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Description of periodic and weakly periodic ground states for the modified SOS model on the Cayley tree

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Abstract. This paper investigates the periodic and weakly periodic ground states on the Cayley tree of order two and three. Utilizing techniques from statistical mechanics, algebraic graph theory, and group theory, we determine the conditions under which these ground states can exist. Our results demonstrate that the unique structure of the Cayley tree supports a variety of periodic and weakly periodic ground states, each with distinct physical properties. These findings enhance our understanding of phase transitions and critical phenomena in physical systems.

Keywords: Cayley tree, configuration, modified SOS model, translation-invariant ground state, periodic ground state, weakly periodic ground state.

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1. Introduction

It is considered that a phase diagram of the Gibbs measures for a Hamiltonian has been close to the phase diagram of isolated (stable) ground states of this Hamiltonian. A periodic ground state is compatible with a periodic Gibbs measure at the low temperatures, (see [5, 9, 11, 16, 17, 18]). It encourages us to investigate the problem of description of periodic and weakly periodic ground states. The notion of a weakly periodic ground state is introduced in [18].

For the Ising model with competing interactions, weakly periodic ground states are described in [9, 18]. The authors of [18] also explore weakly periodic ground states for the normal subgroups of index 2 and 4. For the Ising model with competing interactions in [17] ground states are described constructively on a Cayley tree of order $k \geq 1$. In [12] translation-invariant ground states for the Ising model with translation-invariant external field and some periodic ground states for the Ising model with periodic external field are described. In [11], ground states are investigated for the Ising model with competing interactions and a nonzero external field on the Cayley tree of order two.

In [4], for the three-state Potts model with competing interactions on the Cayley tree of order $k = 2$, all periodic ground states are studied. Weakly periodic ground states for the Potts model for the normal subgroups of index 2 are examined in [10, 13]. For the Potts model with competing interactions, such states for the normal subgroups of index 4 are explored in [14]. In [3], periodic ground states are studied for the Potts model with competing interactions and countable spin values on a Cayley tree of order three.

In [7], translation-invariant ground states for the solid-on-solid (SOS) model with a translation-invariant external field, as well as several periodic ground states for the SOS model with a periodic external field, are delineated. The authors of [8] examines periodic and weakly periodic ground states for the SOS model with competing interactions on Cayley trees of orders two and three. Additionally, periodic and weakly periodic ground states for the SOS model with competing interactions on a Cayley tree of order two are examined for the normal subgroups of index 2 in [1].

Modified models in statistical mechanics extend classical frameworks to capture richer behaviors on complex structures. A recent study by Akin [2] investigates a modified q-state Potts model on the Cayley tree, introducing cosine-modulated interactions that lead to phase transitions exclusively in the antiferromagnetic region which is a notable departure from classical models. Using the cavity method and recurrence relations, the work constructs Gibbs measures and analyzes phase transitions via fixed-point dynamics. These methods, rooted in the self-similarity of Cayley trees, align with foundational approaches by Rozikov [15], and contribute to the growing literature on modified models on graphs [2].

In this paper, we study periodic and weakly periodic ground states for the modified SOS model with competing interactions on the Cayley trees of order two and three. The main concepts are presented in the second section. In the third section, periodic and weakly periodic ground states are investigated on the Cayley tree of order two. Furthermore, in the fourth section, periodic and weakly periodic ground states are meticulously examined on the Cayley tree of order three.

2. Preliminaries

Let $\Gamma^k = (V, L)$ be the Cayley tree of order k, i.e. an infinite tree such that exactly $k+1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of Γ^k. Let G_k denote the free product of $k+1$ cyclic groups $\{e; a_i\}$ of order two with generators $a_1, a_2, a_3, ...a_{k+1}$, i.e., $a_i^2 = e$ (see [6, 15]).

There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order k and the group G_k, (see [6, 15]).

For an arbitrary vertex $x^0 \in V$, we put

$$W_n = \{x \in V : \ d(x, x^0) = n\}, \quad V_n = \{x \in V : \ d(x, x^0) \leq n\}, \tag{2.1}$$

where $d(x, y)$ is the distance between x and y on the Cayley tree, i.e., the number of edges of the path between x and y. For each $x \in G_k$, let $S(x)$ denote the set of direct successors of x, i.e., if $x \in W_n$ then

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x) = \{y \in G_k : \langle x, y\rangle \in L\}$. The set $S_1(x)\backslash S(x)$ is a singleton. Let $x_\downarrow$ denote the (unique) element of this set.

Let assume that the spin values belong to the set $\Phi = \{-1, 0, 1\}$. A function $\sigma : x \in V \to \sigma(x) \in \Phi$ is called configuration on V. The set of all configurations coincides with the set $\Omega = \Phi^V$.

Consider the quotient group $G_k/G_k^* = \{H_1, H_2, ..., H_r\}$, where G_k^* is a normal subgroup of index r with $r \geq 1$.

Definition 2.1. A configuration σ is called G_k^**-periodic*, if $\sigma(x) = \sigma_i$ for all $x \in G_k$ with $x \in H_i$. A G_k-periodic configuration is called translation-invariant.

The period of a periodic configuration is the index of the corresponding normal subgroup.

Definition 2.2. A configuration σ is called G_k^*-weakly periodic, if $\sigma(x) = \sigma_{ij}$ for all $x \in G_k$ with $x_\downarrow \in H_i$ and $x \in H_j$.

The Hamiltonian of the modified SOS model with competing interactions has a form:

$$H(\sigma) = J_1 \sum_{\langle x,y\rangle \in L} |\sigma(x) - \sigma(y)| \cos[\pi(\sigma(x) - \sigma(y))] + J_2 \sum_{\substack{x,y \in V:\\ d(x,y)=2}} |\sigma(x) - \sigma(y)| \cos[\pi(\sigma(x) - \sigma(y))], \tag{2.2}$$

where $J_1, J_2 \in \mathbb{R}$.

Unlike the classical SOS model, which depends solely on absolute spin differences $|\sigma(x) - \sigma(y)|$, our modified Hamiltonian includes an oscillatory factor $\cos\big(\pi(\sigma(x) - \sigma(y))\big)$, introducing parity-sensitive interactions. This modification promotes alternating spin configurations and reflects competing, antiferromagnetic-like behavior. As a result, the model enables the study of richer phase structures and boundary effects, especially on non-amenable graphs like the Cayley tree.

3. Ground States on the Cayley Tree of Order Two

In this section, we study periodic ground states corresponding to the normal subgroups of index two and four for the model (2.2). Moreover, H_A-weakly periodic ground states are examined. Note that H_A is a normal subgroup of index two in G_k (see [6]). For the classical SOS model, the ground states were studied in the papers [1, 7, 8].

Let M be the set of all unit balls with vertices in V, i.e. $M = \{\{x\} \cup S_1(x) : \forall x \in V\}$. A restriction of a configuration σ to the ball $b \in M$ is a bounded configuration and it is denoted by σ_b. We let c_b denote the center of the unit ball b.

We define the energy of the configuration σ_b on b by the following formula

$$U(\sigma_b) = \frac{1}{2} J_1 \sum_{x \in S_1(c_b)} |\sigma(x) - \sigma(c_b)| \cos[\pi(\sigma(x) - \sigma(c_b))] + J_2 \sum_{\substack{x,y \in b;\\ d(x,y)=2}} |\sigma(x) - \sigma(y)| \cos[\pi(\sigma(x) - \sigma(y))], \tag{3.1}$$

where $J_1, J_2 \in \mathbb{R}$.

We consider the case $k = 2$. It is easy to prove the following:

Lemma 3.1. *For all $b \in M$ and any $\sigma \in \Omega$ we have*

$$U(\sigma_b) \in \{U_0, U_1, U_2, ..., U_{11}\},$$

where

$$U_0 = 0, \quad U_1 = -\frac{3J_1}{2}, \quad U_2 = -2J_2, \quad U_3 = -J_1 - 2J_2,$$

$$U_4 = 3J_1, \quad U_5 = \frac{3J_1}{2} - 2J_2, \quad U_6 = 2J_1 + 4J_2, \quad U_7 = -\frac{J_1}{2} - 2J_2,$$

$$U_8 = J_1 + 4J_2, \quad U_9 = \frac{J_1}{2}, \quad U_{10} = -J_1, \quad U_{11} = -\frac{3J_1}{2} + 4J_2.$$

Definition 3.2. A configuration φ is called a ground state of the Hamiltonian (2.2), if

$$U(\varphi_b) = \min\{U_0, U_1, U_2, ..., U_{11}\},$$

for all $b \in M$.

For a fixed $m = 0, 1, 2, ..., 11$, we set

$$A_m = \{(J_1, J_2) \in \mathbb{R}^2 : U_m = \min\{U_0, U_1, U_2, ..., U_{11}\}\}, \tag{3.2}$$

and we find the sets:

$$A_0 = A_9 = A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 = 0\},$$

$$A_1 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad 0 \leq J_2 \leq \frac{J_1}{4}\},$$

$$A_2 = A_7 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 \geq 0\},$$

$$A_3 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 \geq \frac{J_1}{4}\},$$

$$A_4 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad \frac{J_1}{4} \leq J_2 \leq -\frac{3J_1}{4}\},$$

$$A_5 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad J_2 \geq \frac{-3J_1}{4}\},$$

$$A_6 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad J_2 \leq \frac{J_1}{4}\},$$

$$A_8 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 \leq 0\},$$

$$A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 \leq 0\}$$

and $\bigcup_{m=0}^{11} A_m = \mathbb{R}^2$.

We put

$$C_i = \{\sigma_b : U(\sigma_b) = U_i\}, i = 0, ..., 11$$

and

$$B^{(t)} = | \{x \in S_1(c_b) : \varphi_b(x) = t\} |,$$

for $t \in \Phi$.

The following theorem describes translation-invariant ground states.

Theorem 3.3. *If $(J_1, J_2) \notin A_0$, then there is no translation-invariant ground state for the model (2.2) on the Cayley tree of order two.*

Proof. Let $\sigma(x) = l, l \in \Phi$, $\forall x \in V$. According to (3.1), we have $U_0 = 0$, $\forall b \in M$ and then we can easily see from (3.2) that

$$A_0 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 = 0\}.$$

This finishes the proof of Theorem 3.3. □

Remark 3.4. If $\cos[\pi(\sigma(x) - \sigma(c_b))] = -1$, where $x \in S_1(c_b)$ and $\cos[\pi(\sigma(x) - \sigma(y))] = -1$, where $x, y \in b; d(x, y) = 2$, then Hamiltonian (2.2) coincides with Hamiltonian of SOS model with competing interactions which is studied in (see [1, 8]).

3.1. H_A**-Periodic Ground States.** Let $A \subset \{1, 2, 3\}$ and $H_A = \{x \in G_2 : \sum_{i \in A} w_x(a_i) - \text{even}\}$, where $w_x(a_i)$ is the number of a_i in the word x.

Note that H_A is a normal subgroup of index two in G_k (see [6]). Let $G_2/H_A = \{H_A, G_2 \setminus H_A\}$ be a quotient group. We set $H_1 := H_A, H_2 := G_2 \setminus H_A$.

We shall study H_A-periodic ground states. Note that each H_A-periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, \text{ if } x \in H_1, \\ \sigma_2, \text{ if } x \in H_2, \end{cases} \tag{3.3}$$

where $\sigma_i \in \Phi = \{-1, 0, 1\}, i = 1, 2$.

In the sequel, we write (σ_1, σ_2) for such an H_A-periodic configuration (3.3).

Theorem 3.5. *Let $|A| = p, p \in \{1, 2, 3\}$. For the modified SOS model with competing interactions given by (2.2) the following statements hold on the Cayley tree of order two:*
a) If $(J_1, J_2) \in A_{p^2-7p+13}$ and $|\sigma_1 - \sigma_2| = 1$, then the number of H_A-periodic ground states is 4 and in the forms $(1, 0), (0, 1), (-1, 0), (0, -1)$.
b) If $(J_1, J_2) \in A_{10-2p}$ and $|\sigma_1 - \sigma_2| = 2$, then the number of H_A-periodic ground states is 2 and in the forms $(1, -1), (-1, 1)$.

Proof. We prove the theorem for $p = 1$.
a) Let us consider the following configuration

$$\varphi(x) = \begin{cases} m, \text{ if } \quad x \in H_1, \\ j, \text{ if } \quad x \in H_2, \end{cases}$$

where $|m - j| = 1, m, j \in \Phi$.
1) Assume that $c_b \in H_1$

$$\varphi_b(c_b) = m, B^{(m)} = 2, B^{(j)} = 1.$$

Consequently, $\varphi_b \in C_7$.
2) Let $c_b \in H_2$, then one has

$$\varphi_b(c_b) = j, B^{(j)} = 2, B^{(m)} = 1.$$

Consequently, $\varphi_b \in C_7$.
We conclude that, if $(J_1, J_2) \in A_7$ then the periodic configuration φ is an H_A-periodic ground state.
b) Let us consider the following configuration

$$\varphi(x) = \begin{cases} m, \text{ if } \quad x \in H_1, \\ j, \text{ if } \quad x \in H_2, \end{cases}$$

where $|m - j| = 2$.
1) Assume that $c_b \in H_1$

$$\varphi_b(c_b) = m, B^{(m)} = 2, B^{(j)} = 1.$$

Consequently, $\varphi_b \in C_8$.
2) Let $c_b \in H_2$, then one has

$$\varphi_b(c_b) = j, B^{(j)} = 2, B^{(m)} = 1.$$

Consequently, $\varphi_b \in C_8$.
We conclude that, if $(J_1, J_2) \in A_8$ then the periodic configuration φ is an H_A-periodic ground state. □

Other cases are proved similar. This finishes the proof of Theorem 3.5.

Remark 3.6. If $|A| = p, p \in \{1, 3\}$ and $|\sigma_1 - \sigma_2| = 1$, the number of H_A-periodic ground states obtained in the Theorem 3.5 is equal to the Theorem 2 of [1]. However, the sets containing these ground states differ. If $|A| = 2$ and $|\sigma_1 - \sigma_2| = 1$, the paper [1] does not have the H_A-periodic ground state, but it is found that the H_A-periodic ground state exists by contrast. Furthermore, if $|A| = p, p \in \{1, 2, 3\}$ and $|\sigma_1 - \sigma_2| = 2$, the number of H_A-periodic ground states obtained in Theorem 3.5 is equal to the Theorem 2 of [1]. However, the sets containing these ground states differ. The H_A-periodic ground state in $p = 3$ is $G_2^{(2)}$-periodic ground state, where $G_2^{(2)} = \{x \in G_2 : |x| - \text{even}\}$. If $|\sigma_1 - \sigma_2| = 0$, then H_A-periodic ground state is translation-invariant ground state. Furthermore, the differences in the translation-invariant ground states studied in Theorem 3.3 and the paper [1] shows that the SOS model does not overlap with model (2.2).

3.2. **$G_2^{(4)}$-Periodic Ground States.** Let $A \subset \{1, 2, 3\}$, $G_2^{(4)} = \{x \in G_2 : \sum_{j \in A} w_j(x) - \text{even}, |x| - \text{even}\}$.

Note that $G_2^{(4)}$ is a normal subgroup of index four in G_2 (see[6]). In fact, there are other forms of normal subgroup of index four as well. But for the sake of convenience we consider the following quotient group $G_2/G_2^{(4)} = \{H_1, H_2, H_3, H_4\}$, where

$$H_1 := G_2^{(4)},$$

$$H_2 := \{x \in G_2 : \sum_{j \in A} w_j(x) - \text{even}, |x| - \text{odd}\},$$

$$H_3 := \{x \in G_2 : \sum_{j \in A} w_j(x) - \text{odd}, |x| - \text{even}\},$$

$$H_4 := \{x \in G_2 : \sum_{j \in A} w_j(x) - \text{odd}, |x| - \text{odd}\}.$$

We study $G_2^{(4)}$-periodic ground states. We note that each $G_2^{(4)}$-periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, \text{ if } x \in H_1, \\ \sigma_2, \text{ if } x \in H_2, \\ \sigma_3, \text{ if } x \in H_3, \\ \sigma_4, \text{ if } x \in H_4, \end{cases} \tag{3.4}$$

where $\sigma_i \in \Phi = \{-1, 0, 1\}, i = 1, 2, 3, 4$.

In the sequel, we write $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ for such a $G_2^{(4)}$-periodic configuration (3.4).

Theorem 3.7. *For the modified SOS model with competing interactions given by (2.2) the following statements holds on a Cayley tree of order two:*
1. Let $|A| = 1$.
1.*a*) *If* $(J_1, J_2) \in A_1 \cap A_{11}$ *and* $|\sigma_1 - \sigma_2| = 1$, *then the configuration* $\varphi = (\sigma_1, \sigma_2, -\sigma_1, -\sigma_2)$ *is* $G_2^{(4)}$-*periodic ground state.They have the following forms* $(\pm 1, 0, \mp 1, 0)$ *and* $(0, \pm 1, 0, \mp 1)$.
1.*b*) *If* $(J_1, J_2) \in A_2 \cap A_3$ *and* $|\sigma_1 - \sigma_2| = 1$, *then the configuration* $\varphi = (\sigma_1, \sigma_2, -\sigma_2, -\sigma_1)$ *is* $G_2^{(4)}$-*periodic ground state. They have the following forms* $(\pm 1, 0, 0, \mp 1)$ *and* $(0, \pm 1, \mp 1, 0)$.
1.*c*) *If* $(J_1, J_2) \in A_5 \cap A_7$ *and* $|\sigma_1 - \sigma_2| = 1$, *then the configuration* $\varphi = (\sigma_1, -\sigma_1, \sigma_2, -\sigma_2)$ *is* $G_2^{(4)}$-*periodic ground state. They have the following forms* $(\pm 1, \mp 1, 0, 0)$ *and* $(0, 0, \pm 1, \mp 1)$.
1.*d*) *Other* $G_2^{(4)}$-*periodic ground states correspond to* H_A-*periodic ground states.*

2. Let $|A| = 2$.
2.a) If $(J_1, J_2) \in A_2 \cap A_3$ and $|\sigma_1 - \sigma_2| = 1$, then the configuration $\varphi = (\sigma_1, -\sigma_1, \sigma_2, -\sigma_2)$ is $G_2^{(4)}$-periodic ground state. They have the following forms $(\pm 1, \mp 1, 0, 0)$ and $(0, 0, \pm 1, \mp 1)$.
2.b) If $(J_1, J_2) \in A_5 \cap A_7$ and $|\sigma_1 - \sigma_2| = 1$, then the configuration $\varphi = (\sigma_1, \sigma_2, -\sigma_2, -\sigma_1)$ is $G_2^{(4)}$-periodic ground state. They have the following forms $(\pm 1, 0, 0, \mp 1)$ and $(0, \pm 1, \mp 1, 0)$.
2.c) Other $G_2^{(4)}$-periodic ground states correspond to H_A-periodic ground states.
3. Let $|A| = 3$.
3.a) All $G_2^{(4)}$-periodic ground states correspond to $G_2^{(2)}$-periodic ground states.

Proof. 1.*a*) Let us consider the following configuration

$$\varphi_1(x) = \begin{cases} 1, \text{ if } & x \in H_1, \\ 0, \text{ if } & x \in H_2, \\ -1, \text{ if } & x \in H_3, \\ 0, \text{ if } & x \in H_4. \end{cases}$$

1) Assume that $c_b \in H_1$

$$\varphi_b(c_b) = 1, B^{(1)} = 0, B^{(-1)} = 0, B^{(0)} = 3.$$

Consequently, $\varphi_b \in C_1$.
2) Let $c_b \in H_2$, then one has

$$\varphi_b(c_b) = 0, B^{(0)} = 0, B^{(1)} = 2, B^{(-1)} = 1.$$

Consequently, $\varphi_b \in C_{11}$.
3) Let $c_b \in H_3$, then one has

$$\varphi_b(c_b) = -1, B^{(-1)} = 0, B^{(1)} = 0, B^{(0)} = 3.$$

Consequently, $\varphi_b \in C_1$.
4) Let $c_b \in H_4$, then one has

$$\varphi_b(c_b) = 0, B^{(0)} = 0, B^{(-1)} = 2, B^{(1)} = 1.$$

Consequently, $\varphi_b \in C_{11}$.
We conclude that, if $(J_1, J_2) \in A_1 \cap A_{11}$, then corresponding periodic configuration φ_1 is a $G_2^{(4)}$-periodic ground state. Other cases are proved similarly.
b) and *c*) are proved similar to *a*). This finishes the proof of Theorem 3.7. □

Remark 3.8. If $|\sigma_1 - \sigma_2| = 1$ and (J_1, J_2) takes a value different from the sets in the above Theorem 3.7, the $G_2^{(4)}$-periodic ground states coincide with the case a) of Theorem 3.5. If $|\sigma_1 - \sigma_2| \neq 1$, the $G_2^{(4)}$-periodic ground states coincide with H_A-periodic ground states which is studied the case b) of Theorem 3.5.

3.3. **Weakly Periodic Ground States.** Weakly periodic ground states have been studied and shown to be exist for other models, including the Ising and Potts models. For the Ising model with competing interactions, weakly periodic ground states are described in [9, 18]. In [18], the authors also explore weakly periodic ground states for the normal subgroups of index 2 and 4. Such states for the Potts model with competing interactions for the normal subgroups of index 2 are examined in [10, 13]. For this model weakly periodic ground states for the normal subgroups of index 4 are explored in [14]. It should be noted that the set of all weakly periodic ground states include all periodic ground states. In the above works, non-periodic weakly periodic ground states were found for the considered models. Therefore, the question arises whether there are non-periodic weakly periodic ground states for model (2.2) as well. The answer to this question is given in this subsection.

Let $A \in \{1, 2, 3\}$. In this subsection, we describe H_A-weakly periodic ground states, where H_A is a normal subgroup of index two in G_2 (see [6]). Let $G_2/H_A = \{H_A, G_2 \setminus H_A\}$ be the quotient group.

We set $H_1 := H_A, H_2 := G_2 \setminus H_A$. Due to the Definition 2.2, we infer that each H_A-weakly periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_{11}, \text{ if } x_\downarrow \in H_1, \ x \in H_1, \\ \sigma_{12}, \text{ if } x_\downarrow \in H_1, \ x \in H_2, \\ \sigma_{21}, \text{ if } x_\downarrow \in H_2, \ x \in H_1, \\ \sigma_{22}, \text{ if } x_\downarrow \in H_2, \ x \in H_2, \end{cases} \tag{3.5}$$

where $\sigma_{ij} \in \Phi, i, j = 1, 2$.

In the sequel, we write $\sigma = (\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22})$ for such a weakly periodic configuration $\sigma(x), x \in G_k$.

Theorem 3.9. *Let $k = 2$ and $|A| = j, j = 1, 2$. If $(J_1, J_2) \notin A_0$, then for the modified SOS model (2.2) there is no H_A-weakly periodic (non periodic) ground state.*

Proof. Let $|A| = 1$. If $\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22}$, then corresponding configurations are translation-invariant. Translation-invariant ground states for this case are studied in Theorem 3.3. It is easy to see that in the case $\sigma_{11} = \sigma_{21}$ and $\sigma_{12} = \sigma_{22}$ the H_A-weakly periodic configuration (3.5) are periodic configuration which are studied in Theorem 3.5.

Now we consider the cases $\sigma_{11} \neq \sigma_{21}$ or $\sigma_{12} \neq \sigma_{22}$.
Let

$$\varphi(x) = \begin{cases} -1, & x_\downarrow \in H_1, \ x \in H_1, \\ 0, & x_\downarrow \in H_1, \ x \in H_2, \\ 0, & x_\downarrow \in H_2, \ x \in H_1, \\ 1, & x_\downarrow \in H_2, \ x \in H_2. \end{cases}$$

Let $c_b \in H_1$, then we have the following possible cases:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 1$, $B^{(0)} = 2$, $B^{(1)} = 0$, $\varphi_b \in C_3$.
b) if $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 2$, $B^{(0)} = 1$, $B^{(1)} = 0$, $\varphi_b \in C_7$.
c) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 0$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 1$, $\varphi_b \in C_{11}$.
Let $c_b \in H_2$, then we have the following possible cases:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = 0$, $B^{(-1)} = 1$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{11}$.
b) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 0$, $B^{(0)} = 2$, $B^{(1)} = 1$, $\varphi_b \in C_3$.
c) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 0$, $B^{(0)} = 1$, $B^{(1)} = 2$, $\varphi_b \in C_7$.
We conclude that the configuration φ is a ground state if

$$(J_1, J_2) \in A_3 \cap A_7 \cap A_{11} = \{(J_1, J_2) \in R^2 : J_1 = J_2 = 0\}.$$

Therefore, if $J_1 \neq 0$ and $J_2 \neq 0$ then the weakly periodic configuration φ is not a weakly periodic ground state. All possible configurations can be checked similarly.
By similar way we can prove that all H_A-weakly periodic (non periodic) configurations are not ground states. This finishes the proof of Theorem 3.9. □

Remark 3.10. Due to computational complexity, weakly periodic ground states for the normal subgroups of index 4 have not been considered. Furthermore, all H_A-weakly periodic ground states correspond to H_A-periodic ground state. If $|A| = 1$, H_A-weakly periodic ground states are explored in the [1]. All weakly periodic ground states for the SOS model with competing interactions found to be H_A-periodic ground states but not translation-invariant.

4. Ground States on the Cayley Tree of Order Three

In this section, we are going to continue investigation related to periodic ground states corresponding to the normal subgroups of index two and four for the modified SOS model on the Cayley tree of order three. Moreover, H_A-weakly periodic ground states are examined. Let $k = 3$. It is easy to prove the following lemma.

Lemma 4.1. *For all $b \in M$ and any $\sigma \in \Omega$ we have*

$$U(\sigma_b) \in \{U_0, U_1, U_2, ..., U_{17}\},$$

where

$$U_0 = 0, \quad U_1 = -\frac{J_1}{2} - 3J_2, \quad U_2 = -2J_1, \quad U_3 = -2J_1 + 6J_2,$$

$$U_4 = -\frac{3J_1}{2} - 3J_2, \quad U_5 = -J_1 - 2J_2, \quad U_6 = -J_1 - 4J_2, \quad U_7 = -2J_1 + 8J_2,$$

$$U_8 = J_1 + 6J_2, \quad U_9 = \frac{5J_1}{2} - 3J_2, \quad U_{10} = 3J_1 + 6J_2, \quad U_{11} = 4J_1,$$

$$U_{12} = -2J_2, \quad U_{13} = -\frac{3J_1}{2} + J_2, \quad U_{14} = \frac{J_1}{2} + J_2,$$

$$U_{15} = -\frac{3J_1}{2} + J_2, \quad U_{16} = J_1 - 4J_2, \quad U_{17} = 2J_1 + 8J_2.$$

Definition 4.2. A configuration φ is called a ground state for the Hamiltonian (2.2), if $U(\varphi_b) = \min\{U_0, U_1, U_2, ..., U_{17}\}$, for all $b \in M$.

For a fixed $m = 0, 1, 2, ..., 17$, we set

$$A_m = \{(J_1, J_2) \in \mathbb{R}^2 : U_m = \min\{U_0, U_1, U_2, ..., U_{17}\}\}. \tag{4.1}$$

It is easy to check that

$$A_0 = A_1 = A_5 = A_8 = A_{12} = A_{13} = A_{14} = A_{15} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 = 0\},$$

$$A_2 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad 0 \leq J_2 \leq \frac{J_1}{6}\},$$

$$A_3 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 = 0\},$$

$$A_4 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad \frac{J_1}{6} \leq J_2 \leq \frac{J_2}{2}, J_2 \geq 0\},$$

$$A_6 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 \geq \frac{J_1}{10}\},$$

$$A_7 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 \leq 0\},$$

$$A_9 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad -\frac{J_1}{2} \leq J_2 \leq -\frac{3J_1}{2}\},$$

$$A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad \frac{5J_1}{2} \leq J_2 \leq \frac{J_1}{6}\},$$

$$A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad \frac{J_1}{6} \leq J_2 \leq -\frac{J_1}{2}\},$$

$$A_{16} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad J_2 \geq 0, \quad J_2 \geq -\frac{3J_1}{2}\},$$

$$A_{17} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad J_2 \leq \frac{J_1}{2}\}$$

and $\bigcup_{m=0}^{17} A_m = \mathbb{R}^2$.

4.1. H_A**-Periodic Ground States.** Let $A \subset \{1,2,3,4\}$, $H_A = \{x \in G_3 : \sum_{j \in A} w_j(x) - \text{even}\}$, where $w_j(x)$ is the number of letters a_j in the word x. It is obvious that H_A is a normal subgroup of index two. Let $G_3/H_A = \{H_A, G_3 \setminus H_A\}$ be the quotient group. We set $H_1 := H_A, H_2 := G_3 \setminus H_A$.

We study H_A-periodic ground states. Note that each H_A-periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, \text{ if } x \in H_1, \\ \sigma_2, \text{ if } x \in H_2, \end{cases} \tag{4.2}$$

where $\sigma_i \in \Phi = \{-1, 0, 1\}, i = 1, 2$.

Theorem 4.3. *If $(J_1, J_2) \notin A_0$, then there is no translation-invariant ground state for the model (2.2) on the Cayley tree of order three.*

Proof. Let $\sigma(x) = l, l \in \Phi$, $\forall x \in V$. According to (3.1), we have $U_0 = 0$, $\forall b \in M$ and then we can see easily from (4.1) that

$$A_0 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 = 0\}.$$

This finishes the proof of Theorem 4.3. □

In the sequel, we write (σ_1, σ_2) for such an H_A-periodic configuration (4.2).

Theorem 4.4. *Let $|A| = p, p \in \{2,3,4\}$. For the modified SOS model with competing interactions given by (2.2) the following statements hold on the Cayley tree of order three:*
a) If $(J_1, J_2) \in A_{10-2p}$ and $|\sigma_1 - \sigma_2| = 1$, then the number of H_A-periodic ground states is 4 and in the forms $(1,0), (0,1), (-1,0), (0,-1)$.
b) If $(J_1, J_2) \in A_{4p^2-27p+55}$ and $|\sigma_1 - \sigma_2| = 2$, then the number of H_A-periodic ground states is 2 and in the forms $(1,-1), (-1,1)$.

Proof. Let $p = 2$. Now we consider the following configuration

$$\varphi(x) = \begin{cases} m, \text{ if } x \in H_1, \\ j, \text{ if } x \in H_2, \end{cases}$$

where $|m - j| = 1$ or $|m - j| = 2, m, j \in \Phi$.
a) Let $|m - j| = 1$.
1) Assume that $c_b \in H_1$, then

$$\varphi_b(c_b) = m, B^{(m)} = 2, B^{(j)} = 2.$$

Hence, $\varphi_b \in C_6$.
2) Assume that $c_b \in H_2$, then

$$\varphi_b(c_b) = j, B^{(j)} = 2, B^{(m)} = 2.$$

Hence, $\varphi_b \in C_6$.
We conclude that if $(J_1, J_2) \in A_6$, then the periodic configuration φ is an H_A-periodic ground state.
b) Let $|m - j| = 2$.
1) Assume that $c_b \in H_1$, then

$$\varphi_b(c_b) = m, B^{(m)} = 2, B^{(j)} = 2.$$

Hence, $\varphi_b \in C_{17}$.
2) Assume that $c_b \in H_2$, then

$$\varphi_b(c_b) = j, B^{(j)} = 2, B^{(m)} = 2.$$

Hence, $\varphi_b \in C_{17}$.
We conclude that if $(J_1, J_2) \in A_{17}$, then the periodic configuration φ is an H_A-periodic ground state.
By the similar way, we can prove easily other cases of this theorem. This finishes the proof of the Theorem 4.4. □

Remark 4.5. It is known from [8] there exist the H_A-periodic ground states for the SOS model with competing interactions when $|A|=1$. By contrast, by Theorem 4.4 the H_A-periodic ground state does not exist when $|A|=1$ for modified SOS model with competing interactions. H_A-periodic ground state does not exist in [8] when $|A|=p, p\in\{2,3\}$ and $|\sigma_1-\sigma_2|=1$ but the existence of H_A-periodic ground states in the Theorem 4.4 is proved. If $|A|=4$ and $|\sigma_1-\sigma_2|=1$, the number of H_A-periodic ground states obtained in the Theorem 4.4 is similar to the Theorem 2.2 of [8], but the sets containing the ground states are different. In addition, if $|A|=p, p\in\{2,3,4\}$ and $|\sigma_1-\sigma_2|=2$ too, the number of H_A-periodic ground states obtained in the Theorem 4.4 is similar to the Theorem 2.2 of [8], but the sets containing the ground state are different. The H_A-periodic ground state in $p=4$ is $G_3^{(2)}$-periodic ground state which is studied in Theorem 4.4, where $G_3^{(2)}=\{x\in G_3: |x|-\text{even}\}$. If $|\sigma_1-\sigma_2|=0$, then H_A-periodic ground state is translation-invariant ground state but there is no translation-invariant ground state due to Theorem 4.3.

4.2. $G_3^{(4)}$**-Periodic Ground States.** Let $A\subset\{1,2,3,4\}$, $G_3^{(4)}=\{x\in G_3: \sum\limits_{j\in A} w_j(x)-\text{even}, |x|-\text{even}\}$. Note that $G_3^{(4)}$ is a normal subgroup of index four in G_3 (see [6]). In fact, there are other forms of normal subgroup of index four as well. But for the sake of convenience we consider the following quotient group $G_3/G_3^{(4)}=\{H_1,H_2,H_3,H_4\}$, where

$$H_1 := G_3^{(4)},$$

$$H_2 := \{x\in G_3: \sum_{j\in A} w_j(x)-\text{even}, |x|\text{-odd}\},$$

$$H_3 := \{x\in G_3: \sum_{j\in A} w_j(x)-\text{odd}, |x|-\text{even}\},$$

$$H_4 := \{x\in G_3: \sum_{j\in A} w_j(x)-\text{odd}, |x|-\text{odd}\}.$$

We study $G_3^{(4)}$- periodic ground states. We note that each $G_3^{(4)}$-periodic configuration has the following form:

$$\sigma(x)=\begin{cases}\sigma_1, \text{ if } x\in H_1,\\ \sigma_2, \text{ if } x\in H_2,\\ \sigma_3, \text{ if } x\in H_3,\\ \sigma_4, \text{ if } x\in H_4,\end{cases} \tag{4.3}$$

where $\sigma_i\in\Phi=\{-1,0,1\}, i=1,2,3,4$.

In the sequel, we write $\sigma=(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ for such a $G_3^{(4)}$-periodic configuration (4.3).

Theorem 4.6. *Let $|\sigma_1-\sigma_2|=1$. For the modified SOS model with competing interactions given by (2.2) the following statements hold on the Cayley tree of order three:*
a) If $(J_1,J_2)\in A_2\cap A_3$ and $|A|=1$ or $|A|=3$, then the configuration $\varphi=(\sigma_1,\sigma_2,-\sigma_1,-\sigma_2)$ is $G_3^{(4)}$-periodic ground state. They have the following forms $(\pm1,0,\mp1,0)$ and $(0,\pm1,0,\mp1)$.
b) If $(J_1,J_2)\in A_2\cap A_7$ and $|A|=2$, then the configuration $\varphi=(\sigma_1,\sigma_2,-\sigma_1,-\sigma_2)$ is $G_3^{(4)}$-periodic ground state. They have the following forms $(\pm1,0,\mp1,0)$ and $(0,\pm1,0,\mp1)$.
c) If $(J_1,J_2)\in A_6\cap A_{16}$ and $|A|=2$, then the configurations $\varphi=(\sigma_1,-\sigma_1,\sigma_2,-\sigma_2)$ and $\varphi^*=(\sigma_1,-\sigma_2,\sigma_2,-\sigma_1)$ are $G_3^{(4)}$-periodic ground states. They have the following forms $(\pm1,\mp1,0,0)$, $(0,0,\pm1,\mp1)$ and $(\pm1,0,0,\mp1)$, $(0,\pm1,\mp1,0)$.
d) All $G_3^{(4)}$-periodic ground states except the case a),b),c) are H_A-periodic ground states.

Proof. a) Let us consider the following configuration

$$\varphi_1(x)=\begin{cases}1, & \text{if } x\in H_1,\\ 0, & \text{if } x\in H_2,\\ -1, & \text{if } x\in H_3,\\ 0, & \text{if } x\in H_4.\end{cases}$$

1) Assume that $c_b \in H_1$

$$\varphi_b(c_b) = 1, B^{(1)} = 0, B^{(-1)} = 0, B^{(0)} = 4.$$

Consequently, $\varphi_b \in C_2$.
2) Let $c_b \in H_2$, then one has

$$\varphi_b(c_b) = 0, B^{(0)} = 0, B^{(1)} = 3, B^{(-1)} = 1.$$

Consequently, $\varphi_b \in C_3$.
3) Let $c_b \in H_3$, then one has

$$\varphi_b(c_b) = -1, B^{(-1)} = 0, B^{(1)} = 0, B^{(0)} = 4.$$

Consequently, $\varphi_b \in C_2$.
4) Let $c_b \in H_4$, then one has

$$\varphi_b(c_b) = 0, B^{(0)} = 0, B^{(-1)} = 3, B^{(1)} = 1.$$

Consequently, $\varphi_b \in C_3$.
We conclude that, if $(J_1, J_2) \in A_2 \cap A_3$ then the periodic configuration φ_1 is a $G_3^{(4)}$- periodic ground state. Other cases are proved similarly. This finishes the proof of Theorem 4.6. □

Remark 4.7. When $|\sigma_1 - \sigma_2| = 1$ and (J_1, J_2) takes a value different from the sets in the above Theorem 4.6, the $G_3^{(4)}$-periodic ground states coincide with the case a) of Theorem 4.4. When $|\sigma_1 - \sigma_2| \neq 1$, the $G_3^{(4)}$-periodic ground states coincide with H_A-periodic ground states which is studied the case b) of Theorem 4.4. If $|A| = 4$, $G_3^{(4)}$-periodic ground states correspond to $G_3^{(2)}$-periodic ground states.

4.3. **Weakly Periodic Ground States.** In this subsection we describe H_A-weakly periodic ground states, where H_A is a normal subgroup of index two.

Theorem 4.8. *Let $k = 3$. The following statements hold:*
1. *Let $|A| = 2$.*
1.*a) If $(J_1, J_2) \in A_{10} \cap A_{17}$, then for the modified SOS model with competing interactions there are H_A-weakly periodic (non periodic) ground states which are the following forms:*

$$\varphi_{1,2} = \pm \begin{cases} 1, & \textit{if} \ \ x_\downarrow \in H_1, \ x \in H_1, \\ -1, & \textit{if} \ \ x_\downarrow \in H_1, \ x \in H_2, \\ -1, & \textit{if} \ \ x_\downarrow \in H_2, \ x \in H_1, \\ 1, & \textit{if} \ \ x_\downarrow \in H_2, \ x \in H_2. \end{cases}$$

1.*b) If $(J_1, J_2) \in A_4 \cap A_6$, then for the modified SOS model with competing interactions there are H_A-weakly periodic (non periodic) ground states which are the following forms:*

$$\varphi_{3,4} = \pm \begin{cases} 1, & \textit{if} \ \ x_\downarrow \in H_1, \ x \in H_1, \\ 0, & \textit{if} \ \ x_\downarrow \in H_1, \ x \in H_2, \\ 0, & \textit{if} \ \ x_\downarrow \in H_2, \ x \in H_1, \\ 1, & \textit{if} \ \ x_\downarrow \in H_2, \ x \in H_2, \end{cases} \qquad \varphi_{5,6} = \pm \begin{cases} 0, & \textit{if} \ \ x_\downarrow \in H_1, \ x \in H_1, \\ 1, & \textit{if} \ \ x_\downarrow \in H_1, \ x \in H_2, \\ 1, & \textit{if} \ \ x_\downarrow \in H_2, \ x \in H_1, \\ 0, & \textit{if} \ \ x_\downarrow \in H_2, \ x \in H_2. \end{cases}$$

2. *Let $|A| = p, p \in \{1, 3\}$. If $(J_1, J_2) \notin A_0$, then for the modified SOS model with competing interactions there is no H_A-weakly periodic (non periodic) ground state.*

Proof. 1.a) Let us prove the case $|A| = 2$ without loss of generality which is H_A-weakly periodic (non periodic) ground state on the set $A_{10} \cap A_{17}$.
Let

$$\varphi_1 = \begin{cases} 1, & \text{if} \ \ x_\downarrow \in H_1, \ x \in H_1, \\ -1, & \text{if} \ \ x_\downarrow \in H_1, \ x \in H_2, \\ -1, & \text{if} \ \ x_\downarrow \in H_2, \ x \in H_1, \\ 1, & \text{if} \ \ x_\downarrow \in H_2, \ x \in H_2. \end{cases}$$

Let $c_b \in H_1$, then we have the following possible cases:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 3$, $B^{(0)} = 0$, $B^{(1)} = 1$, $\varphi_b \in C_{10}$.
b) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{17}$.
c) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 1$, $B^{(0)} = 0$, $B^{(1)} = 3$, $\varphi_b \in C_{10}$.
d) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{17}$.
Let $c_b \in H_2$, then we have the following possible cases:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = -1$,$B^{(-1)} = 1$, $B^{(0)} = 0$, $B^{(1)} = 3$, $\varphi_b \in C_{10}$.
b) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{17}$.
c) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 3$, $B^{(0)} = 0$, $B^{(1)} = 1$, $\varphi_b \in C_{10}$.
d) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{17}$.
We conclude that the configuration φ_1 is a weakly periodic ground state if $(J_1, J_2) \in A_{10} \cap A_{17}$. Remaining cases can be proved analogously. This finishes the proof of Theorem 4.8. □

Remark 4.9. Theorem 4.8 says that if $|A| = p, p \in \{1, 3\}$, then for the modified SOS model with competing interactions there is no H_A-weakly periodic (non periodic) ground state. Furthermore, in the paper [8], it is proved that there are no weakly periodic (non periodic) ground states for the SOS model when $|A| = 1$.

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