$$

In 1977, K.Yano and M.Kon discussed anti-invariant submanifolds of Sasakian space forms [24]. In 1985, H.B Kumar studied anti-invariant submanifolds of almost paracontact manifolds [11]. In 2020, P. Karmakar and A. Bhattacharyya investigated anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds [9].
Let $\phi : M \to \widetilde{M}$ be a differentiable manifold, and let the dimension of the manifold $M, \widetilde{M}$ be n, m respectively. If at each point p of M, $(\phi^*)_p$ is a 1-1 map, that is, if rank $\phi = n$, then ϕ is called an immersion of M *into* M. f ϕ is 1-1, i.e., if $\phi(p') \neq \phi(q')$ for $p' \neq q'$ then ϕ is called an embedding of M into $\widetilde{M}$. The manifold M is called a submanifold of $\widetilde{M}$, if it satisfies the following conditions

(i) $M \subset \widetilde{M}$

(ii) The inclusion map i from M into $\widetilde{M}$ is an embedding of M into $\widetilde{M}$.

A submanifold M is called anti-invariant if $X \in T_x(M) \Rightarrow \phi X \in T_x^{\perp}(M), \forall\ x \in M$, where $T_x(M)$, $T_x^{\perp}(M)$ are respectively the tangent space and the normal space at $x \in M$. Thus, in an anti-invariant submanifold M, we have $\forall\ X, Y \in \chi(M)$,

$$g(X, \phi Y) = 0. \tag{1.11}$$

2. Preliminaries

Let $\widetilde{M}$ be a differentiable manifold of odd dimension with a metric structure (g, ϕ, ξ, η), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and a Riemannian metric g satisfying the following relations-

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta\ o\ \phi = 0,\ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y),\ \eta(X) = g(X, \xi)\ \ \forall X, Y \in \chi(\widetilde{M}). \tag{2.3}$$

An odd-dimensional almost contact metric manifold $\widetilde{M}(\phi, \xi, \eta, g)$ is called a trans-Sasakian manifold of type (p, q), where $p,\ q$ are smooth functions on $\widetilde{M}$ if $\forall\ X, Y \in \chi(M)$ [4].

$$(\nabla_X \phi)Y = p\,[g(X, Y)\xi - \eta(Y)X] + p\,[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

$$\nabla_X \xi = -\,p\,\phi X + q\,[X - \eta(X)\xi]. \tag{2.5}$$

In a $(2n+1)$-dimensional trans-Sasakian manifold of type (p, q), we have the following relations [4]

$$(\nabla_X \eta)\,Y = -p\,g(\phi X, Y) + q\,[g(X, Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$\begin{aligned} R(X, Y)\xi = {}& \left(p^2 - q^2\right)[\eta(Y)X - \eta(X)Y] + 2pq\,[\eta(Y)\phi X - \eta(X)\phi Y] \\ & + \left[(Yp)\phi X - (Xp)\phi Y + (Yq)\phi^2 X - (Xq)\phi^2 Y\right], \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(\xi, Y)X = {}& \left(p^2 - q^2\right)[g(X, Y)\xi - \eta(X)Y] + 2pq\,[g(\phi X, Y)\xi + \eta(X)\phi Y] \\ & +(Xp)\phi Y + g(\phi X, Y)(grad\ p) - g(\phi X, \phi Y)(grad\ q) + (Xq)\,[Y - \eta(Y)\xi], \end{aligned} \tag{2.8}$$

$$S(X, \xi) = \left[2n\left(p^2 - q^2\right) - \xi q\right]\eta(X) - (\phi X)p - (2n-1)(Xq), \tag{2.9}$$

$$Q\xi = \left[2n\left(p^2 - q^2\right) - \xi q\right]\xi + \phi(grad\ q) - (2n-1)(grad\ q). \tag{2.10}$$

Lemma 2.1. [17] *In a $(2n+1)$ dimensional trans-Sasakian manifold of type (p, q), if $\phi(grad\ p) = (2n-1)(grad\ q)$, then $\xi q = 0$.*

3. Generalized Symmetric non-metric connection in trans-Sasakian manifolds

In a trans-Sasakian manifold $\widetilde{M}$, let ∇^G be a linear connection, and ∇ be the Levi-Civita connection and $X, Y \in \chi(M)$. We define a linear connection ∇^G on $\widetilde{M}$ by

$$\nabla^G_X Y = \nabla_X Y + u(X, Y), \tag{3.1}$$

where $u(X, Y)$ is a tensor of type $(1, 2)$ and ∇^G represents a GSNM connection on a trans-Sasakian manifold as-

$$u(X, Y) = \frac{1}{2}[(T(X, Y) + T^{'}(X, Y) + T^{'}(Y, X)] \tag{3.2}$$

where T is the torsion tensor of ∇^G and

$$g(T^{'}(X, Y), Z) = g(T(Z, X), Y). \tag{3.3}$$

Plugging (1.4) in (3.3), we get

$$T^{'}(X,Y)=\alpha\left\{\eta(Y)X-g(X,Y)\xi\right\}+\beta\left\{\eta(Y)\phi X-g(\phi X,Y)\xi\right\} \tag{3.4}$$

Substituting (1.4) and (3.4) in (3.3), we get

$$u(X,Y)=\alpha\left\{\eta(Y)X-g(X,Y)\xi\right\}+\beta\eta(Y)\phi X \tag{3.5}$$

Substituting (3.5) in (3.1), we obtain a generalized symmetric non-metric connection ∇^{G} of type (α,β) in a trans-Sasakian manifold $\widetilde{M}$ as

$$\nabla^{G}_{X}Y=\nabla_{X}Y+\alpha\left\{\eta(Y)X-g(X,Y)\xi\right\}+\beta\eta(Y)\phi X \tag{3.6}$$

Conversely, from (3.7), the torsion tensor with respect to the connection ∇^{G} is defined as

$$T(X,Y)=\nabla^{G}_{X}Y-\nabla^{G}_{Y}X-[X,Y]=\alpha\left\{\eta(Y)X-\eta(X)Y\right\}+\beta\left\{\eta(Y)\phi X-\eta(X)\phi Y\right\}. \tag{3.7}$$

Hence, this shows that the connection ∇^{G} of type (α,β) in a trans-Sasakian manifold $\widetilde{M}$ a generalized symmetric connection. Now, we have

$$\left(\nabla^{G}_{X}g\right)(X,Y)=Xg(Y,Z)-g\left(\nabla^{G}_{X}Y,Z\right)-g\left(Y,\nabla^{G}_{X}Z\right)=-\beta\left\{\eta(Y)g(\phi X,Z)+\eta(Z)g(\phi X,Y)\right\}. \tag{3.8}$$

From (3.8) and (3.9), we obtained that ∇^{G} is a generalized symmetric non-metric connection of type (α,β) in a trans-Sasakian manifold $\widetilde{M}$.
Now, setting $Y=\xi$ in (3.7) and using (2.5), we obtain

$$\nabla^{G}_{X}\xi=(\alpha+q)\left\{X-\eta(x)\xi\right\}+(\beta-p)\phi X. \tag{3.9}$$

In a trans-Sasakian manifold $\widetilde{M}$ with respect to the generalized symmetric non-metric connection of type (α,β), we get

$$(\nabla^{G}_{X}\eta)Y=(\alpha+q)\left\{g(X,Y)-\eta(X)\eta(Y)\right\}-p\;g(Y,\phi X) \tag{3.10}$$

In a trans-Sasakian manifold $\widetilde{M}$, we define its curvature tensor with respect to the generalized symmetric non-metric connection of type (α,β) by

$$R^{G}(X,Y)Z=\nabla^{G}_{X}\nabla^{G}_{Y}Z-\nabla^{G}_{Y}\nabla^{G}_{X}Z-\nabla^{G}_{[X,Y]}Z. \tag{3.11}$$

Using (3.7), we obtain

$$\begin{aligned}\nabla^{G}_{X}\nabla^{G}_{Y}Z=\nabla^{G}_{X}\nabla_{Y}Z+\alpha\Big\{\Big(\;\nabla^{G}_{X}(\eta(Z)\Big)\,Y+\eta(Z)\nabla^{G}_{X}Y-(\nabla^{G}_{X}\left(g(Y,Z)\right)\xi-g(Y,Z)\nabla^{G}_{X}\xi\Big\}\\ +\beta\Big\{(\nabla^{G}_{X}\eta(Z))\phi Y+\eta(Z)\nabla^{G}_{X}(\phi Y)\Big\}.\end{aligned} \tag{3.12}$$

Now, using (3.10) & (3.7), we get

$$\nabla^{G}_{X}\left(\eta(Z)\right)=\eta\left(\nabla_{X}Z\right)-pg(Z,\phi X)+q\left\{g(X,Z)-\eta(X)\eta(Z)\right\}. \tag{3.13}$$

Using (2.1), (2.4) & (3.7), we obtain

$$\nabla^{G}_{X}(\phi Y)=\phi\nabla_{X}Y+p\left\{g(X,Y)\xi-\eta(Y)X\right\}+q\left\{g(\phi X,Y)\xi-\eta(Y)\phi X\right\}-\alpha\;g(X,\phi Y)\xi. \tag{3.14}$$

Now, applying (3.7), (3.12), (3.13), (3.14) in (3.11), we obtain

$$\begin{aligned}R^{G}(X,Y)Z=R(X,Y)Z&+\left(\alpha q-\beta p+\alpha^{2}\right)\left\{\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\right\}\\ &+\left(\alpha p-\beta q-\;\alpha\beta\right)\left\{g(Y,Z)\phi X-g(X,Z)\phi Y\right\}+\left(2\alpha q+\alpha^{2}\right)\left\{g(X,Z)Y-g(Y,Z)X\right\}\\ &+\alpha p\left\{g(Z,\phi Y)X-g(Z,\phi X)Y\right\}+\left(\alpha q+\alpha^{2}\right)\left\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\right\}\\ &+\beta p\left\{g(Z,\phi Y)\phi X-g(Z,\phi X)\phi Y\right\}+2(\alpha\beta+\beta q)\eta(Z)g(\phi X,Y)\xi\\ &+\alpha\beta\left\{\eta(Y)\eta(Z)\phi X-\eta(X)\eta(Z)\phi Y\right\}.\end{aligned} \tag{3.15}$$

$$S^G(Y,Z) = S(Y,Z) + \left\{2n\left(\alpha q - \beta p + \alpha^2\right) - \alpha q - \beta p - \alpha^2\right\}\eta(Y)\eta(Z) \\ + \left\{\alpha q + \beta p + \alpha^2 - 2n\left(2\alpha q + \alpha^2\right)\right\} g(Y,Z) + \left\{(2n-1)\alpha p + \beta q + \alpha\beta\right\} g(\phi Y, Z). \tag{3.16}$$

$$Q^G Y = QY + \left\{2n\left(\alpha q - \beta p + \alpha^2\right) - \alpha q - \beta p - \alpha^2\right\}\eta(Y)\xi \\ + \left\{\alpha q + \beta p + \alpha^2 - 2n\left(2\alpha q + \alpha^2\right)\right\} Y + \left\{(2n-1)\alpha p + \beta q + \alpha\beta\right\}\phi Y. \tag{3.17}$$

$$r^G = r - 8n^2\alpha q - 2n(2n-1)\alpha^2. \tag{3.18}$$

Setting $Z = \xi$ in (3.16), we get

$$S^G(Y,\xi) = S(Y,\xi) - 2n(\alpha q + \beta p)\eta(Y). \tag{3.19}$$

Again, putting $Y = \xi$ in (3.17), we obtain

$$Q^G\xi = Q\xi - 2n(\alpha q + \beta p)\xi. \tag{3.20}$$

Proposition 3.1. *A generalized symmetric non-metric connection of type (α, β) on an anti-invariant submanifold of a trans-sasakian manifold reduces to a generalized symmetric metric connection of type (α, β).*

4. Almost η-Ricci-Bourguignon solitons on anti-invariant submanifolds and trans-Sasakian manifold with respect to a generalized symmetric non-metric connection

Let, (g, ξ, C, μ, ρ) be an almost η-Ricci-Bourguignon soliton on M with respect to a generalized symmetric non-metric connection ∇^G, then from (1.2) we have $\forall\ Y, Z \in \chi(M)$

$$(L^G_\xi g)(Y,Z) + 2Ric^G(Y,Z) + 2\left(\mu + \rho\, r^G\right) g(Y,Z) \ + 2C\eta(Y)\eta(Z) = 0.$$
$$\Rightarrow g\left(\nabla^G_Y\xi, Z\right) + g\left(\nabla^G_Z\xi, Y\right) + 2Ric^G(Y,Z) + 2\left(\mu + \rho\, r^G\right) g(Y,Z) + 2C\eta(Y)\eta(Z) = 0.$$

Using (3.9) in the above equation, we get

$$Ric^G(Y,Z) = \left(\alpha + q - \mu - \rho\, r^G\right) g(Y,Z) + (\alpha + q - C)\eta(Y)\eta(Z). \tag{4.1}$$

Theorem 4.1. *Let (g, ξ, μ, ρ, C) be almost η-Ricci-Bourguignon solitons on a $(2n+1)$ dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p, q), with respect to the GSNM connection of type (α, β), then $\widetilde{M}^{2n+1}$ is η-Einstein manifold.*

Setting $Y = Z = \xi$ in (4.1), we get

$$Ric^G(\xi,\xi) = 2(\alpha + q) - \mu - \rho r^G - C. \tag{4.2}$$

Putting $X = \xi$ in (2.9), we obtain

$$S(\xi,\xi) = 2n\left(p^2 - q^2\right) - 2n(\xi q). \tag{4.3}$$

Now, comparing (4.2) and (4.3), we get

$$2(\alpha + q) - \mu - \rho r^G - C = 2n\left(p^2 - q^2\right) - 2n(\xi q).$$

If $\xi q = 0$, then from the above equation, we have

$$\mu = 2(\alpha + q) - \rho r^G - 2n\left(p^2 - q^2\right) - C.$$

In particular, if we take $\mu = 0,\ C = 0$, then, we yield

$$r^G = \frac{2}{\rho}\left\{\alpha + q - n\left(p^2 - q^2\right)\right\}. \tag{4.4}$$

Theorem 4.2. *If a $(2n+1)$-dimensional of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits almost Ricci-Bourguignon soliton with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r^G > \frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\ r^G = \frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\quad or\ \ r^G < \frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},$$

respectively.

Corollary 4.3. *If a $(2n+1)$-dimensional of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits almost Ricci-Bourguignon soliton with respect to the generalized symmetric metric connection of type $\alpha,\beta)$, provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r^G > \frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\ r^G = \frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\quad or\ \ r^G < \frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},$$

respectively.

Now, if $(\alpha,\beta)=(1,0)$, then from (4.4), we get

$$r^G = \frac{2}{\rho}\left\{1+q-n\left(p^2-q^2\right)\right\}.$$

Corollary 4.4. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits an almost Ricci-Bourguignon soliton with respect to the generalized symmetric metric connection of type (α,β), provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r^G > \frac{2}{\rho}\left\{1+q-n\left(p^2\ -\ q^2\right)\right\},\ r^G = \frac{2}{\rho}\left\{1+q-n\left(p^2\ -\ q^2\right)\right\},\quad or\ \ r^G < \frac{2}{\rho}\left\{1+q-n\left(p^2\ -\ q^2\right)\right\},$$

respectively.

Now, if $(\alpha,\beta)=(0,1)$, then from (4.4), we get

$$r^G = \frac{2}{\rho}\left\{q-n\left(p^2-q^2\right)\right\}.$$

Corollary 4.5. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits an almost Ricci-Bourguignon soliton with respect to the generalized quarter-symmetric metric connection of type (α,β), provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r^G > \frac{2}{\rho}\left\{q-n\left(p^2\ -\ q^2\right)\right\},\ r^G = \frac{2}{\rho}\left\{q-n\left(p^2\ -\ q^2\right)\right\},\quad or\ \ r^G < \frac{2}{\rho}\left\{q-n\left(p^2\ -\ q^2\right)\right\},$$

respectively.

Corollary 4.6. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits a traceless Ricci soliton with respect to the generalized symmetric metric connection of type (α,β), provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then the soliton is expanding, steady, or shrinking according as*

$$\begin{aligned}
&r^G > 2(2n+1)\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\\
&r^G = 2(2n+1)\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\ or\\
&r^G < 2(2n+1)\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\ respectively.
\end{aligned}$$

Corollary 4.7. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits the Schouten soliton with respect to the generalized symmetric metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then the soliton is expanding, steady, or shrinking according as*

$$r^G > 4n\left\{\alpha + q - n\left(p^2 - q^2\right)\right\},$$
$$r^G = 4n\left\{\alpha + q - n\left(p^2 - q^2\right)\right\}, \text{ or}$$
$$r^G < 4n\left\{\alpha + q - n\left(p^2 - q^2\right)\right\}, \text{ respectively.}$$

5. Almost η - Ricci-Bourguignon solitons on Ricci flat anti-invariant submanifolds

In this section, we characterize an almost η-Ricci-Bourguignon soliton on a Ricci flat $(2n+1)$ anti-invariant submanifold with respect to the generalized symmetric non-metric connection of type (α,β). Let (g,ξ,μ,ρ,C) be an almost η-Ricci-Bourguignon soliton on M, then from (1.3) we have $\forall\ Y,Z \in \chi(M)$.

$$L_\xi g(Y,Z) + 2S(Y,Z) + 2(\mu+\rho r)g(Y,Z) + 2C\eta(Y)\eta(Z) = 0$$
$$\Rightarrow g(\nabla_Y\xi,Z) + g(\nabla_Z\xi,Y) + 2S(Y,Z) + 2(\mu+\rho r)g(Y,Z) + 2C\eta(Y)\eta(Z) = 0.$$

Using (2.5), and then applying (1.11), in the above equation, we get

$$S(Y,Z) = (\mu+\rho r-q)g(Y,Z) + (C+q)\eta(Y)\eta(Z). \tag{5.1}$$

Setting $Z=\xi$ in (5.1), we obtain

$$S(Y,\xi) = (\mu+\rho r+C)\eta(Y). \tag{5.2}$$

Now, if $\xi q = 0$ and M is Ricci flat with respect to ∇^G, then from (3.16) we have

$$S(Y,Z) = \left\{\alpha q+\beta p+\alpha^2 - 2n\left(\alpha q-\beta p+\alpha^2\right)\right\}\eta(Y)\eta(Z)$$
$$- \left\{\alpha q+\beta p+\alpha^2 - 2n\left(2\alpha q+\alpha^2\right)\right\}g(Y,Z) - \left\{(2n-1)\alpha p+\beta q+\alpha\beta\right\}g(\phi Y,Z).$$

Using (1.11) in the above equation, we obtain

$$S(Y,Z) = \left\{\alpha q+\beta p+\alpha^2 - 2n\left(\alpha q-\beta p+\alpha^2\right)\right\}\eta(Y)\eta(Z) - \left\{\alpha q+\beta p+\alpha^2 - 2n\left(2\alpha q+\alpha^2\right)\right\}g(Y,Z).$$

Setting $Z=\xi$ in the above equation, we obtain

$$S(Y,\xi) = 2n(\alpha q+\beta p)\eta(Y). \tag{5.3}$$

Equating (5.2) and (5.3), we obtain

$$\mu = 2n(\alpha q+\beta p) - \rho r - C. \tag{5.4}$$

Now, if $\mu = 0$, $C=0$, then from (5.4), we get

$$r = \frac{2n}{\rho}(\alpha q+\beta p). \tag{5.5}$$

Theorem 5.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is Ricci flat with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{2n}{\rho}(\alpha q+\beta p),\ r = \frac{2n}{\rho}(\alpha q+\beta p),\ \text{or } r < \frac{2n}{\rho}(\alpha q+\beta p),\ \text{respectively.}$$

Now, from (5.5), if $(\alpha,\beta)=(0,1)$, and $(\alpha,\beta)=(1,0)$, then we get $r=\frac{1}{\rho}(2np)$ and $r=\frac{1}{\rho}(2nq)$, respectively. We get the following results.

Corollary 5.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is Ricci flat with respect to the generalized quarter- symmetric non-metric connection of type $(0,1)$, provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r>\frac{1}{\rho}(2np),\ r=\frac{1}{\rho}(2np),\ or\ r<\frac{1}{\rho}(2np),\ respectively.$$

Corollary 5.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is Ricci flat with respect to the generalized semi- symmetric metric connection of type $(1,0)$, provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r>\frac{1}{\rho}(2nq),\ r=\frac{1}{\rho}(2nq),\ or\ r<\frac{1}{\rho}(2nq),\ respectively.$$

6. Almost η-Ricci-Bourguignon solitons on concircularly flat anti-invariant submanifolds

In this section, we have discussed an almost η-Ricci-Bourguignon soliton with respect to the GSNM connection of type (α,β), α-semi-symmetric connection and β-quarter-symmetric connection on $(2n+1)$-dimensional concircularly flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q).
Since M is concircularly flat with respect to ∇^G, from (1.6), we have

$$R^G(X,Y)Z=\frac{r^G}{2n(2n+1)}\left[g(Y,Z)X-g(X,Z)Y\right].$$

Contracting the above equation, we get

$$S^G(Y,Z)=\frac{r^G}{2n+1}g(Y,Z). \tag{6.1}$$

Let $\xi q=0$, hence using (3.16), (3.18) and (1.11) in (6.1), we obtain

$$\begin{aligned}S(Y,Z)=&\left[\frac{r^G}{2n+1}-\left\{\alpha q+\beta p+\alpha^2-2n\left(2\alpha q+\alpha^2\right)\right\}\right]g(Y,Z)\\&-\left\{2n\left(\alpha q-\beta p+\alpha^2\right)-\alpha q-\beta p-\alpha^2\right\}\eta(Y)\eta(Z).\end{aligned} \tag{6.2}$$

Setting $Z=\xi$ in (6.2), we obtain

$$S(Y,\xi)=\left\{\frac{r^G}{2n+1}+2n(\alpha q+\beta p)\right\}\eta(Y). \tag{6.3}$$

Comparing (5.2) and (6.3), we get

$$\mu=\frac{r^G}{2n+1}+2n(\alpha q+\beta p)-\rho r-C. \tag{6.4}$$

Now, if $\mu=0$, $C=0$, then from (6.4), we get

$$r=\frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p)-2n(2n-1)\alpha^2-8n^2\alpha q\right]. \tag{6.5}$$

Theorem 6.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is concircularly flat with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \text{ respectively.}$$

Now, from (6.5), if $(\alpha,\beta) = (0,1)$, and $(\alpha,\beta) = (1,0)$, then we get $r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right]$, and $r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right]$. We get the following results-

Corollary 6.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is concircularly flat with respect to the generalized quarter-symmetric non-metric connection of type $(0,1)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \text{ respectively.}$$

Corollary 6.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is concircularly flat with respect to the generalized semi-symmetric metric connection of type $(1,0)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \text{ respectively.}$$

7. Almost η-Ricci-Bourguignon solitons on M-projectively flat anti-invariant submanifolds

This section deals with the study of almost η-Ricci-Bourguinon soliton with respect to the generalized symmetric non-metric connection of type (α,β), α-semi-symmetric connection, and β-quarter-symmetric connection on a (2n + 1)-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q), are investigated.
Since M is M-projectively flat with respect to a generalized symmetric non-metric connection ∇^G. From equation (1.7), we get

$$R^G(X,Y)Z = \frac{1}{4n}\left[S^G(Y,Z)X - S^G(X,Z)Y\right] + \frac{1}{4n}\left[g(Y,Z)Q^G X - g(X,Z)Q^G Y\right]. \tag{7.1}$$

Contracting (7.1) with respect to X, we get

$$S^G(Y,Z) = \frac{r^G}{2n+1}g(Y,Z).$$

Which is the same as equation (6.1). Hence, we get the following results:

Theorem 7.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is M-projectively flat with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \text{ respectively.}$$

Corollary 7.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is M-projectively flat with respect to the generalized quarter-symmetric non-metric connection of type $(0,1)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \text{ respectively.}$$

Corollary 7.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is M-projectively flat with respect to the generalized semi-symmetric metric connection of type $(1,0)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \text{ respectively.}$$

8. Almostη-Ricci-Bourguignon solitons on pseudo-projectively flat anti-invariant submanifolds

This section deals with the study of almost η-Ricci-Bourguignon soliton, α-semi-symmetric contraction and β-quarter-symmetric connection on a $(2n+1)$-dimensional pseudo-projectively flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) with respect to the generalized symmetric non-metric connection of type (α,β).
Let M be pseudo-projectively flat with respect to a generalized symmetric non-metric connection ∇^G. Now, from equation (1.9), we get

$$AR^G(X,Y)Z = -\,B\left[S^G(Y,Z)X - S^G(X,Z)Y\right] - c\, r^G\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{8.1}$$

Contracting (8.1) with respect to X, we have

$$(A+2nB)S^G(Y,Z) = -2cg(Y,Z). \tag{8.2}$$

Applying (1.10), (1.11), (3.16) and (3.18) in (8.2), we get

$$\begin{aligned} S(Y,Z) = &\left[\frac{r^G}{2n+1} - \left\{\alpha q + \beta p + \alpha^2 - 2n\left(2\alpha q + \alpha^2\right)\right\}\right] g(Y,Z) \\ &- \left\{2n\left(\alpha q - \beta p + \alpha^2\right) - \alpha q - \beta p - \alpha^2\right\}\eta(Y)\eta(Z). \end{aligned} \tag{8.3}$$

The above equation is the same as the equation (6.2). Hence, we get the following results:

Theorem 8.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is pseudo-projectively flat with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \text{ respectively.}$$

Corollary 8.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is pseudo-projectively flat with respect to the generalized quarter-symmetric non-metric connection of type $(0,1)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \text{ respectively.}$$

Corollary 8.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is pseudo-projectively flat with respect to the generalized semi-symmetric metric connection of type $(1,0)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \text{ respectively.}$$

9. Conclusion

In 1981, the notion of Ricci-Bourguignon flow as a generalization of Ricci flow [8] was introduced by J. P. Bourguignon [1], and the short-time existence and uniqueness of the solution of this geometric flow have been proved in [3]. In this study, we discuss the geometric properties of an almost η-Ricci-Bourguignon soliton on an anti-invariant submanifold of trans-Sasakian manifolds admitting a generalized symmetric non-metric connection of type (α,β), and we deduce several conditions at which the soliton is expanding, steady, and shrinking.

Acknowledgment

The authors would like to thank the anonymous reviewers and the Editor for their valuable suggestions to improve our manuscript.

References

[1] Bourguignon J. P.; Ricci curvature and Einstein metrics. Global differential geometry and global analysis. Lecture notes in Math.838(1981),42-63.

[2] Bartolotti E.; Sulla geometria della variata a connection affine. Ann. Mat. 1930, 4, 53101.

[3] Catino G., Cremaschi L., Djadli Z., Mantegazza C., Mazzieri L.; The Ricci-Bourguignon flow. Pacific J.Math. 287(2015). 337-370.

[4] De U. C., Shaikh A. A.; Complex Manifolds and Contact Manifolds, Narosa Publishing House, 2009.

[5] Friedmann A., Schouten J.A.; ber die Geometrie der halbsymmetrischen bertragungen. Math Z 21, 211223 (1924).

[6] Golab S.; On semi-symmetric and quarter-symmetric linear connection. Tensor 1975, 29, 249254.

[7] Hayden H.A.; Subspaces of space with torsion. Proc. Lond. Math. Soc. 1932, 34, 2750.

[8] Hamilton R.S.; The Ricci flow on surfaces, Contemp. Mathematics, 71, 237-261, (1988).

[9] Karmakar P., Bhattacharyya A.; Anti-invariant submanifolds of some indefinite almost contact and para-contact manifolds. Bull. Calcutta Math. Soc. 112, 2 (2020), 95-108.

[10] Ojha R. H.; M-projectively flat Sasakian manifolds, Indian J. Pure Appl. Math. 17, 4(1986), 481-484.

[11] Pandey H.B.; A.Kumar, Anti-invariant submanifolds of almost para-contact manifolds, Indian J. Pure Appl. Math. 20, 11 (1989), 1119-1125.

[12] Prakasha D.G.; On ϕ-symmetric Kenmotsu manifolds with regard to quarter-symmetric metric connection. Int. Electron. J. Geom. 2011, 4, 8896

[13] Prakasha D.G., Mirji K.; On the M-projective curvature tensor of a (k, μ)- contact metric manifold. Ser. Math. Inform. 32, 1 (2017), 117-128.

[14] Prakasha D.G., Vanli A.T., Bagewadi C.S., Patil, D.A.; Some classes of Kenmotsu manifolds with regard to semi-symmetric metric connection. Acta Math. Sin. Engl. Ser. 2013, 29, 13111322.

[15] Prasad B.; A pseudo projective curvature tensor on a Riemannian manifold, Bull. Calcutta Math. Soc. 94, 3 (2002), 163-166.

[16] Rastogi S.C.; On a quarter-symmetric metric connection. CR Acad. Sci. Bulg. 1978, 31, 811814.

[17] Shukla S. S., Singh D. D.; On (ϵ)-trans-Sasakian manifolds, Int. J. Math. Anal. 4, 49 (2010), 2401-2414.

[18] Sular S., Ozgur C., De U.C.; Quarter-symmetric metric connection in a Kenmotsu manifold. SUT J. Math. 2008, 44, 297306.

[19] Traore M., Tastan H.M., Gerdan Aydn S.; On almost η-Ricci-Bourguignon solitons, Miskolc. Math. Notes, 25(1), 493508, (2024).

[20] Traore M., Tastan H.M., On sequential warped product η-Ricci-Bourguignon solitons, Filomat. 38(19) 67856797, (2024).

[21] Traore M., Tastan H.M., Gerdan Aydn S.; Some characterizations on Gradient Almost η-Ricci-Bourguignon Solitons. Bol. Soc. Parana. Mat. 43. (2025). 1-12.

[22] Tripathi M.M.; On a semi-symmetric metric connection in a Kenmotsu manifold. J. Pure Math. 1999, 16, 6771.

[23] Yano K.; Concircular geometry I. concircular transformations, Proc.Imp. Acad. Tokyo 16, 6 (1940), 195-200.

[24] Yano K., Kon M.; Anti-invariant submanifolds of Sasakian space forms I, Tohoku Math. J. 29, 1 (1977), 9-23.

[25] Yano K.; On semi-symmetric metric connections, Revue Roumaine De Math. Pures Appl. 1970, 15, 15791586.

Mishra Rajesh Kumar,
Department of Applied Sciences and Humanities,
United College of Engineering and Research,
Naini, Prayagraj, INDIA
email: rrsimt.rajesh@gmail.com

Yadav Sunil Kumar,
Department of Applied Sciences and Humanities,
United College of Engineering and Research,
Naini, Prayagraj, INDIA
email: prof_sky16@yahoo.com