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Vanishing of the second cohomology groups of certain solvable Lie algebras

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Abstract. In this article, we consider cohomology groups of maximal solvable Lie algebras. We give necessary and sufficient conditions for cohomological rigidity of certain solvable Lie algebras.
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1. Introduction

An extensive study of Lie algebras has yielded many beautiful results and generalizations. In the classical theory of finite-dimensional Lie algebras, it is known that any Lie algebra over a field of characteristic zero decomposes into a semidirect sum of its solvable radical and its semisimple subalgebra, according to Levi's theorem. Thanks to the results of Malcev and Mubarakzjanov, the study of non-nilpotent solvable Lie algebras is reduced to the analysis of nilpotent algebras and their derivations [14], [11]. Consequently, the focus of research on finite-dimensional Lie algebras has shifted to nilpotent algebras and the representations of semisimple algebras. Numerous studies have been dedicated to solvable Lie algebras with given nilradicals [10], [13], [4], [6], [15]. Maximal solvable Lie algebras, which form an important class of Lie algebras, can be analyzed through their cohomological properties. This research explores the conditions under which the second cohomology group of such algebras becomes trivial, offering insights into their deeper structure and potential applications in representation theory and algebraic topology.

The investigation of cohomology groups of Lie algebras has been a central focus in the fields of mathematical physics and algebraic geometry. Cohomology problems have been explored in many papers [1], [2], [3], [5]. In particular, the second cohomology group, $H^2(\mathcal{L}, K)$, where $\mathcal{L}$ is a Lie algebra and K is a field, plays a significant role in understanding the structure and classification of Lie algebras. This article specifically examines when the second cohomology group of certain maximal solvable Lie algebras vanishes. The goal of this work is to establish specific criteria that determine when the second cohomology groups of these algebras are trivial, under a variety of necessary and sufficient conditions. F.Leger and E.Luks [9] provided some necessary conditions for cohomological rigidity in certain solvable algebras. We build on their work by generalizing these conditions and providing additional sufficient conditions for cohomological rigidity under specific restrictions.

Unless otherwise stated, any Lie algebra considered in this work is finite-dimensional and $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$. Here, $\mathcal{T}$ is the maximal torus of $\mathcal{N}$, $\mathcal{N} = \mathcal{N}_{\alpha_1} \oplus \mathcal{N}_{\alpha_2} \oplus \cdots \oplus \mathcal{N}_{\alpha_n}$, $\mathcal{N}_{\alpha_i}$ with representing the root subspaces with respect to the maximal torus in $\mathcal{N}$. Additionally, $\dim \mathcal{N}_\alpha = 1$ for all $\alpha \in W$, $rank(\mathcal{N}) = s$ and zero is not in W.

2. Preliminaries

Definition 2.1. A vector space with a bilinear bracket $(\mathcal{L}, [-,-])$ over a field of $\mathbb{F}$ is called a Lie algebra if for any $x, y, z \in \mathcal{L}$ the following identities hold:

$$[x, y] = -[y, x]$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

For a given Lie algebra $\mathcal{L}$ we define the *descending central sequence* and the *derived sequence* in the following recursive way:

$$\mathcal{L}^1 = \mathcal{L},\ \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}],\ \text{and}\ \mathcal{L}^{[1]} = \mathcal{L},\ \mathcal{L}^{[k+1]} = [\mathcal{L}^{[k]}, \mathcal{L}^{[k]}], \quad k \geq 1,\ \text{respectively.}$$

Definition 2.2. A Lie algebra $\mathcal{L}$ is called nilpotent (respectively, solvable) if there exists $s \in \mathbb{N}$ (respectively, $k \in \mathbb{N}$) such that $\mathcal{L}^s = 0$ (respectively, $\mathcal{L}^{[k]} = 0$.)

For a given Lie algebra $\mathcal{R}$, let $C^k(\mathcal{R}, M)$ be the space of all alternating F-linear homogeneous mappings $\wedge^n\mathcal{R} \to M,\ k \geq 0$ and $C^0(\mathcal{R}, M) = M$. Let $d^k : C^k(\mathcal{R}, M) \to C^{k+1}(\mathcal{R}, M)$ be an F-homomorphism defined by

$$\begin{aligned}(d^k f)(x_1, \dots, x_{k+1}) : &= \sum_{i=1}^{k+1}(-1)^{i+1}[x_i, f(x_1, \dots, \widehat{x}_1, \dots, x_{k+1})]+ \\ &+ \sum_{1\leq i<j\leq k+1}(-1)^{i+j} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots \widehat{x}_j, \dots, x_{k+1}),\end{aligned}$$

where $d^k f \in Z^k(R, M)$ and $x_i \in \mathcal{R}$. Since the derivative operator $d = \sum\limits_{i\geq 0} d^i$ satisfies the property $d \circ d = 0$, the k-th cohomology group well defined and

$$H^k = Z^k(\mathcal{R}, M)/B^k(\mathcal{R}, M),$$

where elements of $Z^k(R, M) := Kerd^k$ and $B^k(\mathcal{R}, M) :=\ Imd^{k-1}$ are called k-cocycles and k-coboundaries, respectively.

Hoschild-Serre factorization theorem simplifies computations of cohomology groups for semidirect sums of algebras [12].

Theorem 2.3. *If $\mathcal{R} = \mathcal{N} \oplus \mathcal{Q}$ is a solvable Lie algebra such that $\mathcal{Q}$ is Abelian and operators $ad_{\mathcal{R}}t$ ($t \in \mathcal{Q}$) are diagonal, then the adjoint cohomology $H^p(\mathcal{R}, \mathcal{R})$ satisfies the following isomorphism*

$$H^p(\mathcal{R}, \mathcal{R}) \cong \sum_{a+b=p} H^a(\mathcal{Q}, \mathbb{K}) \otimes H^b(\mathcal{N}, \mathcal{R})^{\mathcal{Q}},$$

where

$$H^b(\mathcal{N}, \mathcal{R})^{\mathcal{Q}} = \{\varphi \in H^b(\mathcal{N}, \mathcal{R})|\ (t.\varphi) = 0,\ t \in \mathcal{Q}\} \tag{2.1}$$

is the space of $\mathcal{Q}$-invariant cocycles of $\mathcal{N}$ with values in $\mathcal{R}$, the invariance being defined by:

$$(t.\varphi)(z_1, z_2, \dots, z_b) = [t, \varphi(z_1, z_2, \dots, z_b)] - \sum_{s=1}^{b} \varphi(z_1, \dots, [t, z_s], \dots, z_b).$$

Let consider

$$H^a(\mathcal{Q}, \mathbb{K}) = \frac{Ker\ (d^a)}{Im\ (d^{a-1})},$$

where $d^a : C^a(\mathcal{Q}, \mathbb{K}) \to C^{a+1}(\mathcal{Q}, \mathbb{K})$. Since $\varphi : \mathcal{Q} \times \dots \times \mathcal{Q} \to \mathbb{K}$ and $[t_i, t_j] = 0,\ t_i, t_j \in \mathcal{Q}$ we get $(d^a\varphi)(t_1, \dots, t_{a+1}) = 0$. It implies that $Im\ d^a = 0$ and $Ker\ d^a = C^a(\mathcal{Q}, \mathbb{K})$, i.e., $H^a(\mathcal{Q}, \mathbb{K}) = \wedge^a\mathcal{Q}$. Therefore, the cohomology groups $H^k(\mathcal{R}, \mathcal{R})$ vanish if and only if the space of $\mathcal{Q}$-invariant cocycles $H^b(\mathcal{N}, \mathcal{R})^{\mathcal{Q}}$ vanish. On the other hand, $H^p(\mathcal{R}, \mathcal{R}) = 0$ implies that $H^b(\mathcal{N}, \mathcal{R})^{\mathcal{Q}} = 0$ for all $0 \leq b \leq p$.

Definition 2.4. A torus on a Lie algebra $\mathcal{L}$ is a commutative subalgebra of $Der(\mathcal{L})$ (the set of all derivations of $\mathcal{L}$) consisting of semisimple endomorphisms. A torus is said to be maximal if it is not strictly contained in any other torus. We denote by $\mathcal{T}_{max}$ a maximal torus of a Lie algebra $\mathcal{L}$.

We should note that if $dim\mathcal{T} = dim(\mathcal{N}/\mathcal{N}^2)$, then $\mathcal{N}$ is called nilpotent Lie algebra of maximal rank.

Definition 2.5. A solvable Lie algebra $\mathcal{R}_{\mathcal{T}} = \mathcal{N} \rtimes \mathcal{T}$ is said to be of maximal rank, if $dim\mathcal{T} = dim(\mathcal{N}/N^2)$.

The dimension of a maximal torus of a nilpotent Lie algebra is denoted by $rank(\mathcal{N})$.

Denote by $W = \{\alpha \in \mathcal{T}^* : \mathcal{N}_\alpha \neq 0\}$ the roots system of $\mathcal{N}$ associated to $\mathcal{T}$, and by $\Psi_1 = \{\alpha_1, \dots, \alpha_s\}$ the set of primitive roots such that any non-primitive root can be expressed by a linear combination of them. In fact, any root $\alpha \in W$ we have $\alpha = \sum\limits_{\alpha_i\in\Psi_1} p_i\alpha_i$ with $p_i \in \mathbb{Z}$.

We should note that $\mathcal{Q}$ in the Theorem 2.3. is nothing else but torus of $\mathcal{N}$. In the following steps, we consider maximal solvable extensions of nilpotent Lie algebras which are constructed by their maximal torus, i.e. $\mathcal{Q} = \mathcal{T}_{max}$. We have the following remark.

Remark 2.6. It should be noted that if $\mathcal{H}(\mathcal{R},\mathcal{R})^p = 0$, where $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$ is a solvable Lie algebra such that $\mathcal{T}$ is Abelian and operators $ad_{\mathcal{R}}t$ $(t \in \mathcal{T})$ are diagonal, then $\mathcal{H}(\mathcal{R},\mathcal{R})^i = 0$ for any $0 \leq i \leq p-1$.

Let consider $b = 1$ on (2.1):

$$H^1(\mathcal{N},\mathcal{R})^{\mathcal{T}} = \{\varphi \in H^1(\mathcal{N},\mathcal{R}) |\ (t.\varphi) = 0,\ t \in \mathcal{T}\}.$$

Hence,

$$H^1(\mathcal{N},\mathcal{R}) = H^1(\mathcal{N}_{\alpha_1} \oplus \cdots \oplus \mathcal{N}_{\alpha_n}, \mathcal{N}_{\alpha_1} \oplus \cdots \oplus \mathcal{N}_{\alpha_n} \oplus \mathcal{T}).$$

Let $\varphi \in H^1(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, then

$$(t.\varphi)(n_\alpha) = [t, \varphi(n_\alpha)] - \varphi([t, n_\alpha]) = [t, \varphi(n_\alpha)] + \alpha(t)\varphi(n_\alpha) = 0.$$

It implies that

$$[\varphi(n_\alpha), t] = \alpha(t)\varphi(n_\alpha) \Rightarrow \varphi(\mathcal{N}_\alpha) \subseteq \mathcal{N}_\alpha.$$

Conditions of the triviality of the first cohomology groups of maximal solvable Lie algebras are obtained in [8].

Theorem 2.7. [8] *A solvable Lie algebra is complete if and only if it is the maximal solvable extension of a d-locally diagonalizable nilpotent Lie algebra.*

Let $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$ be maximal solvable extensions of nilpotent Lie algebra of maximal rank. Then by applying Theorem 4.9. in [7], we can take multiplications table of $\mathcal{R}$ as follows:

$$\mathcal{R}: \begin{cases} [\mathcal{N},\mathcal{N}]; \\ [n_{\alpha_i}, t_i] = \alpha_i n_{\alpha_i}, & 1 \leq i \leq k, \\ [n_{\alpha_i}, t_j] = \alpha_{i,j} n_{\alpha_i}, & 1 \leq j \leq k, \quad k+1 \leq j \leq n, \end{cases}$$

where $\mathcal{N} = \mathcal{N}_{\alpha_1} \oplus \cdots \oplus \mathcal{N}_{\alpha_k} \oplus \cdots \oplus \mathcal{N}_{\alpha_n}$ and $\mathcal{N}_{\alpha_i}$ are root subspaces with respect to the maximal torus on $\mathcal{N}$. Since any Lie algebra is also a Lie superalgebra, we can rewrite the following corollary (see, Corollary 4.11. in [7]).

Corollary 2.8. [7] *A maximal solvable extension of a nilpotent Lie algebra of maximal rank has trivial center and it admits only inner derivations.*

Due to Theorem 2.3 and Corollary 2.8, we can obtain the following remark.

Remark 2.9. Let $\mathcal{R}$ be a maximal solvable extension of a nilpotent Lie algebra of maximal rank. $H^2(\mathcal{R},\mathcal{R}) = 0$ if and only if $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$.

For the elements $\varphi \in Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, we have the following:

$$0 = (t.\varphi)(n_{\alpha_i}, n_{\alpha_j}) = [t, \varphi(n_{\alpha_i}, n_{\alpha_j})] - \varphi([t, n_{\alpha_i}], n_{\alpha_j}) - \varphi(n_{\alpha_i}, [t, n_{\alpha_j}]) \Rightarrow$$
$$[t, \varphi(n_{\alpha_i}, n_{\alpha_j})] = \varphi([t, n_{\alpha_i}], n_{\alpha_j}) + \varphi(n_{\alpha_i}, [t, n_{\alpha_j}]) = (\alpha_i + \alpha_j)(t)\varphi(n_{\alpha_i}, n_{\alpha_j}).$$

It implies that $\varphi(n_{\alpha_i}, n_{\alpha_j}) \in \mathcal{N}_{\alpha_i+\alpha_j}$. Since $B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} \subseteq Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, for any element $\psi \in B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$ we have $\psi(n_{\alpha_i}, n_{\alpha_j}) \in \mathcal{N}_{\alpha_i+\alpha_j}$.

Now, we present a proposition mentioned in the paper by F. Leger and E. Luks in [9]. We will formulate this proposition using our notation:

Let $\mathcal{N}$ be a nilpotent Lie algebra over a field F, of characteristic not 2, $\mathcal{T}$ is a subalgebra of $Der(\mathcal{N})$, $\mathcal{R}$ is semi-direct sum $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$ and we shall assume the following properties (i)-(iv) hold.
(**i**) $\mathcal{T}$ is diagonalizable over F and $dim\mathcal{T} = dim(\mathcal{N}/\mathcal{N}^2)$.
Let W denote the set of weights of $\mathcal{T}$ in $\mathcal{N}$ and for each α in W, denote by $\mathcal{N}_\alpha$ the weight space for α.
(**ii**) For $\alpha \in W$, $dim\mathcal{N}_\alpha = 1$ and if $\alpha, \beta, \alpha+\beta$ are all in W, $[\mathcal{N}_\alpha, \mathcal{N}_\beta] = \mathcal{N}_{\alpha+\beta}$. We fix $\alpha_1, \alpha_2, ..., \alpha_n$ in W and $e_1, e_2, ..., e_n$ in $\mathcal{N}$ so that $\mathcal{N} = \sum F e_i + \mathcal{N}^2$ (vector space direct sum) and $t \cdot e_j = \alpha_j(t) e_j$ for t in $\mathcal{T}$. The weights $\alpha_1, \alpha_2, ..., \alpha_n$ will be called primitive. Every weight in W has the form $\sum\limits_{i=1}^{n} r_i \alpha_i$, where the r_i are non-negative integers.

(**iii**) If characteristic of F is $p \neq 0$ we assume further, for every α in W, that $0 \leq r_i < p/2$ for each i. Since each weight in $\mathcal{N}^2$ is the sum of two weights, it follows from (iii) that zero is not in W.
(**iv**) If $\alpha, \beta, \gamma, \delta, \alpha+\gamma, \beta+\delta$ are all in W with α, β primitive and unequal and with $\alpha+\gamma = \beta+\delta$, then there is some μ in W such that $\delta = \alpha + \mu,\ \gamma = \beta + \mu$ and at least one of the following is satisfied:

Case 1. $\alpha + \beta$ is not in W.

Case 2. $\alpha + \beta$ is in W but $\alpha + 2\beta$ is not in W and $\mu = \beta + \nu$ for some ν in W.

Case 3. $\alpha + \beta$ is in W but $2\alpha + \beta$ is not in W and $\mu = \alpha + \nu$ for some ν in W.

Proposition A. Suppose $\mathcal{T}$, $\mathcal{N}$ are as (i)-(iv). Let $\mathcal{R}$ be an $\mathcal{T}+\mathcal{N}$ module such that the representation of $\mathcal{T}$ on $\mathcal{R}$ is toroidal and the weights of $\mathcal{T}$ in $\mathcal{R}$ are in W. Then $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}} = 0$.

A linear Lie algebra K is called *toroidal* if it can be diagonalized over an algebraic closure of the base field; a representation ρ of K is called *toroidal* if $\rho(K)$ is *toroidal.* From this, we can conclude that in our case, the representation of $\mathcal{T}$ on $\mathcal{R}$ is toroidal.

3. Main part.

Theorem 3.1. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank such that $\mathcal{N}^3 = 0$. Then, $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$ is cohomologically rigid, i.e., $H^2(\mathcal{R}, \mathcal{R}) = 0$.*

Proof. Due to $\mathcal{N}^3 = 0$, we can express non-zero products using only primitive roots:

$$\mathcal{N} : \left\{ \ [n_{\alpha_i}, n_{\alpha_j}] = A_{i,j} n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s, \right.$$

where $n_{\alpha_k} \in \mathcal{N}_{\alpha_k}$. If $A_{i,j} = 0$ for all $1 \leq i \neq j \leq s$, then $\mathcal{N}$ is abelian. In this case, it is clear that $\mathcal{R}$ can be written as a direct sum of two dimensonal solvable Lie algebras $\mathcal{R}_i$, which have one dimensional abelian nilradical. Since, they are all rigid, we can assume $\mathcal{R}$ is cohomologically rigid(see, Proposition 2, in [3]). Assume $A_{i,j} \neq 0$, for some $1 \leq i \neq j \leq s', (s' \leq s)$, then without loss of generalities we can take as $A_{i,j} = 1$ for $1 \leq i \neq j \leq s'$:

$$\mathcal{N} : \left\{ \ [n_{\alpha_i}, n_{\alpha_j}] = n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s'. \right.$$

Remark 2.9 implies that it is enough to show that $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}} = 0$. Let suppose $\varphi \in Z^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}}$. Since $\varphi(n_{\alpha_i}, n_{\alpha_j}) \in \mathcal{N}_{\alpha_i+\alpha_j}$, we can denote as:

$$\left\{ \ \varphi(n_{\alpha_i}, n_{\alpha_j}) = D_{i,j} n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s', \right.$$

where $D_{i,j}$ are scalars.

We now show that $\varphi \in B^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}}$ too. In other words, we prove existence of $f \in C^1(\mathcal{N}, \mathcal{R})$ such that $\varphi = d^1 f$, satisfying the following identity:

$$\varphi(e_\alpha, e_\beta) = [f(e_\alpha), e_\beta] + [e_\alpha, f(e_\beta)] - f([e_\alpha, e_\beta]) \quad \text{for all} \quad \alpha, \beta \in W.$$

Let denote $f(n_\alpha) = \mu(n_\alpha) n_\alpha$, where $\mu(n_\alpha)$ are scalars. Then

$$\left\{ \ \varphi(n_{\alpha_i}, n_{\alpha_j}) = \Big(\mu(n_{\alpha_i}) + \mu(n_{\alpha_j}) - \mu(n_{\alpha_i+\alpha_j})\Big) n_{\alpha_i+\alpha_j} = D_{i,j} n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s', \right.$$

Then we come to

$$\left\{ \ \mu(n_{\alpha_i}) + \mu(n_{\alpha_j}) - \mu(n_{\alpha_i+\alpha_j}) - D_{i,j} = 0, \quad 1 \leq i \neq j \leq s', \right. \tag{3.1}$$

system with variables $\mu(n_{\alpha_i}), \mu(n_{\alpha_j}), \mu(n_{\alpha_i+\alpha_j}),\ \ 1 \leq i \neq j \leq s'$.
System (3.1) has the following solution:

$$\begin{cases} \mu(n_{\alpha_i}) = \mu(n_{\alpha_j}) = 1, & 1 \leq i \neq j \leq s', \\ \mu(n_{\alpha_i+\alpha_j}) = 2 - D_{i,j}, & 1 \leq i \neq j \leq s'. \end{cases}$$

Therefore, there exists $f \in C^1(\mathcal{N}, \mathcal{R})$ such that $d^1 f = \varphi$, i.e., $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}} = 0$. By Remark 2.9 we obtain that $H^2(\mathcal{R}, \mathcal{R}) = 0$. □

We now consider the case $\mathcal{N}^4 = 0$.

Theorem 3.2. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^4 = 0$ and $rank(\mathcal{N}) = 2$. Then $\mathcal{R}$ is cohomologically rigid, i.e., $H^2(\mathcal{R},\mathcal{R}) = 0$.*

Proof. Without loss of generalities we can take the following products for $\mathcal{N}$:

$$\mathcal{N}: \begin{cases} [n_{\alpha_1}, n_{\alpha_2}] = A_1 n_{\alpha_1+\alpha_2}, \\ [n_{\alpha_1+\alpha_2}, n_{\alpha_1}] = A_2 n_{2\alpha_1+\alpha_2}, \\ [n_{\alpha_1+\alpha_2}, n_{\alpha_2}] = A_3 n_{\alpha_1+2\alpha_2}. \end{cases}$$

where $n_\alpha \in \mathcal{N}_\alpha$. If $A_1 = 0$, then we come to the case $\mathcal{N}$ is abelian, then similar to the proof of Theorem 1, we can prove that $H^2(\mathcal{R},\mathcal{R}) = 0$. Let $A_1 \neq 0$, then we can assume $A_1 = 1$. Remark 2.9 implies that it is enough to show $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$. Let suppose $\varphi \in Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$.

Take the following denotions:

$$\begin{cases} \varphi(n_{\alpha_1}, n_{\alpha_2}) = B_1 n_{\alpha_1+\alpha_2}, \\ \varphi(n_{\alpha_1+\alpha_2}, n_{\alpha_1}) = (1-\delta_{A_2,0}) B_2 n_{2\alpha_1+\alpha_2}, \\ \varphi(n_{\alpha_1+\alpha_2}, n_{\alpha_2}) = (1-\delta_{A_3,0}) B_3 n_{\alpha_1+2\alpha_2}, \end{cases} \tag{3.2}$$

where B_1, B_2, B_3 are scalars. We now show $\varphi \in B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$ too. In other words, we prove the existence of $f \in C^1(\mathcal{N},\mathcal{R})$ such that $\varphi = d^1 f$, satisfying the following identity:

$$\varphi(e_\alpha, e_\beta) = [f(e_\alpha), e_\beta] + [e_\alpha, f(e_\beta)] - f([e_\alpha, e_\beta]) \quad \text{for all} \quad \alpha, \beta \in W.$$

Let denote $f(n_\alpha) = \mu(n_\alpha) n_\alpha$, where $\mu(n_\alpha)$ are scalars. Then the system (3.2) has the following non-zero solution:

$$\begin{cases} \mu(n_{\alpha_1}) = \mu(n_{\alpha_2}) = 1, \\ \mu(n_{\alpha_1+\alpha_2}) = 2 - B_1, \\ \mu(n_{2\alpha_1+\alpha_2}) = \begin{cases} 3 - B_1 - \frac{B_2}{A_2}, \text{ if } A_2 \neq 0, \\ 0, \text{ otherwise}, \end{cases} \\ \mu(n_{\alpha_1+2\alpha_2}) = \begin{cases} 3 - B_1 - \frac{B_3}{A_3}, \text{ if } A_3 \neq 0, \\ 0, \text{ otherwise}. \end{cases} \end{cases}$$

Therefore, there exists $f \in C^1(\mathcal{N},\mathcal{R})$ such that $d^1 f = \varphi$, i.e., $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$. □

Now, we consider the case $rank(\mathcal{N}) \geq 3$ and $\mathcal{N}^4 = 0$.

Lemma 3.3. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^4 = 0$ and $rank(\mathcal{N}) \geq 3$. If $\alpha_{i_0} + \alpha_{j_0} + \alpha_{k_0} \in W$, then the element $n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}} \in \mathcal{N}_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}$ can be represented in at least two different ways.*

Proof. It is enough to show that the following equality holds:

$$\alpha_{i_0} + (\alpha_{j_0} + \alpha_{k_0}) = \alpha_{j_0} + (\alpha_{i_0} + \alpha_{k_0}), \tag{3.3}$$

if $\alpha_{i_0} + \alpha_{j_0} + \alpha_{k_0} \in W$ for unequal primitive roots $\alpha_{i_0}, \alpha_{j_0}, \alpha_{k_0}$.

Suppose contrary

$$\alpha_{i_0} + (\alpha_{j_0} + \alpha_{k_0}) \neq \alpha_{j_0} + (\alpha_{i_0} + \alpha_{k_0}) \tag{3.4}$$

for the triple $\{i_0, j_0, k_0\}$. Because of $\alpha_{i_0} + \alpha_{j_0} + \alpha_{k_0} \in W$ and (3.4), without loss of generalities, we can suppose

$$[n_{\alpha_{j_o}+\alpha_{k_0}}, n_{\alpha_{i_0}}] \neq 0, \quad [n_{\alpha_{i_o}+\alpha_{k_0}}, n_{\alpha_{j_0}}] = 0, \quad [n_{\alpha_{i_o}+\alpha_{j_0}}, n_{\alpha_{k_0}}] = 0.$$

(if this $[n_{\alpha_{i_o}+\alpha_{j_0}}, n_{\alpha_{k_0}}]$ product is nonzero too, then we can write the equality(3.3) as

$$\alpha_{i_0} + (\alpha_{j_0} + \alpha_{k_0}) = \alpha_{k_0} + (\alpha_{i_0} + \alpha_{j_0}),$$

which contradicts our assumption). However, this Jacoby identity

$$\underbrace{[n_{\alpha_{i_o}},[n_{\alpha_{j_o}},n_{\alpha_{k_o}}]]}_{\neq 0}+\underbrace{[n_{\alpha_{j_o}},[n_{\alpha_{k_o}},n_{\alpha_{i_o}}]]}_{0}+\underbrace{[n_{\alpha_{k_o}},[n_{\alpha_{i_o}},n_{\alpha_{j_o}}]]}_{0}=0$$

leads to contradiction.

□

Now, we provide criteria for the triviality of the second cohomology groups of maximal solvable extensions in the case $\mathcal{N}^4=0$ and $rank(\mathcal{N})\geq 3$. We consider it in two cases:

- $\alpha_i+\alpha_j+\alpha_k\notin W$ for all unequal primitive roots $\alpha_i,\alpha_j,\alpha_k$.
- $\alpha_i+\alpha_j+\alpha_k\in W$ for some unequal primitive roots $\alpha_i,\alpha_j,\alpha_k$.

Theorem 3.4. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^4=0$ and $rank(\mathcal{N})\geq 3$. If $\alpha_i+\alpha_j+\alpha_k\notin W$ for all unequal primitive roots $\alpha_i,\alpha_j,\alpha_k$, then $H^2(\mathcal{R},\mathcal{R})=0$.*

Proof. In the multiplication table of $\mathcal{N}$, without loss of generalities, we can write the following products:

$$\mathcal{N}:\begin{cases}[n_{\alpha_i},n_{\alpha_j}]=A_{i,j}n_{\alpha_i+\alpha_j}, & 1\leq i\neq j\leq s,\\ [n_{\alpha_i+\alpha_j},n_{\alpha_i}]=A_{i+j,i}n_{2\alpha_i+\alpha_j} & 1\leq i\neq j\leq s,\\ [n_{\alpha_i+\alpha_j},n_{\alpha_j}]=A_{i+j,j}n_{\alpha_i+2\alpha_j} & 1\leq j\neq i\leq s.\end{cases}\tag{3.5}$$

Because of $\varphi(n_{\alpha_i},n_{\alpha_j})\in\mathcal{N}_{\alpha_i+\alpha_j}$ property for $\varphi\in Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, we can denote as:

$$\begin{cases}\varphi(n_{\alpha_i},n_{\alpha_j})=(1-\delta_{A_{i,j},0})B_{i,j}n_{\alpha_i+\alpha_j}, & 1\leq i\neq j\leq s,\\ \varphi(n_{\alpha_i+\alpha_j},n_{\alpha_i})=(1-\delta_{A_{i,j},0})(1-\delta_{A_{i+j,i},0})B_{i+j,i}n_{2\alpha_i+\alpha_j}, & 1\leq i\neq j\leq s,\\ \varphi(n_{\alpha_i+\alpha_j},n_{\alpha_j})=(1-\delta_{A_{i,j},0})(1-\delta_{A_{i+j,j},0})B_{i+j,j}n_{\alpha_i+2\alpha_j}, & 1\leq i\neq j\leq s.\end{cases}\tag{3.6}$$

We now show $\varphi\in B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$ too. In other words, we prove the existence of $f\in C^1(\mathcal{N},\mathcal{R})$ such that $\varphi=d^1f$, satisfying the following identity:

$$\varphi(e_\alpha,e_\beta)=[f(e_\alpha),e_\beta]+[e_\alpha,f(e_\beta)]-f([e_\alpha,e_\beta])\quad\text{for all}\quad \alpha,\beta\in W.$$

Let denote $f(n_\alpha)=\mu(n_\alpha)n_\alpha$, where $\mu(n_\alpha)$ are scalars. Then, we come the following system of linear equations $(1\leq i\neq j\leq s)$:

$$\begin{cases}A_{i,j}\Big(\mu(n_{\alpha_i})+\mu(n_{\alpha_j})-\mu(n_{\alpha_i+\alpha_j})\Big)=(1-\delta_{A_{i,j},0})B_{i,j},\\ A_{i+j,i}\big(\mu(n_{\alpha_i+\alpha_j})+\mu(n_{\alpha_i})-\mu(n_{2\alpha_i+\alpha_j})\big)=(1-\delta_{A_{i,j},0})(1-\delta_{A_{i+j,i},0})B_{i+j,i},\\ A_{i+j,j}\big(\mu(n_{\alpha_i+\alpha_j})+\mu(n_{\alpha_j})-\mu(n_{\alpha_i+2\alpha_j})\big)=(1-\delta_{A_{i,j},0})(1-\delta_{A_{i+j,j},0})B_{i+j,j}.\end{cases}\tag{3.7}$$

Then the system (3.7) has the following solution:

$$\begin{cases}\mu(n_{\alpha_i})=\mu(n_{\alpha_j})=1, \quad 1\leq i\neq j\leq s,\\ \mu(n_{\alpha_i+\alpha_j})=\begin{cases}2-\frac{B_{i,j}}{A_{i,j}}, \text{ if } A_{i,j}\neq 0 \quad 1\leq i\neq j\leq s,\\ 0, \text{ otherwise}, \quad 1\leq i\neq j\leq s,\end{cases}\\ \mu(n_{2\alpha_i+\alpha_j})=\begin{cases}3-\frac{B_{i,j}}{A_{i,j}}-\frac{B_{i+j,i}}{A_{i+j,i}}, \text{ if } A_{i+j,i}\neq 0 \quad 1\leq i\neq j\leq s,\\ 0, \text{ otherwise}, \quad 1\leq i\neq j\leq s,\end{cases}\\ \mu(n_{\alpha_i+2\alpha_j})=\begin{cases}3-\frac{B_{i,j}}{A_{i,j}}-\frac{B_{i+j,j}}{A_{i+j,j}}, \text{ if } A_{i+j,j}\neq 0 \quad 1\leq i\neq j\leq s,\\ 0, \text{ otherwise}, \quad 1\leq i\neq j\leq s.\end{cases}\end{cases}$$

Therefore, there exists $f\in C^1(\mathcal{N},\mathcal{R})$ such that $d^1f=\varphi$, i.e., $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}=0$. Then by Remark 2.9 we obtain that $H^2(\mathcal{R},\mathcal{R})=0$. □

An example that satisfies the conditions of Theorem 3.4:

$$\mathcal{N}: \left\{ [e_1, e_2] = e_4, \quad [e_1, e_3] = e_5, \quad [e_2, e_3] = e_6, \quad [e_5, e_1] = e_7. \right.$$

$\mathcal{N}$ can be decomposed to the following root subspaces:

$\mathcal{N}_{\alpha_1} = \{e_1\},\ \mathcal{N}_{\alpha_2} = \{e_2\},\ \mathcal{N}_{\alpha_3} = \{e_3\},\ \mathcal{N}_{\alpha_1+\alpha_2} = \{e_4\},\ \mathcal{N}_{\alpha_1+\alpha_3} = \{e_5\},\ \mathcal{N}_{\alpha_2+\alpha_3} = \{e_6\},\ \mathcal{N}_{2\alpha_1+\alpha_3} = \{e_7\}.$

Its maximal solvable extension $\mathcal{R}(\mathcal{N}) = \mathcal{N} \rtimes \mathcal{T}_{max}$ has the following products:

$$\mathcal{R}(\mathcal{N}): \left\{ \begin{array}{l} [e_i, t_1] = e_i,\ i = 1, 4, 5, \quad [e_7, t_1] = 2e_7, \\ [e_i, t_2] = e_i,\ i = 2, 4, 6, \quad [e_i, t_3] = e_i,\ i = 3, 6, 7, \quad [\mathcal{N}, \mathcal{N}]. \end{array} \right.$$

This algebra satisfies the conditions of Theorem 3.4, i.e., $\alpha_1 + \alpha_2 + \alpha_3 \notin W$ and one can check that $H^2(\mathcal{R}(\mathcal{N}), \mathcal{R}(\mathcal{N})) = 0$.

An example that does not satisfy the conditions of Theorem 3.4:

$$\mathcal{M}: \left\{ \begin{array}{llll} [e_1, e_2] = e_4, & [e_2, e_3] = e_6, & [e_4, e_3] = e_8, & [e_6, e_1] = e_8, \\ [e_1, e_3] = e_5, & [e_5, e_1] = e_7, & [e_5, e_2] = 2e_8. & \end{array} \right.$$

$\mathcal{M}$ can be decomposed to the following root subspaces:

$$\mathcal{M}_{\alpha_1} = \{e_1\},\ \ \mathcal{M}_{\alpha_2} = \{e_2\},\ \ \mathcal{M}_{\alpha_3} = \{e_3\},\ \ \mathcal{M}_{\alpha_1+\alpha_2} = \{e_4\},\ \mathcal{M}_{\alpha_1+\alpha_3} = \{e_5\},\ \ \ \mathcal{M}_{\alpha_2+\alpha_3} = \{e_6\},$$

$$\mathcal{M}_{2\alpha_1+\alpha_3} = \{e_7\},\ \ \mathcal{M}_{\alpha_1+\alpha_2+\alpha_3} = \{e_8\}.$$

Its maximal solvable extension $\mathcal{R}(\mathcal{M}) = \mathcal{M} \rtimes \mathcal{T}_{max}$ has the following products:

$$\mathcal{R}(\mathcal{M}): \left\{ \begin{array}{l} [e_i, t_1] = e_i,\ i = 1, 4, 5, 8, \quad [e_7, t_1] = 2e_7, \\ [e_i, t_2] = e_i,\ i = 2, 4, 6, 8, \quad [e_i, t_3] = e_i,\ i = 3, 5, 6, 7, 8, \quad [\mathcal{M}, \mathcal{M}]. \end{array} \right.$$

This algebra does not satisfy the conditions of Theorem 3.4, i.e., $\alpha_1+\alpha_2+\alpha_3 \in W$, and one can verify that $\dim H^2(\mathcal{R}(\mathcal{M}), \mathcal{R}(\mathcal{M})) = 1$.

We should note that in Theorem 3.4, the case $\alpha_{i_p} + \alpha_{j_p} + \alpha_{k_p} \notin W$ for all unequal primitive roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ has been considered. It has been proven that all the maximal solvable Lie algebras in this case are cohomologically rigid. Therefore, we present the following theorem in the case where $\alpha_{i_p} + \alpha_{j_p} + \alpha_{k_p} \in W$ for some roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ in Ψ_1, where $1 \le p \le C_s^3$, and s is the number of different roots in Ψ_1.

Theorem 3.5. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^4 = 0$ and rank$(\mathcal{N}) \ge 3$. Then $H^2(\mathcal{R}, \mathcal{R}) = 0$ if and only if for any unequal primitive roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ such that $\alpha_{i_p}+\alpha_{j_p}+\alpha_{k_p} \in W$, the equalities $\alpha_{i_p} + (\alpha_{j_p} + \alpha_{k_p}) = \alpha_{j_p} + (\alpha_{i_p} + \alpha_{k_p})$ imply $\alpha_{i_p} + \alpha_{j_p} \notin W$ for $1 \le p \le C_s^3$.*

Proof. By Lemma 2.3, from $\alpha_{i_p} + \alpha_{j_p} + \alpha_{k_p} \in W$, we can write

$$\alpha_{i_p} + (\alpha_{j_p} + \alpha_{k_p}) = \alpha_{j_p} + (\alpha_{i_p} + \alpha_{k_p}). \tag{3.8}$$

Because of Remark 2.9, instead of $H^2(\mathcal{R}, \mathcal{R})$, we consider $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}}$. Let $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}} = 0$, we now show that for any unequal primitive roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ such that $\alpha_{i_p} + \alpha_{j_p} + \alpha_{k_p} \in W$, these equalities

$$\alpha_{i_p} + (\alpha_{j_p} + \alpha_{k_p}) = \alpha_{j_p} + (\alpha_{i_p} + \alpha_{k_p}), \quad 1 \le p \le C_s^3,$$

imply

$$\alpha_{i_p} + \alpha_{j_p} \notin W, \quad 1 \le p \le C_s^3.$$

Assume contrary, let $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}} = 0$, but there exists a triple of primitive roots $\{\alpha_{i_0}, \alpha_{j_0}, \alpha_{k_0}\}$ such that $\alpha_{i_0} + \alpha_{j_0} + \alpha_{k_0} \in W$, and this equality

$$\alpha_{i_0} + (\alpha_{j_0} + \alpha_{k_0}) = \alpha_{j_0} + (\alpha_{i_0} + \alpha_{k_0})$$

implies

$$\alpha_{i_0}+\alpha_{j_0}\in W.$$

Then, without loss of generality, we can write the products of $\mathcal{N}$ as follows:

$$\mathcal{N}:\begin{cases}[n_{\alpha_{i_0}},n_{\alpha_{j_0}}]=n_{\alpha_{i_0}+\alpha_{j_0}},\\ [n_{\alpha_{i_0}},n_{\alpha_{k_0}}]=n_{\alpha_{i_0}+\alpha_{k_0}},\\ [n_{\alpha_{j_0}},n_{\alpha_{k_0}}]=n_{\alpha_{j_0}+\alpha_{k_0}},\\ [n_{\alpha_{i_0}+\alpha_{j_0}},n_{\alpha_{k_0}}]=n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\\ [n_{\alpha_{i_0}+\alpha_{k_0}},n_{\alpha_{j_0}}]=An_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\quad A\neq 0,\\ [n_{\alpha_{j_0}+\alpha_{k_0}},n_{\alpha_{i_0}}]=(A-1)n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\\ [\mathcal{N}_\alpha,\mathcal{N}_\beta],\quad \{\alpha,\beta\}\neq\{\alpha_{i_0},\alpha_{j_0}\}\neq\{\alpha_{i_0},\alpha_{k_0}\}\neq\{\alpha_{j_0},\alpha_{k_0}\},\\ [\mathcal{N}_{\alpha+\beta},\mathcal{N}_\gamma],\quad \{\alpha,\beta\}\neq\{\alpha_{i_0},\alpha_{j_0}\}\neq\{\alpha_{i_0},\alpha_{k_0}\}\neq\{\alpha_{j_0},\alpha_{k_0}\},\\ [\mathcal{N}_{2\alpha},\mathcal{N}_\beta],\ [\mathcal{N}_\alpha,\mathcal{N}_{2\beta}].\end{cases}$$

Let us consider the following map, defined as:

$$\begin{cases}\varphi(n_{\alpha_{i_0}+\alpha_{k_0}},n_{\alpha_{j_0}})=n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\\ \varphi(n_{\alpha_{j_0}+\alpha_{k_0}},n_{\alpha_{i_0}})=n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\\ \varphi(n_\alpha,n_\beta)=0,\quad \text{otherwise}.\end{cases}\tag{3.9}$$

For 2-cocycles, we have the following identity:

$$\varphi(a,[b,c])-\varphi([a,b],c)+\varphi([a,c],b)+[a,\varphi(b,c)]-[\varphi(a,b),c]+[\varphi(a,c),b]=0,\tag{3.10}$$

for any elements $a,b,c\in\mathcal{N}$.

Now, we show that φ, defined as in (3.9), belongs to $Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$. Let us consider the identity (3.10) for the triple $\{n_\alpha,n_\beta,n_\gamma\}$ in the case where one of α,β,γ is non-primitive. Then, due to $\mathcal{N}^4=0$ and the way φ is defined, all terms in (3.10) simultaneously equal zero. Therefore, it remains to check the triples $\{n_\alpha,n_\beta,n_\gamma\}$ where all α,β,γ are primitive. If at least one element in the triple $\{n_\alpha,n_\beta,n_\gamma\}$ differs from the triple $\{n_{\alpha_{i_0}},n_{\alpha_{j_0}},n_{\alpha_{k_0}}\}$, then, due to the way φ is defined, all six summands in the 2-cocycle identity are simultaneously equal to zero.

Now, it is enough to check 3.10 for the triples $\{n_{\alpha_{i_0}},n_{\alpha_{j_0}},n_{\alpha_{k_0}}\}$. By applying the 2-cocycle identity to the triple $\{n_{\alpha_{i_0}},n_{\alpha_{j_0}},n_{\alpha_{k_0}}\}$, we obtain the following result:

$$\underbrace{[n_{\alpha_{i_0}},\varphi(n_{\alpha_{j_0}},n_{\alpha_{k_0}})}_{0}-\underbrace{[\varphi(n_{\alpha_{i_0}},n_{\alpha_{j_0}}),n_{\alpha_{k_0}}]}_{0}+\underbrace{[\varphi(n_{\alpha_{i_0}},n_{\alpha_{k_0}}),n_{\alpha_{j_0}}]}_{0}+$$

$$+\underbrace{\varphi(n_{\alpha_{i_0}},[n_{\alpha_{j_0}},n_{\alpha_{k_0}}])}_{-n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}}-\underbrace{\varphi([n_{\alpha_{i_0}},n_{\alpha_{j_0}}],n_{\alpha_{k_0}})}_{0}+\underbrace{\varphi([n_{\alpha_{i_0}},n_{\alpha_{k_0}}],n_{\alpha_{j_0}})}_{n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}}=0.$$

Therefore, $\varphi\in Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$. Let assume $\varphi\in B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, then suppose there exists $f\in C^1(\mathcal{N},\mathcal{R})$ such that $\varphi=d^1f$, and we have the following identity:

$$\varphi(n_\alpha,n_\beta)=[f(n_\alpha),n_\beta]+[n_\alpha,f(n_\beta)]-f([n_\alpha,n_\beta])\quad\text{for all}\quad \alpha,\beta\in W.$$

$$\begin{cases}\varphi(n_{\alpha_{i_0}},n_{\alpha_{j_0}})=(\mu(n_{\alpha_{i_0}})+\mu(n_{\alpha_{j_0}})-\mu(n_{\alpha_{i_0}+\alpha_{j_0}}))n_{\alpha_{i_0}+\alpha_{j_0}}=0,\\ \varphi(n_{\alpha_{i_0}},n_{\alpha_{k_0}})=(\mu(n_{\alpha_{i_0}})+\mu(n_{\alpha_{k_0}})-\mu(n_{\alpha_{i_0}+\alpha_{k_0}}))n_{\alpha_{i_0}+\alpha_{k_0}}=0,\\ \varphi(n_{\alpha_{i_0}+\alpha_{j_0}},n_{\alpha_{k_0}})=(\mu(n_{\alpha_{i_0}+\alpha_{j_0}})+\mu(n_{\alpha_{k_0}})-\mu(n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}))n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}=0,\\ \varphi(n_{\alpha_{i_0}+\alpha_{k_0}},n_{\alpha_{j_0}})=A(\mu(n_{\alpha_{i_0}+\alpha_{k_0}})+\mu(n_{\alpha_{j_0}})-\mu(n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}))n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}=n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\end{cases}$$

Then we come to

$$\mu(n_{\alpha_{i_0}})+\mu(n_{\alpha_{j_0}})+\mu(n_{\alpha_{k_0}})=\mu(n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}})=-\tfrac{1}{A}+\mu(n_{\alpha_{i_0}})+\mu(n_{\alpha_{j_0}})+\mu(n_{\alpha_{k_0}}),\quad A\neq 0.$$

Thus, $\varphi \notin B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$ and it is a contradiction for triviality of $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$.

Let assume for any unequal primitive roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ such that $\alpha_{i_p}+\alpha_{j_p}+\alpha_{k_p} \in W$, the equalities $\alpha_{i_p}+(\alpha_{j_p}+\alpha_{k_p}) = \alpha_{j_p}+(\alpha_{i_p}+\alpha_{k_p})$ imply $\alpha_{i_p}+\alpha_{j_p} \notin W, \quad 1 \le p \le C_s^3$. Let us show that all the conditions of Proposition A hold true for $\mathcal{N}$.

(i) holds true, because of $\mathcal{N}$ is a nilpotent Lie algebra of maximal rank i.e., $dim\mathcal{T} = dim(\mathcal{N}/\mathcal{N}^2)$;
(ii) holds true, because in our case $\dim \mathcal{N}_\alpha = 1$ for all $\alpha \in W$;
(iii) holds true, because we are considering zero is not in W case;
(iv) holds true, because, if we take as

$$\alpha = \alpha_{i_p}, \quad \beta = \alpha_{j_p}, \quad \gamma = \alpha_{j_p}+\alpha_{k_p}, \quad \delta = \alpha_{i_p}+\alpha_{k_p}, \quad 1 \le p \le C_s^3,$$

then $\alpha+\gamma = \beta+\delta$, which is what we need to show.

Finally, the condition $\alpha+\beta \notin W$ implies that we are in **Case1.** Therefore, by Proposition A, we obtain $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$. □

An example that satisfies the conditions of Theorem 3.5:

$$\mathcal{L}: \Big\{ \; [e_1,e_2]=e_4, \quad [e_1,e_3]=e_5, \quad [e_4,e_3]=e_6, \quad [e_5,e_2]=e_6.$$

$\mathcal{L}$ can be decomposed to the following root subspaces:

$$\mathcal{L}_{\alpha_1}=\{e_1\}, \quad \mathcal{L}_{\alpha_2}=\{e_2\}, \quad \mathcal{L}_{\alpha_3}=\{e_3\}, \quad \mathcal{L}_{\alpha_1+\alpha_2}=\{e_4\}, \; \mathcal{L}_{\alpha_1+\alpha_3}=\{e_5\}, \mathcal{L}_{\alpha_1+\alpha_2+\alpha_3}=\{e_6\}.$$

Its maximal solvable extension $\mathcal{R}(\mathcal{L}) = \mathcal{L} \rtimes \mathcal{T}_{max}$ has the following products:

$$\mathcal{R}(\mathcal{L}): \Big\{ \; [e_i,t_1]=e_i, \; i=1,4,5,6, \quad [e_i,t_2]=e_i, \; i=2,4,6, \quad [e_3,t_i]=e_i, \; i=3,5,6, \quad [\mathcal{L},\mathcal{L}].$$

If we take as $\alpha_1 = \alpha_{i_0}, \; \alpha_2 = \alpha_{j_0}, \; \alpha_3 = \alpha_{k_0}$, then $\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0} \in W$. Due to Lemma 3.3, we can write $\alpha_{j_0}+(\alpha_{k_0}+\alpha_{i_0}) = \alpha_{k_0}+(\alpha_{i_0}+\alpha_{j_0})$, and we have $\alpha_{j_0}+\alpha_{k_0} \notin W$. One can check that $H^2(\mathcal{R}(\mathcal{L}),\mathcal{R}(\mathcal{L})) = 0$.

An example that does not satisfy the conditions of Theorem 3.5:

Let consider the algebra $\mathcal{M}$ as we already mentioned above:

$$\mathcal{M}: \begin{cases} [e_1,e_2]=e_4, \quad [e_2,e_3]=e_6, \quad [e_4,e_3]=e_8, \quad [e_6,e_1]=e_8, \\ [e_1,e_3]=e_5, \quad [e_5,e_1]=e_7, \quad [e_5,e_2]=2e_8. \end{cases}$$

$\mathcal{M}$ can be decomposed to the following root subspaces:

$$\mathcal{M}_{\alpha_1}=\{e_1\}, \quad \mathcal{M}_{\alpha_2}=\{e_2\}, \quad \mathcal{M}_{\alpha_3}=\{e_3\}, \quad \mathcal{M}_{\alpha_1+\alpha_2}=\{e_4\}, \; \mathcal{M}_{\alpha_1+\alpha_3}=\{e_5\}, \quad \mathcal{M}_{\alpha_2+\alpha_3}=\{e_6\},$$

$$\mathcal{M}_{2\alpha_1+\alpha_3}=\{e_7\}, \quad \mathcal{M}_{\alpha_1+\alpha_2+\alpha_3}=\{e_8\}.$$

This algebra does not satisfy the conditions of Theorem 3.5. Indeed, $\alpha_1+\alpha_2+\alpha_3 \in W$ and for all identity $\alpha_i+(\alpha_j+\alpha_k) = \alpha_j+(\alpha_k+\alpha_i), 1 \le i \ne j \ne k \le 3$ we have $\alpha_i+\alpha_j \in W$. One can check that for its maximal solvable extension, we have $dimH^2(\mathcal{R}(\mathcal{M}),\mathcal{R}(\mathcal{M})) = 1$.

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