Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.201-216
DOI: 10.29229/uzmj.2025-4-20

# Differential game with a "life-line" for nonlinear motion dynamics of players

**Turgunboeva M.A., Soyibboev U.B.**

**Abstract.** We investigate the interception problem in a differential game with non-inertial players (a pursuer and an evader) who move in dynamic flow fields with various influences. Throughout the paper, we solve the pursuit and "life-line" game problems. To solve the pursuit, the strategy of parallel pursuit (**Π**-strategy for short) is defined and used. With the help of the **Π**-strategy and applying the Grönwall-Bellman inequality, sufficient pursuit condition is determined. In order to solve the "life-line" game to the advantage of the pursuer, we build the set of meeting points of the players and prove that this set monotonically decreases with regard to inclusion relative to time. The "life-line" game to the advantage of the evader is solved by constructing evader's attainability domain where it reaches without being caught for an arbitrary control of the pursuer.

**Keywords:** Differential game, Caratheodory's conditions, Lipschitz's condition, players, geometric constraint, pursuit, "life-line", **Π**-strategy, Grönwall-Bellman inequality

**MSC (2020):** 49N70, 49N75, 91A23, 91A24

## 1. Introduction

Differential games are a special kind of problems for dynamic systems particularly moving objects. In 1965, this theory was studied systematically by R. Isaacs and published in the form of monograph [21], in which numerous examples were examined and theoretical questions were only touched upon. The foundation of the modern theory of the differential game was settled by mathematicians R. Isaacs [21], L.D. Bercovitz [8], R.J. Elliot, N.J. Kalton [14], A. Friedman [15], O. Hajek [17], Y. Ho, A. Bryson, S. Baron [18], L.S. Pontryagin [27], N.N. Krasovskii [22], L.A. Petrosyan [26], B.N. Pshenichnii [28], A.A. Chikrii [12].

In the differential games theory, in accordance with the basic approaches proposed by L.S. Pontryagin [27] and N.N. Krasovskii [22], a differential game is explored as a control problem from the viewpoint of either the chasing player (pursuer) or the escaping player (evader). Under this framework, the game can be stated as either a pursuit or an evasion problem. Pursuit-evasion differential game have been extensively studied in the literature [8, 12, 16, 19, 20, 32] with significant contributions addressing theoretical foundations, optimal strategies, and real-world applications.

The book [21] by R. Isaacs covers several game problems that were explored thoroughly and proposed for further investigation. One of these is named "life-line" problem that was initially stated and examined for specific cases in [21] (Problem 9.5.1). When control functions of both players meet geometric restrictions, the stated game has been rather comprehensively considered in [26] by L.A. Petrosyan. The Π-strategy, which was introduced in [26, 28] for a simple pursuit game with geometric restrictions, functioned as the starting point for the development of the effective method in pursuit games with multiple pursuers (see [3, 4, 11, 12], [29]–[30]).

There are numerous studies on nonlinear differential games that have found key conditions for successful capture and the optimality of capture time. For example, in work [33], a differential game of the stationary nonlinear system was studied, and the optimality of capture time was analyzed for a specific case on a plane, where the pursuer applied a counter-strategy. Similarly, A. Azamov, in [2], considered the pursuit differential game, where the dynamics were governed by a nonlinear system of differential equations of a specific form, through a positional counter-strategy on a plane, and also presented clear examples that illustrate the explicit characteristics of the game. In [24], a two-player nonlinear differential game with an integral quality criterion was investigated at the time interval divided into two segments. Necessary and sufficient conditions were obtained for the existence of a saddle point for a general two-person zero-sum differential game when one or both players use suboptimal control laws of specified form (referred to as piecewise control laws). Additionally, the work of K.A. Shchelchkov [34] was concerned with the problems of stabilization to zero under disturbance in terms of a differential pursuit game described by a nonlinear autonomous system of differential equations. The sufficient conditions for the existence of a neighborhood of zero from each point of which a capture occurs in the indicated sense were derived.

Some optimal control problem formulations have taken into account the effect of an external flow field. For example, in [23], the authors considered the problem of optimal guidance of a Dubin's vehicle [13] to a specified position under the influence of an external flow. The minimum-time guidance problem for an isotropic rocket in the presence of wind has been studied in [7]. The problem of minimizing the expected time to steer a Dubin's vehicle to a target set in a stochastic wind field has also been discussed in [1]. However, the same level of attention in the literature has not been devoted to pursuit-evasion or "life-line" games with two (or more) competing agents under the influence of external disturbances (e.g., winds or currents). In papers [35]–[36], a multi-pursuer and one-evader for the pursuit-evasion game in an external dynamic flow field is considered. Due to the generality of the external flow, Isaacs's approach is not readily applicable [21]. Instead, in [36], a different approach is used and, the optimal trajectories of the players through a reachable set method are found.

This work studies the differential game with a "life-line" when players (a pursuer and an evader) move in dynamic flow fields with various influences. Throughout the paper, we solve the pursuit problem and the game with a "life-line". The obtained results are based on Krasovskii-Pontryagin's formalization ([22, 27]), Pshenichnii-Chikrii's method of resolving functions ([12, 28]), the **Π**-strategy ([3]–[6], [11, 26], [29]–[31]) and the properties of the multi-valued mapping [10].

## 2. Statement of problems

Let two controllable players $P$ and $E$ be given in Euclidean space $\mathbb{R}^n$. The first player $P$ called a Pursuer chases the second player $E$ called an Evader. Suppose, $x$ signifies the position of the Pursuer, and $y$ signifies that of the Evader in $\mathbb{R}^n$. Then the players perform their motions in accordance with the equations

$$P: \quad \dot{x} = u + F_P(t, x), \qquad x(0) = x_0, \tag{2.1}$$

$$E: \quad \dot{y} = v + F_E(t, y), \qquad y(0) = y_0, \tag{2.2}$$

appropriately, where $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$, $t \in \mathbb{R}_+ := [0, +\infty)$; $F_P : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ ($F_E : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$) is an effective flow field for the Pursuer (for the Evader); $x_0, y_0$ are the players' initial positions. It is considered that $x_0 \neq y_0$.

In (2.1), the parameter $u$ denotes as the Pursuer's control, and it is hereafter selected as a measurable function $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ complying with

$$|u(t)| \leq \alpha \text{ for almost all } t \geq 0, \tag{2.3}$$

where $\alpha$ is a positive constant.

Likewise, in (2.2), the parameter $v$ denotes as the Evader's control, and it is henceforth chosen as a measurable function $v(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ complying with

$$|v(t)| \leq \beta \text{ for almost all } t \geq 0, \tag{2.4}$$

where $\beta$ is a non-negative constant.

In the Theory of Differential Games, inequalities (2.3) and (2.4) are generally called *geometric constraints* (briefly, **G**-constraints) for the control functions.

Henceforward, the considered game (2.1)–(2.4) is referred as the *nonlinear differential game (2.1)–(2.4)* or briefly, *NDG (2.1)–(2.4)*.

**Definition 2.1.** A measurable function $u : \mathbb{R}_+ \to \mathbb{R}^n$, $|u(t)| \leq \alpha$, $t \geq 0$, (respectively, $v : \mathbb{R}_+ \to \mathbb{R}^n$, $|v(t)| \leq \beta$, $t \geq 0$) is called an *admissible control* of the Pursuer (respectively, Evader).

Let $U_{\mathbf{G}}$ (respectively, $V_{\mathbf{G}}$) denote the set of all admissible controls of the Pursuer (respectively, Evader).

**Assumption 2.2.** (Caratheodory's conditions) Let the functions $F_P(t,x)$ and $F_E(t,y)$ be defined on the domain $D := \mathbb{R}_+ \times \mathbb{R}^n$ and let they satisfy the conditions given below: 1) $F_P(t,x)$ and $F_E(t,y)$ are continuous in $x$ and $y$ for each fixed $t$; 2) $F_P(t,x)$ and $F_E(t,y)$ are measurable functions in $t$ for each fixed $x$ and $y$; 3) for each compact subset $Q$ of $D$, there can be found Lebesgue-integrable functions $h_P(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and $h_E(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, where $\sup_{t\geq 0} h_P(t) = h_P^*$, $0 \leq h_P^* < \infty$, $\sup_{t\geq 0} h_E(t) = h_E^*$, $0 \leq h_E^* < \infty$, such that $|F_P(t,x)| \leq h_P(t)$ and $|F_E(t,y)| \leq h_E(t)$ for all $(t,x),(t,y) \in Q$.

In equations (2.1), (2.2), the functions $F_P(t,x)$ and $F_E(t,y)$ represent the exogenous dynamic flows, but they may also represent the endogenous drift owing to the nonlinear dynamics of the players [35]–[36]. It is reasonable to suppose that the magnitude of these flows (e.g. winds or currents) is bounded from above by some functions $h_P(t)$ and $h_E(t)$ in the third condition of Assumption 2.2. B.T. Samatov et al. [31] considered the intercept problem when objects move in the same type external dynamic flow field. Unlike this work, in our study, the players are assumed to move within different influence zones, and the pursuit problem is solved for the "life-line" game also. In other words, our work can be regarded as a logical continuation of the study [31].

**Assumption 2.3.** (Lipshitz's conditions) For each compact subsets $Q_P$ and $Q_E$ of $D$, there exist Lebesgue-integrable functions $k_P(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and $k_E(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, where $\sup_{t\geq 0} k_P(t) = k_P^*$, $0 \leq k_P^* < \infty$, $\sup_{t\geq 0} k_E(t) = k_E^*$, $0 \leq k_E^* < \infty$, such that

$$|F_P(t,x_1) - F_P(t,x_2)| \leq k_P(t)|x_1 - x_2|,$$

$$|F_E(t,y_1) - F_E(t,y_2)| \leq k_E(t)|y_1 - y_2|$$

for all $(t,x_1),(t,x_2) \in Q_P$ and $(t,y_1),(t,y_2) \in Q_E$.

**Proposition 2.4.** *If Assumptions 2.2 and 2.3 are valid, then*

$$|F_P(t,x) - F_E(t,y)| \leq p(t)|x-y| + q(t) \tag{2.5}$$

*is true for any $x, y \in \mathbb{R}^n$, where*

$$p(t) = k_P(t) + k_E(t), \quad q(t) = h_P(t) + h_E(t),$$

$$\sup_{t\geq 0} p(t) = p = k_P^* + k_E^* < \infty, \quad \sup_{t\geq 0} q(t) = q = h_P^* + h_E^* < \infty. \tag{2.6}$$

*Proof.* Indeed, by using inequalities in Assumptions 2.2 and 2.3, the left side of (2.5) can be estimated as follows:

$$|F_P(t,x) - F_E(t,y)| = |F_P(t,x) - F_P(t,y) + F_P(t,y) - F_E(t,x) + F_E(t,x) - F_E(t,y)|$$

$$\leq |F_P(t,x) - F_P(t,y)| + |F_P(t,y)| + |F_E(t,x)| + |F_E(t,x) - F_E(t,y)|$$

$$\leq k_P(t)|x-y| + h_P(t) + h_E(t) + k_E(t)|x-y| = (k_P(t)+k_E(t))|x-y| + h_P(t) + h_E(t) = p(t)|x-y| + q(t),$$

which is the desired result.

□

In NDG (2.1)–(2.4), the objective of Pursuer $P$ is to catch Evader $E$ (a pursuit game) at some moment $T_*$, $0 < T_* < +\infty$, i.e. to reach the equality

$$x(T_*) = y(T_*),$$

where $x(t)$ and $y(t)$ are trajectories generated during the game. The notion of "trajectories generated during the game" requires clarification. Evader $E$ tries to avoid the meeting with Pursuer $P$ (an evasion game), i.e. to guarantee the relation $x(t) \neq y(t)$ for all $t \geq 0$, and if it is impossible, to prolong the moment of the meeting as far as possible. Naturally, this is the preliminary problems setting.

**Definition 2.5.** If $u(\cdot) \in U_{\mathbf{G}}$ and $v(\cdot) \in V_{\mathbf{G}}$, then Caratheodory's differential equations

$$\dot{x} = u(t) + F_P\big(t, x\big), \quad x(0) = x_0,$$

$$\dot{y} = v(t) + F_E\big(t, y\big), \quad y(0) = y_0$$

give rise to the unique trajectories $x(t) = x\big(t; x_0, u(\cdot)\big)$ and $y(t) = y\big(t; y_0, v(\cdot)\big)$ correspondingly. In the given case, $x(t)$ is called the Pursuer's trajectory, and $y(t)$ is called the Evader's trajectory.

**Definition 2.6.** A control function $\boldsymbol{u}(t, x, y, v) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be a strategy of Pursuer $P$ if: 1) $\boldsymbol{u}(t, x, y, v)$ is continuous with regard to $x, y, v$ for each fixed $t$; 2) $\boldsymbol{u}(t, x, y, v)$ is Lebesgue measurable with regard to $t$ for each fixed $(x, y, v)$ and is Borel measurable with regard to $v$ for each fixed $(t, x, y)$; 3) $\boldsymbol{u}\big(t, x(\cdot), y(\cdot), v(\cdot)\big) \in U_{\mathbf{G}}$ for all $v(\cdot) \in V_{\mathbf{G}}$; 4) there exists such a Lebesgue-integrable function $w(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ that

$$\big|\boldsymbol{u}\big(t, x_1, y, v(t)\big) - \boldsymbol{u}\big(t, x_2, y, v(t)\big)\big| \le w(t)|x_1 - x_2|$$

for any $\big(t, x_1, y, v(t)\big)$, $\big(t, x_2, y, v(t)\big)$ with $t \ge 0$, $x_1, x_2, y \in \mathbb{R}^n$, $v(\cdot) \in V_{\mathbf{G}}$.

Write

$$z(t) = x(t) - y(t), \; z(0) = z_0, \; z_0 = x_0 - y_0. \tag{2.7}$$

**Definition 2.7.** A strategy $\boldsymbol{u}(t, x, y, v)$ is said to be a parallel pursuit strategy or briefly, **Π**-strategy if for all $v(\cdot) \in V_{\mathbf{G}}$, the function $z(t)$ is representable as

$$z(t) = \mathcal{A}\big(t, x(t), y(t), v(\cdot)\big) z_0, \tag{2.8}$$

where $\big(x(t), y(t)\big)$ is the solution of the system of differential equations

$$\begin{cases} \dot{x} = \boldsymbol{u}\big(t, x, y, v(t)\big) + F_P\big(t, x\big), & x(0) = x_0, \\ \dot{y} = v(t) + F_E\big(t, y\big), & y(0) = y_0, \end{cases} \tag{2.9}$$

and $\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)$ is a scalar monotonically decreasing continuous function with respect to $t$, $t \ge 0$, and it is generally called an approach function of the players $P$ and $E$ in the pursuit game.

**Remark 2.8.** It is necessary to state that similar to the definition of **Π**-strategy for the case of simple motions of the players given in works [3], [4], [26], [28]–[31], the following properties are met: a) the vector $z(t)$ in (2.8) joining the positions of the players changes its position in the parallel way to itself during the pursuit; b) depending on the property of the approach function $\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)$ in Definition 2.7, the distance between the players $|z(t)| = |x(t) - y(t)|$ strictly decreases.

**Definition 2.9.** We say that **Π**-strategy guarantees that Pursuer $P$ wins on the time interval $\big[0, T_{\mathbf{G}}\big]$ in NDG (2.1)–(2.4) if for all $v(\cdot) \in V_{\mathbf{G}}$: a) there exists a time moment $T_* \in \big[0, T_{\mathbf{G}}\big]$ at which the vector function $z(t)$, which is defined by the solutions $x(t)$ and $y(t)$ of system (2.9), meets $z(T_*) = 0$; b) $\boldsymbol{u}\big(t, x(\cdot), y(\cdot), v(\cdot)\big) \in U_{\mathbf{G}}$ on $[0, T_*]$. In the given case, the number $T_{\mathbf{G}}$ is called a guaranteed time of the pursuit.

## 3. The obtained results

The given section is devoted to give solutions of the pursuit and "life-line" game problems for NDG (2.1)–(2.4). First of all, in the pursuit game, a **Π**-strategy is set up for Pursuer $P$, and a sufficient pursuit condition is demonstrated. By this strategy, an explicit formula for a set of all the meeting points of the players is generated. Then in the "life-line" game, a reachability domain of Evader $E$ is constructed.

3.1. **The pursuit game solution.** Write the following functions:

$$\omega = \omega(t, x, y, v) := v + F_E(t, y) - F_P(t, x), \tag{3.1}$$

$$r(\omega) = \langle \omega, \hat{z}_0 \rangle + \sqrt{\langle \omega, \hat{z}_0 \rangle^2 + \alpha^2 - |\omega|^2}, \tag{3.2}$$

where $\hat{z}_0 = \frac{z_0}{|z_0|}$, and $\langle \omega, \hat{z}_0 \rangle$ means the scalar product of the vectors $\omega$ and $\hat{z}_0$ in $\mathbb{R}^n$.

**Definition 3.1.** For $\alpha > |\omega|$, the control function

$$\boldsymbol{u}(\omega) := \omega - r(\omega)\hat{z}_0 \tag{3.3}$$

is called the **Π**-strategy of Pursuer $P$ in the pursuit game.

It should be mentioned that the function $r(\omega)$ is mainly called a *resolving function* [12].

**Lemma 3.2.** *If $\alpha > |\omega|$, then the function $r(\omega)$ is continuous and non-negative in $\omega$, $\omega \in \mathbb{R}^n$, and it is bounded as*

$$\alpha - |\omega| \le r(\omega) \le \alpha + |\omega|. \tag{3.4}$$

*Proof.* As $\alpha > |\omega|$, the function $r(\omega)$ is monotonically increasing with regard to $\langle \omega, \hat{z}_0 \rangle$ (see (3.2)). Thus, applying $-|\omega| \le \langle \omega, \hat{z}_0 \rangle \le |\omega|$ to $r(\omega)$ yields (3.4), which ends the proof. □

**Lemma 3.3.** *If $\alpha > |\omega|$, then the function $\boldsymbol{u}(\omega)$ is continuous in $\omega$, $\omega \in \mathbb{R}^n$, and it satisfies*

$$|\boldsymbol{u}(\omega)| = \alpha.$$

*Proof.* Squaring both sides of (3.3) gives

$$|\boldsymbol{u}(\omega)|^2 = |\omega|^2 + r(\omega)\big[r(\omega) - 2\langle \omega, \hat{z}_0 \rangle\big],$$

and replacing (3.2) into the last expression leads to $|\boldsymbol{u}(\omega)|^2 = \alpha^2$, which is our claim. □

**Lemma 3.4.** *If $\alpha > |\omega|$, then there is a Lebesgue-integrable function $g(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies*

$$\big|\boldsymbol{u}(\omega_1) - \boldsymbol{u}(\omega_2)\big| \le g(t)|x_1 - x_2|$$

*for any $\omega_1, \omega_2 \in \mathbb{R}^n$, where*

$$\omega_1 = \omega(t, x_1, y, v) := v + F_E(t, y) - F_P(t, x_1), \quad \omega_2 = \omega(t, x_2, y, v) := v + F_E(t, y) - F_P(t, x_2).$$

*Proof.* Write $c = \langle \omega, \hat{z}_0 \rangle$, $b = \alpha^2 - |\omega|^2$ and introduce a function $\psi(c) = c + \sqrt{c^2 + b}$ from (3.2). Since $\frac{d\psi(c)}{dc} = 1 + \frac{1}{\sqrt{c^2+b}}$, we can assert that the function $\psi(c)$ is continuous on $\big[c_1, c_2\big]$ and is differentiable at each point of the interval $\big(c_1, c_2\big)$. Thus, on the basis of the Lagrange theorem, there is such a point $c^* \in \big(c_1, c_2\big)$ that

$$\begin{aligned} \psi(c_2) - \psi(c_1) &= \psi(c^*)\big(c_2 - c_1\big) = \psi(c^*)\big(\langle \omega_2, \hat{z}_0 \rangle - \langle \omega_1, \hat{z}_0 \rangle\big) \\ &= \psi(c^*)\big\langle \omega_2 - \omega_1, \hat{z}_0 \big\rangle \le \psi(c^*)\big|\omega_1 - \omega_2\big|. \end{aligned} \tag{3.5}$$

Now, with the help of (3.1), (3.3), (3.5) and by Assumptions 2.3, the function $\boldsymbol{u}(\omega)$ can be estimated for any $\omega_1, \omega_2 \in \mathbb{R}^n$ as follows:

$$\big|\boldsymbol{u}(\omega_1) - \boldsymbol{u}(\omega_2)\big| = \big|v + F_E(t, y) - F_P\big(t, x_1\big) - v - F_E(t, y) + F_P\big(t, x_2\big) - r\big(\omega_1\big)\hat{z}_0 + r\big(\omega_2\big)\hat{z}_0\big|$$

$$\le \big|F_P\big(t, x_1\big) - F_P\big(t, x_2\big)\big| + \big|r\big(\omega_1\big) - r\big(\omega_2\big)\big| = \big|F_P\big(t, x_1\big) - F_P\big(t, x_2\big)\big| + \big|\psi(c_1) - \psi(c_2)\big|$$

$$\le \big|F_P\big(t, x_1\big) - F_P\big(t, x_2\big)\big| + \psi\big(c^*\big)\big|\omega_1 - \omega_2\big| = \big(\psi\big(c^*\big) + 1\big)\big|F_P\big(t, x_1\big) - F_P\big(t, x_2\big)\big| \le g(t)|x_1 - x_2|,$$

where $g(t) = \big(\psi\big(c^*\big) + 1\big)k_P(t)$. This completes the proof. □

**Lemma 3.5.** *(The Grönwall-Bellman inequality* [25, pp. 13, Theorem 1.3.2]*) Let $\eta(t)$ be a real valued continuous function, and let $\phi(t)$ be a non-negative integrable function in respect to $t$, $t \geq 0$. If the integral inequality*

$$|z(t)| \leq \eta(t) + \int_0^t \phi(s)|z(s)|ds$$

*is valid, then*

$$|z(t)| \leq \eta(t) + \int_0^t \phi(s)\eta(s)\exp\left(\int_s^t \phi(\tau)d\tau\right)ds$$

*holds.*

**Theorem 3.6.** *Let $\alpha > \beta + q + p|z_0|$. Then $\mathbf{\Pi}$-strategy (3.3) guarantees that Pursuer $P$ wins on the time interval $[0, T_{\mathbf{G}}]$ in the pursuit game, where*

$$T_{\mathbf{G}} = \begin{cases} \frac{1}{p}\ln\frac{\alpha-\beta-q}{\alpha-\beta-q-p|z_0|}, & \text{if } p > 0, \\ \frac{|z_0|}{\alpha-\beta-q}, & \text{if } p = 0. \end{cases}$$

*Proof.* Suppose, Evader $E$ makes use of an arbitrary control function $v(\cdot) \in V_{\mathbf{G}}$ and Pursuer $P$ realizes $\mathbf{\Pi}$-strategy (3.3). Then by means of (2.9), (3.1), (3.3), we derive the system of Caratheodory's differential equations

$$\begin{cases} \dot{x} = v(t) + F_E(t, y(t)) - r(\omega(t, x(t), y(t), v(t)))\hat{z}_0, & x(0) = x_0, \\ \dot{y} = v(t) + F_E(t, y(t)), & y(0) = y_0, \end{cases} \tag{3.6}$$

where the equations in (3.6) pose the unique trajectories $x(t) := x(t; x_0, \boldsymbol{u}(\cdot))$ and $y(t) := y(t; y_0, v(\cdot))$ of the players $P$ and $E$, respectively. On the basis of (2.7), it proceeds from (3.6) that

$$\dot{z} = -r(\omega(t, x(t), y(t), v(t)))\hat{z}_0, \ z(0) = z_0. \tag{3.7}$$

Integrating equation (3.7), we attain the solution

$$z(t) = \mathcal{A}(t, x(t), y(t), v(\cdot))z_0, \tag{3.8}$$

where

$$\mathcal{A}(t, x(t), y(t), v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t r(\omega(s, x(s), y(s), v(s)))ds. \tag{3.9}$$

By reason of (2.3), (2.4), (3.1), (3.4) and by Proposition 2.4, the function (3.9) is maximized as follows:

$$\mathcal{A}(t, x(t), y(t), v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t r(\omega(s, x(s), y(s), v(s)))ds \leq 1 - \frac{1}{|z_0|}\int_0^t (\alpha - |\omega(s)|)ds$$

$$= 1 - \frac{1}{|z_0|}\left(\alpha t - \int_0^t |v(s) + F_E(s, y(s)) - F_P(s, x(s))|ds\right)$$

$$\leq 1 - \frac{1}{|z_0|}\left((\alpha - \beta)t - \int_0^t |F_P(s, x(s)) - F_E(s, y(s))|ds\right)$$

$$\leq 1 - \frac{1}{|z_0|}\left((\alpha - \beta)t - \int_0^t [p(s)|x(s) - y(s)| + q(s)]ds\right)$$

$$\leq 1 - \frac{1}{|z_0|}\left((\alpha-\beta-q)t - \int_0^t p|z(s)|ds\right),$$

or to sum up,

$$\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big) \leq 1 - \frac{(\alpha-\beta-q)t}{|z_0|} + \int_0^t \frac{p}{|z_0|}|z(s)|ds. \tag{3.10}$$

Combining (3.8) and (3.10) we obtain

$$|z(t)| \leq |z_0| - (\alpha-\beta-q)t + \int_0^t p|z(s)|ds. \tag{3.11}$$

In the right side of (3.11), taking as $\eta(t) = |z_0| - \big(\alpha-\beta-q\big)t$, $\phi(t) = p$ and applying Lemma 3.5 to (3.11) give rise to

$$|z(t)| \leq \mathcal{K}(t), \tag{3.12}$$

where $\mathcal{K}(t) = |z_0| - (\alpha-\beta-q-p|z_0|)\left(e^{pt}-1\right)/p$ if $p > 0$, $\mathcal{K}(t) = |z_0| - (\alpha-\beta-q)t$ if $p = 0$. Since $\alpha > \beta + q + p|z_0|$, substituting the value of $T_{\mathbf{G}}$ (see the theorem) into $\mathcal{K}(t)$ yields $\mathcal{K}\left(T_{\mathbf{G}}\right) = 0$. For this reason and because of (3.12), there is a time value $T_* \in [0, T_{\mathbf{G}}]$ satisfying $z(T_*) = 0$, which is the desired conclusion.

Now let's confirm the admissibility of $\mathbf{\Pi}$-strategy (3.3) for all $t \in [0, T_*]$. To this end, we have to show $\alpha > |\omega(t)|$ on the interval $[0, T_*]$ as specified by Definition 3.1. By means of Proposition 2.4 and by (3.1), we obtain the following estimations from the condition of the theorem:

$$\alpha > \beta + q + p|z_0| \geq |v(t)| + q(t) + p(t)|z(t)|$$

$$\geq |v(t)| + \big|F_E\big(t, y(t)\big) - F_P\big(t, x(t)\big)\big| \geq \big|v(t) + F_E\big(t, y(t)\big) - F_P\big(t, x(t)\big)\big| = |\omega(t)|.$$

This ends the proof of the theorem. □

**Remark 3.7.** If the players $P$ and $E$ move in the same compact subset of $\mathbb{R}^n$, then in (2.5), it is supposed that $p(t) = \min\{k_P(t), k_E(t)\}$ and $q(t) = h_P(t) + h_E(t)$, where $k_P(t)$, $k_E(t)$, $h_P(t)$, $h_E(t)$ are given functions in Assumptions 2.2 and 2.3.

3.2. **The set of meeting points of the players.** In the theory of differential games, after solving the pursuit game problem, it is highly significant that the set of all points, which the players $P$ and $E$ meet, is explicitly constructed.

Let $\mathcal{D}(x, y)$ designate a domain consisting of such all points $d$ that Pursuer $P$ starting its motion from the position $x$ is able to first get through to the point $d$ before Evader $E$ starting its motion from the position $y$, i.e.:

$$\mathcal{D}(x, y) = \big\{d \mid \beta|d-x| \geq \alpha|d-y|\big\}. \tag{3.13}$$

For $\alpha \neq \beta$, the boundary of the domain in (3.13) is represented as

$$\partial\mathcal{D}(x, y) = \big\{d \mid \beta|d-x| = \alpha|d-y|\big\},$$

which is usually said as the *sphere of Apollonius*.

If Theorem 3.6 is satisfied, then, with the help of $\mathbf{\Pi}$-strategy (3.3), Pursuer $P$ can catch Evader $E$ on some point in $\mathbb{R}^n$. For NDG (2.1)–(2.4), we are going to define a meeting domain of the players.

As known, the pair $\big(y_0, v(\cdot)\big)$, $v(\cdot) \in V_{\mathbf{G}}$, produces the Evader's motion trajectory $y(t) := y\big(t; y_0, v(\cdot)\big)$, and the pair $\big(x_0, \boldsymbol{u}(\cdot)\big)$, $\boldsymbol{u}(\cdot) \in U_{\mathbf{G}}$, creates the Pursuer's motion trajectory $x(t) := x\big(t; x_0, \boldsymbol{u}(\cdot)\big)$ for every $t \in \big[0, T_*\big]$, $0 < T_* \leq T_{\mathbf{G}}$, where $T_*$ is the players' meeting time, viz,

$x(T_*) = y(T_*)$ holds at this time. Accordingly, for $(x(t), y(t))$ at each $t \in [0, T_*]$, it makes sense to write the multi-valued mapping

$$\mathcal{D}(x(t), y(t)) = \{d \mid \beta|d - x(t)| \geq \alpha|d - y(t)|\} \tag{3.14}$$

on the interval $[0, T_*]$. Mention that

$$\mathcal{D}(x_0, y_0) = \{d \mid \beta|d - x_0| \geq \alpha|d - y_0|\}.$$

It is apparent that $y(t) \in \mathcal{D}(x(t), y(t))$ is accurate on account of $|z(t)| \geq 0$ on the interval $[0, T_*]$.

**Proposition 3.8.** *It is true to write the multi-valued mapping (3.14) as*

$$\mathcal{D}(x(t), y(t)) = x(t) + \mathcal{A}(t, x(t), y(t), v(\cdot))\,[\mathcal{D}(x_0, y_0) - x_0], \tag{3.15}$$

*where $\mathcal{A}(t, x(t), y(t), v(\cdot))$ is the approach function of the players in (3.9), and*

$$\mathcal{D}(x_0, y_0) = x_0 - \mathcal{C}(z_0) + \mathcal{R}(z_0)\mathcal{B},\ \mathcal{C}(z_0) = \left(\frac{\alpha^2}{\alpha^2 - \beta^2}\right) z_0,\ \mathcal{R}(z_0) = \frac{\alpha\beta|z_0|}{\alpha^2 - \beta^2}, \tag{3.16}$$

$\mathcal{B} = \{b \in \mathbb{R}^n \mid |b| \leq 1\}$.

Set

$$\mathcal{H}(t, x(t), y(t)) = \frac{\alpha^2}{\alpha^2 - \beta^2} \int_0^t \left(F_P(s, x(s)) - F_E(s, y(s))\right) ds$$

$$- \left(\frac{\alpha\beta}{\alpha^2 - \beta^2} \int_0^t \left|F_P(s, x(s)) - F_E(s, y(s))\right| ds\right) \mathcal{B} - \int_0^t F_P(s, x(s)) ds. \tag{3.17}$$

Then, write the multi-valued mapping

$$\mathcal{D}^*(t, x(t), y(t)) = \mathcal{D}(x(t), y(t)) + \mathcal{H}(t, x(t), y(t)) \tag{3.18}$$

**Theorem 3.9.** *$\mathcal{D}^*(t_2, x(t_2), y(t_2)) \subset \mathcal{D}^*(t_1, x(t_1), y(t_1))$ for $t_1 < t_2$ from any $t_1,\ t_2 \in [0, T_*]$.*

*Proof.* First off, let us introduce the following notation for convenience in calculations:

$$\xi(t) = \xi(t, x(t), y(t)) := F_P(t, x(t)) - F_E(t, y(t)). \tag{3.19}$$

Inequality (2.4) can be immediately transformed into the form

$$|v(t)|^2 \leq \frac{\beta^2}{\alpha^2 - \beta^2}\left(\alpha^2 - |v(t)|^2\right). \tag{3.20}$$

Then considering (3.1) and (3.19), inequality (3.20) may be rewritten as

$$|\omega(t) + \xi(t)|^2 \leq \frac{\beta^2}{\alpha^2 - \beta^2}\left(\alpha^2 - |\omega(t) + \xi(t)|^2\right),$$

or from here it is derived that

$$|\omega(t)|^2 + 2\langle\omega(t), \xi(t)\rangle + |\xi(t)|^2 \leq \frac{\beta^2}{\alpha^2 - \beta^2}\left(\alpha^2 - |\omega(t)|^2 - 2\langle\omega(t), \xi(t)\rangle - |\xi(t)|^2\right). \tag{3.21}$$

In accordance with (3.2), it can be readily verified that the following equality holds:

$$\alpha^2 - |\omega(t)|^2 = r(\omega(t))\left(r(\omega(t)) - 2\langle\omega(t), \hat{z}_0\rangle\right) \tag{3.22}$$

Replacing the right-side term of (3.22) into (3.21) leads to the inequality

$$|\omega(t)|^2 + 2\left\langle \omega(t),\ \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 \right\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}|\xi(t)|^2$$

$$\leq \frac{\beta^2}{\alpha^2-\beta^2} r^2(\omega(t)) - 2\left\langle \omega(t),\ \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right\rangle,$$

or

$$|\omega(t)|^2 + 2\left\langle \omega(t),\ \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}|\xi(t)|^2 \leq \frac{\beta^2}{\alpha^2-\beta^2} r^2(\omega(t)). \tag{3.23}$$

We convert both sides of (3.23) into quadratic forms:

$$|\omega(t)|^2 + 2\left\langle \omega(t),\ \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right\rangle + \left| \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right|^2$$

$$\leq \frac{\beta^2}{\alpha^2-\beta^2} r^2(t,\omega(t)) - \frac{\alpha^2}{\alpha^2-\beta^2}|\xi(t)|^2 + \frac{\alpha^4}{(\alpha^2-\beta^2)^2}|\xi(t)|^2 + \frac{\beta^4}{(\alpha^2-\beta^2)^2} r^2(\omega(t))$$

$$+2\left(\frac{\alpha\beta}{\alpha^2-\beta^2}\right)^2 \langle r(\omega(t))\hat{z}_0, \xi(t)\rangle.$$

or we get

$$\left| \omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right|^2$$

$$\leq \left(\frac{\alpha\beta}{\alpha^2-\beta^2}\right)^2 \left( r^2(\omega(t)) + 2\langle r(\omega(t))\hat{z}_0, \xi(t)\rangle + |\xi(t)|^2 \right),$$

that is,

$$\left| \omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right| \leq \frac{\alpha\beta}{\alpha^2-\beta^2} |r(\omega(t))\hat{z}_0 + \xi(t)|. \tag{3.24}$$

It is evident that for any vector $\psi \in \mathbb{R}^n$, with $|\psi| = 1$, the following relation is true:

$$\left\langle \omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t), \psi \right\rangle \leq \left| \omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right|$$

Applying the latter inequality to (3.24), we obtain

$$\left\langle \omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t), \psi \right\rangle \leq \frac{\alpha\beta}{\alpha^2-\beta^2} |r(\omega(t))\hat{z}_0 + \xi(t)|. \tag{3.25}$$

The left-side term of (3.25) may be rewritten as:

$$\left\langle \omega(t) + \left(\frac{\alpha^2}{\alpha^2-\beta^2} - 1\right) r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t), \psi \right\rangle = \langle \omega(t) - r(\omega(t))\hat{z}_0, \psi\rangle +$$

$$+\frac{\alpha^2}{\alpha^2-\beta^2} \langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle = \langle \boldsymbol{u}(\omega(t)), \psi\rangle + \frac{\alpha^2}{\alpha^2-\beta^2} \langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle.$$

From the last equality and from (3.25), it is achieved that

$$\langle \boldsymbol{u}(\omega(t)), \psi\rangle + \frac{\alpha^2}{\alpha^2-\beta^2} \langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle - \frac{\alpha\beta}{\alpha^2-\beta^2} |r(\omega(t))\hat{z}_0 + \xi(t)| \leq 0. \tag{3.26}$$

The multi-valued mapping $\mathcal{D}\big(x(t), y(t)\big)$ is, by and large, regarded as the ball with center and radius changing in time. Thus, a support function $c\Big(\mathcal{D}\big(x(t), y(t)\big), \psi\Big)$ of $\mathcal{D}\big(x(t), y(t)\big)$ can be defined for arbitrary $\psi \in \mathbb{R}^n$, $|\psi| = 1$ (see [10, pp. 68]), and this enables to determine a support function

$$c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big) = \sup_{d \in \mathcal{D}^*(t, x(t), y(t))} \langle d, \psi \rangle$$

of the multi-valued mapping $\mathcal{D}^*\big(t, x(t), y(t)\big)$ as well. Now, we compute the $t$-derivative of $c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big)$ by the properties of a support function (see [10, Property 1, pp. 34; Property 3, pp. 35; Theorem 1, pp. 67]). To do this, from (2.1), (3.3), (3.9), (3.15)–(3.19), (3.26) it is derived that

$$\begin{aligned}
\frac{d}{dt} c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big) &= \frac{d}{dt} c\Big(\mathcal{D}\big(x(t), y(t)\big) + \mathcal{H}\big(t, x(t), y(t)\big), \psi\Big) \\
&= \frac{d}{dt} c\Big(x(t) + \mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)\big[\mathcal{D}(x_0, y_0) - x_0\big], \psi\Big) \\
&+ \frac{d}{dt} c\left(\frac{\alpha^2}{\alpha^2 - \beta^2}\int_0^t \xi(s)ds - \left(\frac{\alpha\beta}{\alpha^2 - \beta^2}\int_0^t |\xi(s)|ds\right)\mathcal{B} - \int_0^t F_P\big(s, x(s)\big)ds, \psi\right) \\
&= \langle \boldsymbol{u}(\omega(t)), \psi \rangle + \langle F_P(t, x(t)), \psi \rangle + \frac{\alpha^2}{\alpha^2 - \beta^2}\langle r(\omega(t))\hat{z}_0, \psi \rangle - \frac{\alpha\beta}{\alpha^2 - \beta^2} r(\omega(t)) \\
&+ \frac{\alpha^2}{\alpha^2 - \beta^2}\langle \xi(t), \psi \rangle - \frac{\alpha\beta}{\alpha^2 - \beta^2}|\xi(t)| - \langle F_P(t, x(t)), \psi \rangle \\
&= \langle \boldsymbol{u}(\omega(t)), \psi \rangle + \frac{\alpha^2}{\alpha^2 - \beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi \rangle - \frac{\alpha\beta}{\alpha^2 - \beta^2}\left(|r(\omega(t))\hat{z}_0| + |\xi(t)|\right) \\
&\leq \langle \boldsymbol{u}(\omega(t)), \psi \rangle + \frac{\alpha^2}{\alpha^2 - \beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi \rangle - \frac{\alpha\beta}{\alpha^2 - \beta^2}|r(\omega(t))\hat{z}_0 + \xi(t)|.
\end{aligned}$$

In consequence, we get

$$\begin{aligned}
\frac{d}{dt} c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big) \leq \langle \boldsymbol{u}(\omega(t)), \psi \rangle &+ \frac{\alpha^2}{\alpha^2 - \beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi \rangle \\
&- \frac{\alpha\beta}{\alpha^2 - \beta^2}|r(\omega(t))\hat{z}_0 + \xi(t)|.
\end{aligned}$$

Referring to (3.26), we conclude that the relation $\frac{d}{dt} c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big) \leq 0$ holds for all $t \in [0, T_*]$ and for any $\psi \in \mathbb{R}^n$, $|\psi| = 1$. The proof is now complete. □

**Lemma 3.10.** *For an arbitrary control $v(\cdot) \in V_{\mathbf{G}}$, the following inclusions are satisfied for all $t \in [0, T_*]$:*
*1) $\mathcal{D}\big(x(t), y(t)\big) \subset \mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big)$;*
*2) $y(t) \in \mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big)$.*

*Proof. 1)* We should say that Theorem 3.9 implies

$$\mathcal{D}^*\big(t, x(t), y(t)\big) \subset \mathcal{D}^*\big(0, x(0), y(0)\big).$$

For this reason, from the views of $\mathcal{D}^*\big(t, x(t), y(t)\big)$ and $\mathcal{H}\big(t, x(t), y(t)\big)$ in (3.17)–(3.18) the following arise:

$$\mathcal{D}\big(x(t), y(t)\big) + \mathcal{H}\big(t, x(t), y(t)\big) \subset \mathcal{D}^*\big(0, x(0), y(0)\big) = \mathcal{D}(x_0, y_0),$$

or we can write

$$\mathcal{D}\big(x(t), y(t)\big) \subset \mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big). \tag{3.27}$$

*2)* It is evident from (3.14) that $y(t) \in \mathcal{D}\big(x(t), y(t)\big)$ for $t \in [0, T_*]$, and accordingly, we see that $y(t) \in \mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big)$ is valid owing to (3.27). □

On the strength of Theorem 3.9 and Lemma 3.10, we define the set of meeting points of the players in the pursuit game.

**Definition 3.11.** For an arbitrary control $v(\cdot) \in V_{\mathbf{G}}$, we call

$$\mathcal{D}_P(x_0, y_0, T_{\mathbf{G}}) = \bigcup_{t=0}^{T_{\mathbf{G}}} \Big(\mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big)\Big) \tag{3.28}$$

the set of meeting points of the players, where $T_{\mathbf{G}}$ is defined in Theorem 3.6.

3.3. **The "life-line" game solution.** Consider a closed subset $\mathcal{L}$, referred to as a "life-line", in the space $\mathbb{R}^n$. The Pursuer aims to capture the Evader before the Evader reaches the set $\mathcal{L}$, meaning that there exists a time $\widehat{T} > 0$ such that their positions coincide, i.e., $x(\widehat{T}) = y(\widehat{T})$. Meanwhile, the Evader's goal is to either to reach the subset $\mathcal{L}$ before being captured or to carry on the inequality $x(t) \neq y(t)$ for all $t \geq 0$. Notably, the Pursuer's motion is not restricted by the subset $\mathcal{L}$. Additionally, it is supposed that the initial positions $x_0$ and $y_0$ are given under the conditions $x_0 \neq y_0$ and $y_0 \neq \mathcal{L}$.

**Definition 3.12.** It is said that $\mathbf{\Pi}$-strategy (3.3) guarantees that Pursuer $P$ wins on the time interval $\big[0, T_{\mathbf{G}}\big]$ in the "life-line" game, if there exists a time $\widehat{T} \in \big[0, T_{\mathbf{G}}\big]$ such that:

***(i)*:** $x\Big(\widehat{T}\Big) = y\Big(\widehat{T}\Big)$;

***(ii)*:** $y(t) \neq \mathcal{L}$ for all $t \in \Big[0, \widehat{T}\Big]$.

**Definition 3.13.** We say that a control $\boldsymbol{v}_{\mathcal{L}}(\cdot) \in V_{\mathbf{G}}$ guarantees that Evader $E$ wins in the "life-line" game if, for any control $u(\cdot) \in U_{\mathbf{G}}$:

***(i)*:** there exists a finite time $T_{\mathcal{L}}$ such that $y(T_{\mathcal{L}}) \in \mathcal{L}$ and $x(t) \neq y(t)$ for all $t \in \big[0, T_{\mathcal{L}}\big)$;

***(ii)*:** $x(t) \neq y(t)$ for all $t \geq 0$.

**Theorem 3.14.** *Let $\alpha > \beta + q + p|z_0|$ and $\mathcal{L} \cap \mathcal{D}_P\big(x_0, y_0, T_{\mathbf{G}}\big) = \varnothing$. Then $\mathbf{\Pi}$-strategy (3.3) guarantees that the Pursuer wins on the time interval $\big[0, T_{\mathbf{G}}\big]$ in the "life-line" game, where $T_{\mathbf{G}}$ is the guaranteed time of the pursuit.*

*Proof.* The proof arises instantly from Theorems 3.6 and 3.9 and from Lemma 3.10. □

Our next concern will be solving the "life-line" game to the advantage of Evader $E$.
First off, define the following set:

$$\overline{\mathcal{D}}(x_0, y_0) = \big\{d \mid \beta|d - x_0| \geq (\alpha + p|z_0| + q)|d - y_0|\big\}. \tag{3.29}$$

**Lemma 3.15.** *In accord with the definitions of $\overline{\mathcal{D}}(x_0, y_0)$ and $\mathcal{D}(x_0, y_0)$, the inclusion*

$$\overline{\mathcal{D}}(x_0, y_0) \subset \mathcal{D}(x_0, y_0)$$

*is satisfied.*

*Proof.* Due to $p \geq 0$, $q \geq 0$ (see Proposition 2.4) and owing to $\frac{\alpha + p|z_0| + q}{\beta} \geq \frac{\alpha}{\beta}$, we have

$$\frac{\alpha + p|z_0| + q}{\beta}|d - y_0| \geq \frac{\alpha}{\beta}|d - y_0|. \tag{3.30}$$

Combining the inequality in (3.29) with (3.30) yields

$$|d - x_0| \geq \frac{\alpha + p|z_0| + q}{\beta}|d - y_0| \geq \frac{\alpha}{\beta}|d - y_0|.$$

From here, it follows

$$|d - x_0| \geq \frac{\alpha}{\beta}|d - y_0|.$$

The lemma is proved. □

**Definition 3.16.** The set

$$\mathcal{D}_E(y_0, T_\mathcal{L}, \boldsymbol{v}_\mathcal{L}) = \{d_E \mid d_E = y(y_0, T_\mathcal{L}, \boldsymbol{v}_\mathcal{L})\}, \tag{3.31}$$

is said to be a reachability domain of the Evader in the "life-line" game.

**Theorem 3.17.** *Let $\alpha > \beta + q + p|z_0|$ and $\mathcal{L} \cap \mathcal{D}_E(y_0, T_\mathcal{L}, \boldsymbol{v}_\mathcal{L}) \neq \varnothing$. Then there exists a control $\boldsymbol{v}_\mathcal{L}(\cdot) \in V_\mathbf{G}$ which guarantees that Evader E wins in the "life-line" game.*

*Proof.* In accordance with the second condition in the theorem, there is at least one point $d_E \in \mathcal{D}_E(y_0, T_\mathcal{L}, \boldsymbol{v}_\mathcal{L}) \cap \mathcal{L}$ that

$$d_E = y(y_0, T_\mathcal{L}, \boldsymbol{v}_\mathcal{L}).$$

From Definition 3.16, if the Evader employs $\boldsymbol{v}_\mathcal{L}$, then it gets to the point $d_E$ at the time $T_\mathcal{L}$.

Let

$$T_\mathcal{L} := T_\mathcal{L}(d) = \frac{|d - y_0|}{\beta}, \quad d \in \overline{\mathcal{D}}(x_0, y_0), \tag{3.32}$$

and

$$\boldsymbol{v}_\mathcal{L} := \boldsymbol{v}_\mathcal{L}(d) = \beta\frac{d - y_0}{|d - y_0|}, \quad d \in \overline{\mathcal{D}}(x_0, y_0). \tag{3.33}$$

Now, we prove that the condition ***(i)*** of Definition 3.13 is satisfied, more precisely, the Evader remains uncaught. Let us suppose the opposite, that is, there exists some control $\tilde{u}(\cdot) \in U_\mathbf{G}$ of the Pursuer giving rise to $x(t) = y(t)$ at a time $\widetilde{T}$ less than $T_\mathcal{L}$, i.e., $\widetilde{T} < T_\mathcal{L}$. Due to denotations (2.7), we generate the initial value problem

$$\dot{z}(t) = \tilde{u}(t) - \boldsymbol{v}_\mathcal{L} + F_P(t, x(t)) - F_E(t, y(t)), \ z(0) = z_0,$$

and integrate both sides of this equation, we obtain

$$z(t) = z_0 + \int_0^t \tilde{u}(s)ds - \int_0^t \boldsymbol{v}_\mathcal{L} ds + \int_0^t \big(F_P(s, x(s)) - F_E(s, y(s))\big)ds. \tag{3.34}$$

Consequently, equation (3.34) allows us to write

$$z\big(\widetilde{T}\big) = z_0 + \int_0^{\widetilde{T}} \tilde{u}(s)ds - \int_0^{\widetilde{T}} \boldsymbol{v}_\mathcal{L} ds + \int_0^{\widetilde{T}} \big(F_P(s, x(s)) - F_E(s, y(s))\big)ds = 0. \tag{3.35}$$

In essence, depending on how the control $\tilde{u}(\cdot) \in U_\mathbf{G}$ is chosen, the Pursuer can chase the Evader along different motion trajectories. In particular, the distance between the players may first increase and then decrease, or in the second case, it may continuously decrease from the initial distance $|z_0|$ at the start of the game. Therefore, the time $\widetilde{T}$ will be less in the second case rather than the first one. For this reason, it suffices to consider the second case, i.e., $|z(t)| \leq |z_0|$ for all $t \in [0, \widetilde{T}]$, to prove the condition ***(i)*** of Definition 3.13. Hence, by dint of (2.3), (2.5), (2.6), we carry out the following estimates in (3.35):

$$\left| z_0 - \int_0^{\widetilde{T}} \boldsymbol{v}_\mathcal{L} ds \right| \leq \int_0^{\widetilde{T}} |\tilde{u}(s)|ds + \int_0^{\widetilde{T}} \big|F_P(s, x(s)) - F_E(s, y(s))\big|ds$$

$$\leq \int_0^{\widetilde{T}} |\tilde{u}(s)|ds + \int_0^{\widetilde{T}} \big(p(s)|z(s)| + q(s)\big)ds \leq (\alpha + p|z_0| + q)\widetilde{T},$$

or

$$\left| z_0 - \boldsymbol{v}_\mathcal{L}\widetilde{T} \right| \leq (\alpha + p|z_0| + q)\widetilde{T}. \tag{3.36}$$

Taking the square of both sides of (3.36) and taking account of $|\boldsymbol{v}_{\mathcal{L}}| = \beta$, we get

$$\big((\alpha + p|z_0| + q)^2 - \beta^2\big)\widetilde{T}^2 + 2\widetilde{T}\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle - |z_0|^2 \geq 0.$$

From the latter, the solution is

$$\widetilde{T} \geq \frac{1}{(\alpha + p|z_0| + q)^2 - \beta^2}\Big(\sqrt{\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2 + |z_0|^2\big((\alpha + p|z_0| + q)^2 - \beta^2\big)} - \langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle\Big). \tag{3.37}$$

In light of the assumption $T_{\mathcal{L}} > \widetilde{T}$ it is derived from (3.32) and (3.37) that

$$\frac{|d - y_0|}{\beta} > \frac{1}{(\alpha + p|z_0| + q)^2 - \beta^2}\Big(\sqrt{\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2 + |z_0|^2\big((\alpha + p|z_0| + q)^2 - \beta^2\big)} - \langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle\Big). \tag{3.38}$$

We obtain the following inequality from (3.29):

$$\beta|d - x_0| \geq (\alpha + p|z_0| + q)|d - y_0|.$$

From this and from $z_0 = x_0 - y_0$ (see (2.7)) it is taken that

$$\beta^2|z_0 - (d - y_0)|^2 \geq (\alpha + p|z_0| + q)^2|d - y_0|^2,$$

and this reduces to the following form after using (3.33) and making some computations:

$$|z_0|^2 \geq \frac{|d - y_0|^2}{\beta^2}\big((\alpha + p|z_0| + q)^2 - \beta^2\big) + \frac{2|d - y_0|}{\beta}\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle. \tag{3.39}$$

Now, firstly, multiplying both sides of (3.39) by $\big((\alpha + p|z_0| + q)^2 - \beta^2\big)$ and then adding $\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2$ to both sides gives

$$\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2 + \big((\alpha + p|z_0| + q)^2 - \beta^2\big)|z_0|^2 \geq \left[\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle + \frac{|d - y_0|}{\beta}\big((\alpha + p|z_0| + q)^2 - \beta^2\big)\right]^2,$$

or

$$\frac{1}{(\alpha + p|z_0| + q)^2 - \beta^2}\left(\sqrt{\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2 + |z_0|^2\big((\alpha + p|z_0| + q)^2 - \beta^2\big)} - \langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle\right) \geq \frac{|d - y_0|}{\beta}. \tag{3.40}$$

In the light of (3.38) and (3.40), we meet a contradiction. This concludes the proof. □

**Corollary 3.18.** *In accord with the definitions of (3.28) and (3.31), it is confirmed that*

$$\mathcal{D}_E\big(y_0, T_{\mathcal{L}}, \boldsymbol{v}_{\mathcal{L}}\big) \subset \mathcal{D}_P\big(x_0, y_0, T_{\mathbf{G}}\big)$$

*is accurate.*

## 4. Examples

**Example 1.** Consider the differential game

$$P: \quad \dot{x} = u + x + z_0\cos^2 t, \quad x(0) = x_0, \quad |u(t)| \leq \alpha, \tag{4.1}$$

$$E: \quad \dot{y} = v + y - z_0\sin^2 t, \quad y(0) = y_0, \quad |v(t)| \leq \beta, \tag{4.2}$$

respectively, where $x, y, u, v, z_0 \in \mathbb{R}^n$, $n \geq 2$. $z_0 = x_0 - y_0$.

For Proposition 2.4, we can take as $p = 1$ and $q = |z_0|$. Thus, the function $\mathcal{K}(t)$ in (3.12) will be as follows: $\mathcal{K}(t) = |z_0| - (\alpha - \beta - 2|z_0|)\,[\exp(t) - 1]$. Then we will give the following result.

**Theorem 4.1.** *Let $\alpha > \beta + 2|z_0|$. Then $\mathbf{\Pi}$-strategy (3.3) guarantees that Pursuer $P$ wins on the time interval $\big[0, T_{\mathbf{G}}\big]$ in the pursuit game (4.1)–(4.2), where $T_{\mathbf{G}} = \ln\frac{\alpha-\beta-|z_0|}{\alpha-\beta-2|z_0|}$.*

**Example 2.** Consider the differential game

$$P: \quad \dot{x} = u + sin(2e^{-t}x), \qquad x(0) = x_0, \quad |u(t)| \leq \alpha, \tag{4.3}$$

$$E: \quad \dot{y} = v + cos\left(\frac{1}{t^2+1}y\right), \quad y(0) = y_0, \quad |v(t)| \leq \beta, \tag{4.4}$$

where $x, y, u, v \in \mathbb{R}^n$. Here it is obvious that

$$|F_P(t,x)| = \left|\sin\left(2e^{-t}x\right)\right| \leq 1 = h_P^*,$$

$$|F_E(t,y)| = \left|\cos\left(\frac{1}{t^2+1}y\right)\right| \leq 1 = h_E^*.$$

To compute Lipschitz constants for $F_P(t,x) = \sin\left(2e^{-t}x\right)$ and $F_E(t,y) = \cos\left(\frac{1}{t^2+1}y\right)$, we use the following statement.

**Lemma 4.2.** *([9]) Let $f : [a,b] \times \mathbb{D} \to \mathbb{R}^m$ be continuous for some domain $\mathbb{D} \subset \mathbb{R}^n$. Suppose that $[\partial f/\partial x]$ exists and is continuous on $[a,b] \times \mathbb{D}$. If, for a convex subset $W \subset \mathbb{D}$, there exists a constant $L \geq 0$ such that $\left|\frac{\partial f}{\partial x}\right| \leq L$ on $[a,b] \times W$, then the Lipschitz condition*

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

*is satisfied for all $t \in [a,b]$, $x, y \in W$.*

According to this property, we get

$$\left|\frac{\partial F_P(t,x)}{\partial x}\right| = \left|2e^{-t}\cos\left(2e^{-t}x\right)\right| \leq 2 = L_1, \quad \left|\frac{\partial F_E(t,y)}{\partial y}\right| = \left|-\frac{1}{t^2+1}\sin\left(\frac{1}{t^2+1}y\right)\right| \leq 1 = L_2.$$

From this and from Assumption 2.3 it follows that

$$|F_P(t,x_1) - F_P(t,x_2)| \leq k_P(t)|x_1 - x_2| = L_1|x_1 - x_2|,$$

$$|F_E(t,y_1) - F_E(t,y_2)| \leq k_E(t)|x_1 - x_2| = L_2|y_1 - y_2|.$$

Consequently, for Proposition 2.4, we find $p = k_P^* + k_E^* = 2 + 1 = 3$ and $q = h_P^* + h_E^* = 1 + 1 = 2$. Thus, the function $\mathcal{K}(t)$ in (3.12) will be as follows: $\mathcal{K}(t) = |z_0| - (\alpha - \beta - 2 - 3|z_0|)\left[\exp(3t) - 1\right]/3$. Then we will give the following result.

**Theorem 4.3.** *Let $\alpha > \beta + 3|z_0| + 2$. Then $\mathbf{\Pi}$-strategy (3.3) guarantees that Pursuer $P$ wins on the time interval $\left[0, T_{\mathbf{G}}\right]$ in the pursuit game (4.3)–(4.4), where $T_G = \frac{1}{3}\ln\frac{\alpha-\beta-2}{\alpha-\beta-3|z_0|-2}$.*

## References


[1] Anderson R.P., Bakolas E., Milutinović D., and Tsiotras P.; Optimal Feedback Guidance of a Small Aerial Vehicle in a Stochastic Wind. Journal of Guidance, Control, and Dynamics,–2013.–36.–No4.–P.975–985. https://doi.org/10.2514/1.59512

[2] Azamov A.; A class of nonlinear differential games. Mathematical notes of the Academy of Sciences of the USSR,–1981.–30.–issue 4.–P.805–808. https://doi.org/10.1007/BF01137812

[3] Azamov A.; About the quality problem for the games pursuit with the restriction. Serdica Bulgarian math.,–1986.–12.–No.2.–P.38–43.

[4] Azamov A., Samatov B.T.; The Π-Strategy: Analogies and Applications. GTM 2010: The fourth international conference on Game Theory and Management, St. Petersburg University,–2010.–No.4.–P.33–47.

[5] Azamov A.A., Samatov B.T., Soyibboev U.B.; The game with a "life-line" for simple garmonic motions of objects. International Game Theory Review,–2024.–IGTR.151.–P.2450009-1–2450009-29. DOI: 10.1142/S0219198924500099

[6] Azamov A.A., Turgunboeva M.A.; The differential game with inertial players under integral constraints on controls. Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes,–2025.–21.–1.–P.122–138.

[7] Bakolas E.; Optimal Guidance of the Isotropic Rocket in the Presence of Wind. Journal of Optimization Theory and Applications.,–2014.–162.–No.3.–P.954–974. doi:10.1007/s10957–013–0504–4

[8] Berkovitz L.D.; Differential game of generalized pursuit and evasion. SIAM J. Contr.,–1986.–24.–No.3.–P.361–373. DOI 10.1137/0324021

[9] Birkhoff G., Rota G.C.; Ordinary differential equations. John Wiley and Sons, New York.–1989.

[10] Blagodatskikh V.I.; Introduction to Optimal Control. Vysshaya Shkola, Moscow.–1979 (in Russian).

[11] Blagodatskikh A.I., Petrov N.N.; Conflict interaction of groups of controlled objects. Izhevsk: Udmurt State University.–2009 (in Russian).

[12] Chikrii A.A.; Conflict-Controlled Processes. Kluwer, Dordrecht.–1997. DOI 10.1007/978-94-017-1135-7

[13] Dubins L.E.; On Curves of Minimal Length with a Constraint on Average Curvature, and with Prescribed Initial and Terminal Positions and Tangents. American Journal of Mathematics,–1957.–79.–No.3.–P.497–516. doi:10.2307/2372560

[14] Elliot R.J., Kalton N.J.; The existence of value in differential games of pursuit and evasion. Journal of Differential Equations,–1972.–12.–No.3.–P.504–523. DOI 10.1016/0022-0396(72)90022-8

[15] Friedman A.; Differential Games (2nd ed.). Pure and Applied Mathematics, Wiley Interscience, New York.–1971.–25.

[16] Grigorenko N.L.; Mathematical Methods of Control for Several Dynamic Processes. Izdat. Mos. Univ., Moscow.–1990.

[17] Hajek O.; Pursuit Games: An Introduction to the Theory and Applications of Differential Games of Pursuit and Evasion. Dove. Pub. New York.–2008.

[18] Ho Y., Bryson A., Baron S.; Differential games and optimal pursuit-evasion strategies. IEEE Trans Autom Control,–1965.–10.–P.385-389. DOI 10.1109/TAC.1965.1098197

[19] Ibragimov G.I.; The optimal pursuit problem reduced to an infinite system of differential equations. J. Appl. Maths Mekhs.–2013.–77.–No.5.–P.470–476. http://dx.doi.org/10.1016/j.jappmathmech.2013.12.002.

[20] Ibragimov G.I.; Optimal pursuit time for a differential game in the Hilbert Space $l_2$. Science Asia,–2013.–39.–No.S.–P.25–30. DOI 10.2306/scienceasia1513-1874.2013.39S.025

[21] Isaacs R; Differential Games. John Wiley and Sons, New York,–1965.

[22] Krasovskii N.N.; Control of a Dynamical System. Nauka, Moscow.–1985 (in Russian).

[23] McNeely R.L., Iyer R.V., Chandler P.R.; Tour Planning for an Unmanned Air Vehicle Under Wind Conditions. Journal of Guidance, Control, and Dynamics,–2007.–3.–No.5.–P.1299–1306. doi:10.2514/1.26055

[24] Natarajan T., Pierre D.A., Naadimuthu G., Lee E.S.; Piecewise suboptimal control laws for differential games. Journal of Mathematical Analysis and Applications,–1984.–104.–issue 1.–P.189–211. https://doi.org/10.1016/0022-247X(84)90042-8

[25] Pachpatte B.G.; Inequalities for Differential and Integral Equations. Mathematics in Science and Engineering Academic, London.–1998.–197.

[26] Petrosyan L.A.; Differential games of pursuit (Series on Optimization). World Scientific Pob., Singapore,–1993.

[27] Pontryagin L.S.; Selected Works. CRC Press, New York,–2004.

[28] Pshenichnii B.N.; Simple pursuit by several objects. Cybernetics and System Analysis,–1976,–12.–No.5.–P.484–485.

[29] Samatov B.T.; Problems of group pursuit with integral constraints on controls of the players I. Cybernetics and Systems Analysis,–2013.–49.–No.5.-P.756–767.

[30] Samatov B.T.; Problems of group pursuit with integral constraints on controls of the players II. Cybernetic and Systems Analysis,–2013.–49.–No.6.–P.907–921.

[31] Samatov B.T., Sotvoldiyev A.I.; Intercept problem in dynamic flow field. Uzbek Mathematical Journal.–2019.–No.2.–P.103–112.

[32] Satimov N.Yu.; Methods to Solve Pursuit Problems in Theory Differential Games. NUU Press, Tashkent,–2003. (in Russian)

[33] Satimov N.; Pursuit problems in nonlinear differential games. Cybernetics.–1973.–9.–issue 3.–P.469–475. https://doi.org/10.1007/BF01069203

[34] Shchelchkov K.A.; On the problem of controlling a nonlinear system by a discrete control under disturbance, Differential Equations,–2024.–60.–issue 1.–P.127–135. https://doi.org/10.1134/S0012266124010105

[35] Sun W., Tsiotras P.; Pursuit evasion game of two players under an external flow field. Proceedings of the American Control Conference. Los Angeles, CA, USA, P.–2015.–P.5617–5622. DOI: 10.1109/ACC.2015.7172219

[36] Sun W., Tsiotras P., Lolla T., Subramani D.N., Lermusiaux P.F.J.; Multiple-pursuit/one-evader pursuit-evasion game in dynamical flow fields. J. of Guidance, Control, and Dynamics.–2017.–40.–No.7.–P.1627–1637. https://doi.org/10.2514/1.G002125

Turgunboeva M.A.,
Department of Mathematics,
Namangan State University,
Namangan, Uzbekistan
email: turgunboyevamohisanam95@gmail.com

Soyibboev U.B.,
Department of Applied Mathematics,
Andijan State University,
Andijan, Uzbekistan
email: ulmasjonsoyibboev@gmail.com