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Optimal quadrature formulas in the Sobolev space of complex-valued functions

Shadimetov Kh. M., Tursunmuradov S.T.

Abstract. In this work, the extremal function of the error functional in the Sobolev space of complex-valued functions is derived. Using the obtained extremal function, the squared norm of the error functional of quadrature formulas is computed. By minimizing the squared norm of the error functional with respect to the coefficients of the quadrature formulas, a system is derived for determining the optimal coefficients of the considered quadrature formula in the Sobolev space. In addition, an analogue of I. Babuška's theorem is proved.

Keywords: Extremal function, error functional, Sobolev space, quadrature formula, optimal coefficient.

MSC (2020): 65D30, 65D32

1. Introduction

One of the important tasks of computational mathematics is the development of new methods for constructing optimal quadrature, cubature, interpolation and difference formulas, and the assessment of their errors in various functional spaces. The construction of optimal quadrature and cubature formulas was first considered in the works of S.M. Nikolsky [1], A. Sard [2], S.L. Sobolev [3, 4]. In the works of M.D. Ramazanov [5, 6, 7] a new algorithm for constructing cubature formulas with a boundary layer was constructed. In the work of M.D. Ramazanov and Kh.M. Shadimetov [8] weighted optimal cubature formulas in the Sobolev space of periodic functions were constructed. In recent years, a number of works have been published on this topic (see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). In this paper, the variational method will be used to construct optimal quadrature formulas in the Sobolev space of complex-valued functions.

Let us consider quadrature formulas of the form

$$\int_0^1 p(x)\,\varphi(x)\,dx \cong \sum_{\beta=0}^{N} C[\beta]\,\varphi[\beta], \tag{1.1}$$

with the error functional

$$\ell_N(x) = \varepsilon_{[0,1]}(x)\,p(x) - \sum_{\beta=0}^{N} C[\beta]\,\delta(x-h\beta). \tag{1.2}$$

Here $p(x)$ is a weight function. It is required that the integral of this function exists in some sense, i.e.

$$\int_0^1 p(x)\,dx < \infty,$$

$\varphi(x) \in H_2^{(m)}(0,1)$, and $H_2^{(m)}(0,1)$ is the Sobolev space of complex-valued functions with the inner product and norm, respectively

$$(\psi,\varphi)_{H_2^{(m)}} = \sum_{k=0}^{m} \binom{m}{k} \frac{1}{\omega^{2k}} \int_0^1 \frac{d^k}{dx^k}\overline{\psi}(x)\,\frac{d^k}{dx^k}\varphi(x)dx, \tag{1.3}$$

$$\|\varphi\|_{H_2^{(m)}} = \left(\sum_{k=0}^{m} \binom{m}{k} \frac{1}{\omega^{2k}} \int_0^1 \frac{d^k}{dx^k}\overline{\varphi}(x)\,\frac{d^k}{dx^k}\varphi(x)dx\right)^{\frac{1}{2}}, \tag{1.4}$$

ω a real number and $\omega \neq 0$, $\overline{\psi}(x)$ is the complex conjugate function of the function $\psi(x)$, $C[\beta]$, $\beta = 0, 1, ..., N$, are coefficients of the quadrature formula (1.1), $\varepsilon_{[0,1]}(x)$ is the characteristic function of the segment $[0,1]$, $\delta(x)$ is the Dirac delta-function, $\binom{m}{k} = \frac{m!}{k!(m-k)!}$, $[\beta] = h\beta$, $h = \frac{1}{N}$, $N = 1, 2, ...$.

The next difference

$$(\ell_N, \varphi) = \int_0^1 p(x)\varphi(x)\,dx - \sum_{\beta=0}^{N} C[\beta]\varphi[\beta] = \int_R \ell_N(x)\varphi(x)\,dx \tag{1.5}$$

is called the error of the quadrature formula (1.1).

From the Cauchy-Schwarz inequality

$$|(\ell_N, \varphi)| \leq \left\| \ell_N | H_2^{(m)*} \right\| \cdot \left\| \varphi | H_2^{(m)} \right\|,$$

it is clear that the absolute value of the error of the quadrature formula (1.1) is estimated using the norm of the error functional (1.2)

$$\left\| \ell_N | H_2^{(m)*} \right\| = \sup_{\varphi,\ \|\varphi\| \neq 0} \frac{|(\ell_N, \varphi)|}{\left\| \varphi | H_2^{(m)} \right\|} \tag{1.6}$$

in the conjugate space $H_2^{(m)*}$. It follows that the error estimate of the quadrature formula (1.1) on complex-valued functions in the Sobolev space $H_2^{(m)}(0,1)$ reduces to finding the norm of the error functional (1.2) in the conjugate space $H_2^{(m)*}(0,1)$. It is evident that the norm of the error functional (1.2) depends on the coefficients $C[\beta]$ of the quadrature formula (1.1). By minimizing the norms of the error functional $\ell_N(x)$ with respect to the coefficients $C[\beta]$, we obtain **the optimal coefficients** of the quadrature formula. The resulting formula is called **the optimal quadrature formula**.

Thus, in order to construct an optimal quadrature formula in the Sobolev space, $H_2^{(m)}(0,1)$ it is necessary to find the following quantity

$$\inf_{C[\beta]} \left\| \ell_N | H_2^{(m)*} \right\|. \tag{1.7}$$

We denote this value by $\left\| \overset{\circ}{\ell}_N | H_2^{(m)*} \right\|$, which we call the norm of the error functional of the optimal quadrature formula, i.e.

$$\left\| \overset{\circ}{\ell}_N | H_2^{(m)*} \right\| = \inf_{\overset{\circ}{C}[\beta]} \left\| \ell_N | H_2^{(m)*} \right\|. \tag{1.8}$$

The coefficients $C[\beta]$ for which the exact lower bound is achieved as in (1.8) are called the optimal coefficients of the quadrature formula and are denoted by $\overset{\circ}{C}[\beta]$, $\beta = 0, 1, \ldots, N$.

So, we need to solve the following problems sequentially.

Problem 2. Find the norm of the error functional $\ell_N(x)$ of the quadrature formula (1.1) on the space $H_2^{(m)}(0,1)$.

Problem 3. Find the coefficients $\overset{\circ}{C}[\beta]$, $\beta = 0, 1, \ldots, N$ that satisfy (1.8).

In the next section, we find the extremal function that helps us to calculate the norm of error functional.

2. An extremal function of quadrature formulas in Sobolev space $H_2^{(m)}(0,1)$ - complex-valued functions

The explicit form of the norm of the error functional $\ell_N(x)$ in the space $H_2^{(m)*}(0,1)$ is obtained by means of the so-called extremal function of this functional [1, 2].

Definition 2.1. (S.L. Sobolev) A function $\psi_{\ell,H}(x) \in H_2^{(m)}(0,1)$ is called an extremal function of a given functional $\ell_N(x)$ if

$$(\ell, \psi_{\ell,H}) = \left\| \ell_N | H_2^{(m)*} \right\| \cdot \left\| \psi_{\ell,H} | H_2^{(m)} \right\|. \tag{2.1}$$

The following is true.

Theorem 2.2. *In the Sobolev space $H_2^{(m)}(0,1)$, the extremal function of the error functional $\ell_N(x)$ is given by the formula*

$$\psi_{\ell,H}(x) = \ell_N(x) * \varepsilon_{m,\omega}(x) = \int_0^1 \overline{p}(y)\, \varepsilon_m(x-y)\, dy - \sum_{\beta=0}^{N} \overline{C}[\beta]\, \varepsilon_{m,\omega}(x - h\beta), \tag{2.2}$$

where

$$\varepsilon_{m,\omega}(x) = \frac{\omega e^{-\omega|x|}}{2^{2m-1}(m-1)!} \sum_{k=0}^{m} \frac{(2m-k-2)!(2\omega)^k |x|^k}{k!\,(2m-k-1)!}. \tag{2.3}$$

$\overline{\ell}_N(x)$ is the complex conjugate functional of the functional $\ell_N(x)$, $\overline{p}(y)$ is the complex conjugate function of the function $p(y)$, $\overline{C}[\beta]$ is complex conjugate coefficient of $C[\beta]$.

Proof. To find an extremal function $\psi_{\ell,H}(x)$, we use the well-known Riesz theorem on the general form of a linear continuous functional. The space $H_2^{(m)}(0,1)$ is a Hilbert space, according to the Riesz theorem for the error functional $\ell_N(x)$ and for any function $\varphi(x) \in H_2^{(m)}(0,1)$ there is a unique function $\psi_{\ell,H}(x) \in H_2^{(m)}(0,1)$ for which the following equality holds

$$(\ell_N, \varphi) = (\psi_{\ell,H}, \varphi)_{H_2^{(m)}}, \tag{2.4}$$

and

$$\left\| \ell_N | H_2^{(m)*} \right\| = \left\| \psi_{\ell,H} | H_2^{(m)} \right\|. \tag{2.5}$$

The right side of (2.4) is determined by formula (1.3).

Now to find $\psi_{\ell,H}(x)$ we solve equation (2.4). Taking into account (1.3), after integrating by parts the right side of (2.4), we get

$$(\ell_N, \varphi) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{\omega^{2k}} \int_0^1 \overline{\psi}_{\ell,H}^{(2k)}(x)\, \varphi(x)\, dx + \sum_{s=1}^{m} \sum_{k=s}^{m} \binom{m}{k} \frac{(-1)^{k-s}}{\omega^{2k}} \overline{\psi}_{\ell,H}^{(2k-s)}(x)\, \varphi^{(s-1)}(x) \Big|_0^1. \tag{2.6}$$

From here, taking account the arbitrariness $\varphi(x)$ and uniqueness of the extremal function, we obtain the following boundary value problem

$$\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{\omega^{2k}} \overline{\psi}_{\ell,H}^{(2k)}(x) = \ell_N(x), \tag{2.7}$$

$$\sum_{s=1}^{m} \sum_{k=s}^{m} \binom{m}{k} \frac{(-1)^{k-s}}{\omega^{2k}} \overline{\psi}_{\ell,H}^{(2k-s)}(x) \Big|_0^1 = 0. \tag{2.8}$$

On the other hand, by Sobolev's theorem (see [1] theorem 11), $C_0^{\infty}(0,1)$ is the space of functions that are infinitely differentiable and finite on the interval [0,1] is dense in L_p for $1 \le p \le \infty$. It follows that the space $C_0^{\infty}(0,1)$ is dense in the Sobolev space $H_2^{(m)}(0,1)$. Further, by virtue of these results it follows that in equality (2.6) the last double sum vanishes. Indeed, from the finiteness of φ we have $\varphi^{(s-1)}(0) = 0$, $\varphi^{(s-1)}(1) = 0$. Then

$$\sum_{s=1}^{m} \sum_{k=s}^{m} \binom{m}{k} \frac{(-1)^{k-s}}{\omega^{2k}} \overline{\psi}_{\ell,H}^{(2k-s)}(x)\, \varphi^{(s-1)}(x) \Big|_0^1 = 0.$$

As a result, to find $\overline{\psi}_{\ell,H}(x)$ it is enough to solve equation (2.7). Now we will deal with solving equation (2.7). We write equation (2.7) in the form

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{\omega^{2k}}\frac{d^{2k}}{dx^{2k}}\overline{\psi}_{\ell,H}(x)=\ell_N(x), \tag{2.9}$$

The operator L_m has the form

$$L_m=\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{\omega^{2k}}\frac{d^{2k}}{dx^{2k}}=\left(1-\frac{d^2}{\omega^2dx^2}\right)^{[m]}=\left(1-\frac{d^2}{\omega^2dx^2}\right)^{[m-1]}\left(1-\frac{d^2}{\omega^2dx^2}\right). \tag{2.10}$$

If we find a fundamental solution of the operator L_m, i.e.

$$L_m\varepsilon_{m,\omega}(x)=\delta(x), \tag{2.11}$$

then the solution to equation (2.9) is given by the following formula for the convolution of two functions

$$\overline{\psi}_{\ell,H}(x)=\ell_N(x)*\varepsilon_{m,\omega}(x). \tag{2.12}$$

Lemma 2.1. A fundamental solution of the operator L_m has the form

$$\varepsilon_{m,\omega}(x)=\frac{\omega e^{-\omega|x|}}{2^{2m-1}(m-1)!}\sum_{k=0}^{m-1}\frac{(2m-k-2)!(2\omega)^k|x|^k}{k!\,(m-k-1)!}. \tag{2.13}$$

Proof. It is not difficult to verify that

$$\left(1-\frac{d^2}{\omega^2dx^2}\right)\varepsilon_{m,\omega}(x)=\varepsilon_{m-1,\omega}(x). \tag{2.14}$$

Then

$$\left(1-\frac{d^2}{\omega^2dx^2}\right)^{[m]}\varepsilon_{m,\omega}(x)=\left(1-\frac{d^2}{\omega^2dx^2}\right)^{[m-1]}\left(1-\frac{d^2}{\omega^2dx^2}\right)\varepsilon_{m,\omega}(x)=\left(1-\frac{d^2}{\omega^2dx^2}\right)^{[m-1]}\varepsilon_{m-1,\omega}(x). \tag{2.15}$$

In the same way, we have

$$\left(1-\frac{d^2}{\omega^2dx^2}\right)^{[m]}\varepsilon_{m,\omega}(x)=\left(1-\frac{d^2}{\omega^2dx^2}\right)^{[m-1]}\varepsilon_{m-(m-1),\omega}(x)=\left(1-\frac{d^2}{\omega^2dx^2}\right)\varepsilon_{1,\omega}(x). \tag{2.16}$$

In case $m=1$ expression (2.13) takes the following form

$$\varepsilon_{1,\omega}(x)=\frac{\omega e^{-\omega|x|}}{2}. \tag{2.17}$$

Now, calculating the generalized derivatives of the function, $\varepsilon_{1,\omega}(x)$ we have

$$\varepsilon_{1,\omega}^{'}(x)=-\frac{\omega^2}{2}e^{-\omega|x|}\mathrm{sign}(x),\qquad \varepsilon_{1,\omega}^{''}(x)=\frac{\omega^3}{2}e^{-\omega|x|}\text{-}\omega^2\delta(x). \tag{2.18}$$

Due to (2.17), (2.18), expression (2.16) takes the following form

$$\left(1-\frac{d^2}{\omega^2dx^2}\right)^{[m]}\varepsilon_{m,\omega}(x)=\left(1-\frac{d^2}{\omega^2dx^2}\right)\varepsilon_{1,\omega}(x)=\varepsilon_{1,\omega}(x)-\frac{\varepsilon_{1,\omega}^{''}(x)}{\omega^2}=\frac{\omega e^{-\omega|x|}}{2}-\frac{\omega}{2}e^{-\omega|x|}+\delta(x)=\delta(x).$$

So, we have proved Lemma 2.1 completely. □

Next, using formulas (1.2), (2.12) and calculating the convolutions, we obtain

$$\overline{\psi}_{\ell,H}(x)=\ell_N(x)*\varepsilon_{m,\omega}(x)=\int\limits_{-\infty}^{\infty}\ell_N(y)\,\varepsilon_{m,\omega}(x-y)\,dy=\int\limits_{-\infty}^{\infty}\Big(\varepsilon_{[0,1]}(y)\,p(y)$$

$$-\sum_{\beta=0}^{N}C[\beta]\,\delta(y-h\beta)\Big)\,\varepsilon_{m,\omega}(x-y)\,dy=\int\limits_0^1 p(y)\,\varepsilon_{m,\omega}(x-y)\,dy-\sum_{\beta=0}^{N}C[\beta]\,\varepsilon_{m,\omega}(x-h\beta).$$

From this, we have

$$\psi_{\ell,H}(x)=\int\limits_0^1\overline{p}(y)\,\varepsilon_{m,\omega}(x-y)\,dy-\sum_{\beta=0}^{N}\overline{C}[\beta]\,\varepsilon_{m,\omega}(x-h\beta).$$

Theorem 2.2 is completely proved. □

3. The square of the norm of the error functional for quadrature formulas

Now, using the expression of the extremal function, $\overline{\psi}_{\ell,H}(x)$ we calculate the norm of the error functional for the quadrature formula (1.1). For this, using formulas (2.1), (2.2) and (2.5), we obtain

$$\left\|\ell_N|H_2^{(m)*}\right\|^2=(\ell_N,\psi_{\ell,H})=\int\limits_{-\infty}^{\infty}\left(\varepsilon_{[0,1]}(x)\,p(x)-\sum_{\gamma=0}^{N}C[\gamma]\,\delta(x-h\gamma)\right)$$

$$\times\left(\int\limits_0^1\overline{p}(y)\,\varepsilon_{m,\omega}(x-y)\,dy-\sum_{\beta=0}^{N}\overline{C}[\beta]\,\varepsilon_{m,\omega}(x-h\beta)\right)dx=\int\limits_0^1\int\limits_0^1 p(x)\overline{p}(y)\,\varepsilon_{m,\omega}(x-y)\,dxdy$$

$$-\sum_{\gamma=0}^{N}C[\gamma]\int\limits_0^1\overline{p}(y)\,\varepsilon_{m,\omega}(h\gamma-y)\,dy-\sum_{\beta=0}^{N}\overline{C}[\beta]\int\limits_0^1 p(x)\,\varepsilon_{m,\omega}(x-h\beta)\,dx+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\overline{C}[\beta]\,C[\gamma]\varepsilon_{m,\omega}(h\beta-h\gamma)\ .$$

So, we have proven the following theorem.

Theorem 3.1. *The square of the norm of the error functional of quadrature formulas of the type* (1.1) *in the Sobolev space* $H_2^{(m)*}(0,1)$ *is expressed by the following formula*

$$\left\|\ell_N|H_2^{(m)*}\right\|^2=\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\overline{C}[\beta]\,C[\gamma]\varepsilon_{m,\omega}(h\gamma-h\beta)-\sum_{\gamma=0}^{N}C[\gamma]\int\limits_0^1\overline{p}(y)\,\varepsilon_{m,\omega}(h\gamma-y)\,dy-$$
$$-\sum_{\beta=0}^{N}\overline{C}[\beta]\int\limits_0^1 p(x)\,\varepsilon_{m,\omega}(x-h\beta)\,dx+\int\limits_0^1\int\limits_0^1 p(x)\overline{p}(y)\,\varepsilon_{m,\omega}(x-y)\,dxdy. \tag{3.1}$$

$C[\gamma]$ are the coefficients of the quadrature formula and $\overline{C}[\beta]$ are the complex conjugate value to them, $\varepsilon_{m,\omega}(x)$ is determined by equality (2.3), $\overline{p}(y)$ is the complex conjugate function to the function $p(y)$.

4. System of equations for determining the optimal coefficients of quadrature formulas

We proceed to minimize the error of quadrature formulas with respect to the coefficients $C[\gamma]$ and $\overline{C}[\beta]$. To do this, by calculating the partial derivatives of the square of the norm of the error functional ℓ_N with respect to $C[\gamma]$ and $\overline{C}[\beta]$. Then we obtain

$$\frac{\partial\left\|\ell_N|H_2^{(m)*}\right\|^2}{\partial C[\gamma]}=\sum_{\beta=0}^{N}\overline{C}[\beta]\,\varepsilon_{m,\omega}(h\gamma-h\beta)-\int\limits_0^1\overline{p}(y)\,\varepsilon_{m,\omega}(h\gamma-y)\,dy, \tag{4.1}$$

$$\frac{\partial\left\|\ell_N|H_2^{(m)*}\right\|^2}{\partial\overline{C}\left[\beta\right]}=\sum_{\gamma=0}^{N}C\left[\gamma\right]\varepsilon_{m,\omega}\left(h\gamma-h\beta\right)-\int_0^1 p\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx. \tag{4.2}$$

Equating these partial derivatives to zero, we have

$$\sum_{\beta=0}^{N}\overline{C}\left[\beta\right]\varepsilon_{m,\omega}\left(h\gamma-h\beta\right)=\int_0^1\overline{p}\left(y\right)\varepsilon_{m,\omega}\left(h\gamma-y\right)dy,\ \gamma=0,1,...,N, \tag{4.3}$$

$$\sum_{\gamma=0}^{N}C\left[\gamma\right]\varepsilon_{m,\omega}\left(h\beta-h\gamma\right)=\int_0^1 p\left(x\right)\varepsilon_{m,\omega}\left(h\beta-x\right)dx,\ \beta=0,1,...,N. \tag{4.4}$$

From these systems it is clear that systems (4.3) and (4.4) are obtained from each other. Therefore, it is enough for us to solve one of them. In what follows, we will solve system (4.4). The solution of this system are the optimal coefficients for the quadrature formula (1.1), which we denoted by $\overset{\circ}{C}[\beta]$, $\beta=0,1,...,N$.

If we know the optimal coefficients of the quadrature formulas, then by virtue of (4.3) the optimal square of the norm of the error functional of the optimal quadrature formula is determined by the equality

$$\left\|\overset{\circ}{\ell}_N|H_2^{(m)*}\right\|^2=\int_0^1\int_0^1 p\left(x\right)\overline{p}\left(y\right)\varepsilon_{m,\omega}\left(x-y\right)dxdy-\sum_{\beta=0}^{N}\overline{\overset{\circ}{C}}\left[\beta\right]\int_0^1 p\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx. \tag{4.5}$$

If we use system (4.4) then we have

$$\left\|\overset{\circ}{\ell}_N|H_2^{(m)*}\right\|^2=\int_0^1\int_0^1 p\left(x\right)\overline{p}\left(y\right)\varepsilon_{m,\omega}\left(x-y\right)dxdy-\sum_{\beta=0}^{N}\overset{\circ}{C}\left[\beta\right]\int_0^1 \overline{p}\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx. \tag{4.6}$$

It follows that

$$\sum_{\beta=0}^{N}\overline{\overset{\circ}{C}}\left[\beta\right]\int_0^1 p\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx=\sum_{\beta=0}^{N}\overset{\circ}{C}\left[\beta\right]\int_0^1 \overline{p}\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx \tag{4.7}$$

or

$$\sum_{\beta=0}^{N}\int_0^1\left[\overline{\overset{\circ}{C}}\left[\beta\right]p\left(x\right)-\overset{\circ}{C}\left[\beta\right]\overline{p}\left(x\right)\right]\varepsilon_{m,\omega}\left(x-h\beta\right)dx=0.$$

Thus, we have proved the following

Theorem 4.1. *The square of the norm of the error functional of optimal quadrature formulas in the space $H_2^{(m)*}\left(0,1\right)$ is determined by the formula*

$$\left\|\overset{\circ}{\ell}_N|H_2^{(m)*}\right\|^2=\int_0^1\int_0^1 p\left(x\right)\overline{p}\left(y\right)\varepsilon_{m,\omega}\left(x-y\right)dxdy-\sum_{\beta=0}^{N}\overset{\circ}{C}\left[\beta\right]\int_0^1 \overline{p}\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx.$$

From system (4.3) we obtain the following theorem, an analogue of the theorem of I. Babuška.

Theorem 4.2. *Let the error functional $\ell_N\left(x\right)$ be defined on the space $H_2^{(m)*}\left(0,1\right)$ and be optimal, i.e., among all functionals of the form*

$$\varepsilon_{[0,1]}\left(x\right)p\left(x\right)-\sum_{\beta=0}^{N}C\left[\beta\right]\delta\left(x-h\beta\right)$$

it has the smallest norm in the space $H_2^{(m)*}(0,1)$. *Then there exists a solution to the equation*

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{\omega^{2k}}\frac{d^{2k}}{dx^{2k}}\overline{\psi}_{\ell,H}(x)=\ell_N(x),$$

which vanishes at points $h\beta,\ \beta=0,1,...,N$ *and belongs to* $H_2^{(m)}(0,1)$.

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Shadimetov Kh. M.
Tashkent state transport university,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
email: kholmatshadimetov@mail.ru
Tursunmurodov S.T.
Tashkent state transport university