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Evasion differential game of one evader and multiple pursuers with changing energies

Ibragimov G.I., Inomiddinov S.N., Ibrokhimjonov I.I.

Abstract. In this paper, we study a simple motion evasion differential game of multiple pursuers and one evader. The control functions of the players are subject to integral constraints, where the energies of the players change linearly. If the state of the evader does not coincide with those of the pursuers forever, we say that evasion is possible in the game. We prove a theorem about evasion and construct a strategy for the evader that guarantees evasion in the game.

Keywords: differential game, integral constraint, pursuer, evader, evasion strategy, pursuit, evasion.
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1. Introduction

During the latter half of the previous century, the theory of differential games experienced remarkable development (see, for example, [1], [2]). Within this field, multiple-pursuer differential games constitute one of the most actively investigated areas. In recent years, a substantial body of research has been devoted to this topic due to its broad range of practical applications. Differential games involving several pursuers are closely connected to real-world situations, such as reach-avoid problems, where multiple pursuers attempt to capture an evader while simultaneously reaching a target region and avoiding specific obstacles or restrictions. They are also relevant to military defense, where missile systems aim to neutralize threats while maintaining the safety of allied assets, and to surveillance and security tasks, where drones monitor a region, protecting restricted zones from intrusion.

Differential games are generally classified according to the nature of constraints imposed on players' control functions. Two fundamental categories are geometric constraint games and integral constraint games. Geometric constraints limit the control parameters to certain subsets of $\mathbb{R}^n$, often representing physical limitations such as velocity or turning rate. Most studies in the literature have concentrated on multi-player games with geometric constraints, which are briefly reviewed below.

A differential game known as the multi-pursuer, single-evader Life-line game was analyzed by Azamov [3], [4]. In this framework, the evader aims to reach a designated "life-line" (safe zone) before capture, while pursuers strive to intercept it beforehand. The author derived a winning condition expressed through Apollonius circles—if the condition is met, the pursuers succeed; otherwise, the evader prevails. Wang et al. [5] studied a differential game involving multiple pursuers and a fast evader, proposing an initial state configuration based on Apollonius circle theory to guarantee capture.

Among the most discussed topics in multi-pursuer, single-evader settings are reach-avoid planar games, examined in works such as those by Deng et al. [6], Garcia et al. [7], and Yan et al. [8]. Von Moll et al. [9] investigated a border defense scenario where multiple unmanned aerial vehicles serve as pursuers against a ground vehicle attempting to cross a boundary. Their study modeled the system as a zero-sum differential game with simple motion dynamics and derived optimal pursuit strategies from geometric principles. Similarly, Pachter [10] employed a geometric framework for single-evader capture, while Souidi et al. [11] introduced a dynamic elimination approach to enhance decision-making in multi-pursuer, multi-evader games.

In evasion games, attention shifts to the evader's strategy to avoid capture by multiple pursuers. Ibragimov [12] analyzed a game involving one fast evader attempting to escape from several pursuers and proposed a new evasion strategy ensuring escape within a region surrounding the ray emanating from the evader's initial position.

Several other contributions [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24] have also explored multiple-pursuer differential games. In particular, Pshenichnii [24] examined a simple-motion differential game with multiple pursuers and a single evader, deriving pursuit and evasion conditions based on the convex hull of the pursuers' initial states, under the assumption of identical dynamic capabilities for all players.

When such games are extended to manifolds, the dynamics of pursuit and evasion acquire richer geometric complexity. A typical example is a group of autonomous robots navigating hilly terrain to capture a target. Kuchkarov et al. [25] studied a simple-motion differential game of multiple pursuers and one evader on manifolds endowed with the Euclidean metric. They established a condition under which pursuit is guaranteed; if the condition fails, the evader succeeds in escaping.

Another important subfield with practical significance concerns multiple-capture game problems, where a group of defenders (pursuers) attempts to capture one or more attackers (evaders) m, $m \geq 1$, times. If fewer than m pursuers reach the attacker, the defenders are considered neutralized. Research on this topic includes works by Blagodatskikh and Petrov [26], Petrov [27], Shaunak and Subhash [28], Sakharov [29], and Petrov and Solov'eva [30]. When the control processes involve limited or exhaustible resources (e.g., fuel or energy), integral constraints naturally arise. A missile constrained by finite fuel reserves or a drone with limited battery capacity exemplifies this class of problems. Integral constraints regulate the total usage of a control input (such as energy or effort) throughout the game. Ibragimov and Salleh [31] investigated an evasion differential game involving multiple pursuers and a single evader under integral constraints, constructing an evasion strategy that exploits the evader's advantage in control resources. Similarly, Salimi and Ferrara [32] considered an optimal approach game with integral constraints.

In another study, Kuchkarov et al. [33] analyzed an optimal approach differential game involving several pursuers and one evader with integral constraints. The game duration was assumed to be fixed, and the payoff function was defined as the minimum distance between the evader and pursuers at the terminal time. An estimate for this payoff functional was obtained.

In the present paper, we study a simple motion evasion differential game of multiple pursuers and one evader. The control functions of the players are subject to integral constraints, where the control resources of the players change linearly. We construct an evasion strategy for the evader, which guarantees evasion in the game.

2. Statement of problem

Let the dynamics of pursuers x_i and evader y be described by the following equations:

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad i = 1, 2, ..., M, \\ &\dot{y} = v, \quad\;\; y(0) = y_0, \end{aligned} \tag{2.1}$$

where $x_i, x_{i0}, u_i, y, y_0, v \in \mathbb{R}^n$, $x_{i0} \neq y_0$, u_i and v are control parameters of the pursuers and evader, respectively.

Definition 2.1. Measurable functions $u_i(t)$ and $v(t)$, $t \geq 0$, are called admissible controls of the pursuer x_i and evader y, respectively, if they satisfy the following integral constraints:

$$\int_0^t |u_i(s)|^2 ds \leq \rho_i^2 t, \; i = 1, ..., M, \quad \int_0^t |v(s)|^2 ds \leq \sigma^2 t. \tag{2.2}$$

Definition 2.2. A function $V = (V_1, \dots, V_n)$,

$$\big(t, y(\cdot)|_{[0,t]}, x_1(\cdot)|_{[0,t]}, \dots, x_M(\cdot)|_{[0,t]}, u_1, \dots, u_M\big) \mapsto V\big(t, y(\cdot)|_{[0,t]}, x_1(\cdot)|_{[0,t]}, \dots, x_M(\cdot)|_{[0,t]}, u_1, \dots, u_M\big),$$

is called a strategy of the evader y if for any admissible controls of the pursuers $u_i = u_i(t)$, $i = 1, \dots, M$, the initial value problem

$$\begin{aligned} &\dot{x}_1 = u_1(t), && x_1(0) = x_1^0, \\ &\vdots && \vdots \\ &\dot{x}_M = u_M(t), && x_M(0) = x_M^0, \\ &\dot{y} = V\big(t, y(\cdot)|_{[0,t]}, x_1(\cdot)|_{[0,t]}, \dots, x_M(\cdot)|_{[0,t]}, u_1, \dots, u_M\big), && y(0) = y^0, \end{aligned} \tag{2.3}$$

has a unique solution $(x_1(t), \ldots, x_M(t), y(t))$, $t \geq 0$, and along this solution

$$\int_0^t |V(s, y(\cdot)|_{[0,s]}, x_1(\cdot)|_{[0,s]}, \ldots, x_M(\cdot)|_{[0,s]}, u_1(s), \ldots, u_M(s))|^2\, ds \leq \sigma^2 t. \tag{2.4}$$

Here, the pursuers apply arbitrary admissible controls, while the evader adheres to a prescribed strategy. By a solution of the initial value problem (2.3), we mean an $(M+1)$-tuple $(x_1(t), \ldots, x_M(t), y(t))$, $t \geq 0$, with absolutely continuous components $x_i(t)$, $i = 1, \ldots, M$, and $y(t)$ that satisfy the initial conditions in (2.3) and differential equations in (2.3) almost everywhere on $[0, \infty)$.

Definition 2.3. Evasion in game (2.1)-(2.2) is said to be achievable if there exists a strategy V for the evader such that, regardless of the pursuers' controls, $x_i(t) \neq y(t)$ holds for all $t \geq 0$, $i = 1, 2, ..., M$.

We can now state the evasion problem.

Problem. Find a condition for the parameters $\rho_1, \ldots, \rho_M$, and σ, which guarantee the evasion in game (2.1)-(2.2).

This is the main problem to be investigated in this paper. It should be noted that, in the evasion game, the pursuers use arbitrary admissible controls $u_1(t), \ldots, u_M(t)$, $t \geq 0$, whereas the evader uses a strategy.

3. Evasion differential game of one evader and multiple pursuers

The main result of the paper is the following statement.

Theorem 3.1. *If*

$$\sum_{i=1}^{M} \rho_i^2 < \sigma^2,$$

then evasion is possible in game (2.1)-(2.2).

We study the game problem in $\mathbb{R}^2$ and the general case where $n \geq 2$ follows from the solution of the problem in $\mathbb{R}^2$.

Without loss of generality, we assume that $y_0 = (0,0)$ in the $\xi_1\xi_2$-coordinate system. Let $I = \{i|\ x_{i2}(0) > 0,\ \ 1 \leq i \leq M\}$. There is no restriction in assuming that $I = \{1, 2, ..., m\}$, which means that initially the first m pursuers, i.e. the points $x_{10}, x_{20}, ..., x_{m0}$ are in the upper half plane.

Let α and a_1 be fixed numbers that satisfy the following conditions

$$0 < \alpha < \min\left\{1, \frac{\sigma - \rho}{\sqrt{2}}\right\}, \quad 0 < a_1 < \min\left\{1, \frac{\rho^2 T}{32}, \max_{1 \leq i \leq M} |y_2^0 - x_{i2}^0|\right\}, \tag{3.1}$$

where $\rho = \left(\sum\limits_{i=1}^{M} \rho_i^2\right)^{1/2}$. Let

$$T_0 = \frac{1}{\alpha} \max_{1 \leq i \leq M} |y_2^0 - x_{i2}^0|, \quad T = T_0 + \frac{2a_1}{\alpha}.$$

Note that this and (3.1) imply that $\alpha a_1 \leq \sigma^2 T$ since

$$\alpha a_1 < \alpha \max_{1 \leq i \leq M} |y_2^0 - x_{i2}^0| = \alpha^2 T_0 < \alpha^2 T < \frac{(\sigma - \rho)^2}{2} T < \sigma^2 T.$$

We define a sequence $a_{k+1} = \beta a_k^4$, $k = 1, 2, ...$, where

$$\beta = \min\left\{\frac{1}{2}, \frac{\alpha^2}{7200\sigma^6 T^3}, \frac{\alpha}{12\sigma^2 T}\right\}.$$

Let $\tau_0 = 0$, and let $\tau_1 > 0$ be the first time that

$$(i) \quad |x_i(\tau_1) - y(\tau_1)| = a_1,$$
$$(ii) \quad x_{i2}(\tau_1) > y_2(\tau_1),$$

for some $i \in I$. Note that such a time τ_1 may not exist. If so we'll have $x_i(t) \neq y(t)$ for all $t \geq 0$ and $i \in I$. If there are several pursuers that satisfy conditions (i) and (ii), then without loss of generality, we can assume, by relabeling if necessary, that one of such pursuers is x_1. We call τ_1 an a_1–approach time.

If τ_1, τ_2, ..., τ_{k-1} are a_1-, a_2-, ..., a_{k-1}-approach times, respectively, then we define $\tau_k > \tau_{k-1}$ to be the a_k-approach time if the following conditions are satisfied:

$$(1) \quad |x_k(\tau_k) - y(\tau_k)| = a_k,$$
$$(2) \quad x_{k2}(\tau_k) > y_2(\tau_k),$$

for some $k \in I$. If there are several pursuers that satisfy the conditions in items (1) and (2) simultaneously, then there is no loss of generality in assuming that one of these pursuers is x_k. In this way, we define the a_k–approach times τ_k, $k = 1, 2, \ldots, m_0$, where m_0 is the number (unspecified) of all approach times. For now, we do not now m_0 is finite or $m_0 = \infty$.

Note that a_k–approach times τ_k will not necessarily be defined for all the pursuers $i \in I$. We will establish that at most one approach time will be defined for each pursuer x_i, $i \in I$, and therefore we will have $m_0 \leq m$.

Let

$$\tau'_k = \tau_k + \frac{2a_k}{\alpha}, \quad k = 1, 2, \ldots, m_0.$$

Note that we have defined τ_k and τ'_k only for $k = 1, 2, \ldots, m_0$.

Let

$$J_k = \bigcup_{j=k}^{m_0} [\tau_j, \tau'_j), \quad k = 1, 2, ..., m_0,$$

where $J_{m_0+1} \neq \emptyset$ if m_0 is finite.

We define the function

$$r : [0, T] \to \{0, 1, 2, \ldots, m_0\},$$

by the following equation

$$r(t) = \begin{cases} 0, & t \in [0, T] \setminus J_1, \\ k, & t \in [\tau_k, \tau'_k) \setminus J_{k+1}, \quad k = 1, 2, \ldots, m_0, \end{cases}$$

for $m_0 > 1$, $\quad k = 1, 2, \ldots, (m_0 - 1)$. It is not difficult to verify that this function has the following properties:

1) $r(t) = k, \quad \tau_k \leq t < \tau'_k, \quad \text{if } \tau_k \leq \tau'_{k+1}$,

2) $r(t) = k, \quad \tau_k \leq t < \tau_{k+1}, \quad \text{if } \tau_{k+1} < \tau'_k$.

3.1. **Strategy of the Evader.** Let $u_i(t)$, $i = 1, \ldots, M$, be arbitrary admissible controls of the pursuers. We construct the strategy of the evader as follows:

$$v(t) = V_0(t) = (0, \alpha + U(t)), \quad t \in [0, T] \setminus J_1, \tag{3.2}$$
$$v(t) = V_r(t) = (V_{r1}(t), \alpha + U(t)), \quad t \in [0, T] \cap J_1, \tag{3.3}$$
$$v(t) = (0, U(t)), \quad t > T, \tag{3.4}$$

where

$$r = r(t), \quad V_k(t) = (V_{k1}(t), \alpha + U(t)), \quad \tau_k < t < \tau'_k, \quad k \in I = \{1, 2, \ldots, m\},$$

$$V_{k1}(t)=\begin{cases}\alpha+|u_{k1}(t)|, & y_1(\tau_k)\geq x_{k1}(\tau_k)\\ -(\alpha+|u_{k1}(t)|), & y_1(\tau_k)<x_{k1}(\tau_k)\end{cases}, \quad U(t)=\left(\sum_{i=1}^{M}u_{i2}^2(t)\right)^{1/2}.$$

To show the admissibility of strategy (3.2)–(3.4), we consider the following functions:

$$f(t)=\begin{cases}(\alpha,\alpha), & t\in[0,T]\setminus J_1,\\ (\alpha,\alpha), & t\in J_1,\\ (0,0), & t>T,\end{cases} \qquad g(t)=\begin{cases}\left(0,\left(\sum\limits_{i=1}^{M}u_{i2}^2(t)\right)^{1/2}\right), & t\in[0,T]\setminus J_1,\\ \left(|u_{r1}(t)|,\left(\sum\limits_{i=1}^{M}u_{i2}^2(t)\right)^{1/2}\right), & t\in J_1,\\ \left(0,\left(\sum\limits_{i=1}^{M}u_{i2}^2(t)\right)^{1/2}\right), & t>T.\end{cases}$$

Then, clearly,

$$\int_0^t|f(s)|^2ds\leq 2\alpha^2t \tag{3.5}$$

and

$$|g(t)|^2=u_{r1}^2(t)+\sum_{i=1}^{M}u_{i2}^2(t)\leq\sum_{i=1}^{M}u_{i1}^2(t)+\sum_{i=1}^{M}u_{i2}^2(t)=\sum_{i=1}^{M}|u_i(t)|^2. \tag{3.6}$$

In view of (3.5) and (3.6) we have

$$\left(\int_0^t|V(s)|^2ds\right)^{1/2}=\left(\int_0^t|f(s)+g(s)|^2ds\right)^{1/2}\leq\left(\int_0^t|f(s)|^2ds\right)^{1/2}+\left(\int_0^t|g(s)|^2ds\right)^{1/2}$$
$$\leq\sqrt{2\alpha^2t}+\left(\int_0^t\sum_{i=1}^{M}|u_i(s)|^2ds\right)^{1/2}\leq\alpha\sqrt{2t}+\left(\sum_{i=1}^{M}\rho_i^2t\right)^{1/2}=\alpha\sqrt{2t}+\rho\sqrt{t}\leq\sigma\sqrt{t}$$

since by choice $\alpha<\frac{\sigma-\rho}{\sqrt{2}}$. Hence, strategy (3.2)–(3.4) is admissible.

Lemma 3.2. *Let the evader apply the strategy* (3.2), (3.3), (3.4).

1) *For any* $k\in\{1,2,3,\dots,m_0\}$, *we have (i)* $\tau_k\leq T_0$, *(ii)* $\tau'_k\leq T$.

2) *If* $y_2^0\geq x_{i_02}^0$ *for some* $i_0\in\{1,2,\dots,M\}$, *then* $y_2(t)>x_{i_02}(t)$, *for all* $t>0$.

Proof. The proof of item 1(i). First, we establish $y_2(T_0)\geq x_{i2}(T_0)$ for all $i=1,2,\dots,M$. Indeed, by (3.2) and (3.3) we have, for any $i\in\{1,2,\dots,M\}$,

$$v_2(t)=\alpha+U(t)\geq\alpha+|u_{i2}(t)|,\quad 0<t\leq T_0<T, \tag{3.7}$$

since

$$U(t)=\left(\sum_{j=1}^{M}u_{j2}^2(t)\right)^{1/2}\geq|u_{i2}(t)|,\quad i\in\{1,2,\dots,M\}.$$

Hence,

$$\frac{d}{dt}(y_2(t)-x_{i2}(t))=v_2(t)-u_{i2}(t)\geq\alpha>0,\quad 0\leq t\leq T,$$

which means $y_2(t)-x_{i2}(t)$ increases strictly.

In view of $T_0 = \frac{1}{\alpha} \max\limits_{1 \le i \le M} |y_2^0 - x_{i2}^0|$ and (3.7), we obtain the following:

$$y_2(T_0) - x_{i2}(T_0) = y_2^0 + \int_0^{T_0} v_2(s)ds - x_{i2}^0 - \int_0^{T_0} u_{i2}(s)ds = y_2^0 - x_{i2}^0 + \int_0^{T_0} (v_2(s) - u_{i2}(s))ds$$
$$= y_2^0 - x_{i2}^0 + \alpha T_0 = y_2^0 - x_{i2}^0 + \max_{1 \le i \le M} |y_2^0 - x_{i2}^0| \ge 0.$$

Consequently, $y_2(T_0) \ge x_{i2}(T_0)$, for all $i = 1, 2, \ldots, M$, which means the evader at the time T_0 is above the horizontal lines passing through the pursuers' states at the time T_0.

Next, since

$$v_2(t) = \alpha + U(t) \ge \alpha + |u_{i2}(t)|, \quad T_0 \le t \le T,$$

and

$$v_2(t) = U(t) \ge |u_{i2}(t)|, \quad t > T,$$

then we have, for all $t > T_0$, that

$$\begin{aligned} y_2(t) - x_{i2}(t) &= y_2(T_0) + \int_{T_0}^{t} v_2(s)ds - x_{i2}(T_0) - \int_{T_0}^{t} u_{i2}(s)ds \\ &= y_2(T_0) - x_{i2}(T_0) + \int_{T_0}^{t} (v_2(s) - u_{i2}(s))ds \\ &\ge \int_{[T_0,t]\cap[T_0,T]} (\alpha + |u_{i2}(s)| - u_{i2}(s))ds \\ &\ge \int_{[T_0,t]\cap[T_0,T]} \alpha ds = \alpha \ \min\{t - T_0, T - T_0\} > 0. \end{aligned}$$

Hence, $y_2(t) > x_{i2}(t)$, for all $t > T_0$ and $i = 1, 2, \ldots, M$. This implies that there is no an approach time τ_k after time T_0 since by the definition of the a_k-approach time of the evader to a pursuer x_i the inequality $x_{i2}(\tau_k) > y_2(\tau_k)$ is required. However, this inequality is not satisfied after time T_0. The proof of the inequality $\tau_k \le T_0$ (item 1(i)) is complete.

Next, to prove the inequality $\tau_k' \le T$, we observe that

$$\tau_k' = \tau_k + \frac{2a_k}{\alpha} \le T_0 + \frac{2a_1}{\alpha} = T,$$

and the proof of item 1(ii) follows.

Finally, to prove item 2, we assume that $y_2^0 \ge x_{i_0 2}^0$ for some $i_0 \in \{1, 2, \ldots, M\}$. Then, clearly, for all $t > 0$,

$$\begin{aligned} y_2(t) - x_{i_0 2}(t) &= y_2^0 - x_{i_0 2}^0 + \int_0^t (v_2(s) - u_{i_0 2}(s))\, ds \\ &\ge \int_{[0,t]\cap[0,T]} (\alpha + |u_{i_0 2}(s)| - u_{i_0 2}(s))\, ds \ge \alpha \ min\{t, T\} > 0. \end{aligned}$$

From this we conclude that $y_2(t) > x_{i_0 2}(t)$ for all $t > 0$, which is the desired result.

From the proved lemma we draw an important conclusion that there is no approach time after T_0 meaning that all the pursuers are "behind the evader" for $t > T_0$, which implies that no pursuer can reach the evader after time T_0.

3.2. **The distance between the pursuer and evader.** Let τ_p be the a_p-approach time of a pursuer x_p, $p \in \{1, 2, \dots, m_0\}$, to the evader y.

A. To estimate the distance between $x_p(t)$ and $y(t)$, we introduce a fictitious evader (FE) whose dynamics is as follows

$$\dot{z}_p = w_p, \quad z_p(\tau_p) = y(\tau_p),$$

on the time interval $[\tau_p, \tau_p')$. Note that the initial state $z_p(\tau_p) = (z_{p1}(\tau_p), z_{p2}(\tau_p))$ of FE coincides with that $y(\tau_p) = (y_1(\tau_p), y_2(\tau_p))$ of the evader. The control w_p of FE is defined by the equation

$$w_p(t) = V_p(t) = (V_{p1}(t), \alpha + U(t)), \quad \tau_p \le t \le \tau_p'.$$

Clearly, $V_{p2}(t) = \alpha + U(t)$, and so, for the point $z_p(t) = (z_{p1}(t), z_{p2}(t))$, we have

$$z_{p2}(t) = z_{p2}(\tau_p) + \int_{\tau_p}^{t} (\alpha + U(s))\, ds = y_2(\tau_p) + \int_{\tau_p}^{t} v_2(s)\, ds = y_2(t), \quad \tau_p \le t \le \tau_p',$$

since $\tau_p' < T$, and so $v_2(s) = \alpha + U(s)$ by (3.2)-(3.3).

Consequently, FE $z_p(t)$, for all $\tau_p \le t \le \tau_p'$, is on the same horizontal line as the real evader $y(t)$.

For definiteness, we assume that $y_1(\tau_p) \ge x_{p1}(\tau_p)$. Then, we have

$$V_{p1}(t) = \begin{cases} \alpha + |u_{p1}(t)|, & y_1(\tau_p) \ge x_{p1}(\tau_p) \\ -(\alpha + |u_{p1}(t)|), & y_1(\tau_p) < x_{p1}(\tau_p) \end{cases} = \alpha + |u_{p1}(t)|.$$

B. We estimate the distance between $x_p(t)$ and $z_p(t)$ for $\tau_p \le t \le \tau_p'$. We estimate this distance in two ways. In the first way, using the inequality $y_1(\tau_p) \ge x_{p1}(\tau_p)$ we have the following estimate for the distance

$$\begin{aligned} |z_p(t) - x_p(t)| \ge z_{p1}(t) - x_{p1}(t) &= y_1(\tau_p) + \int_{\tau_p}^{t} V_{p1}(s)\, ds - x_{p1}(\tau_p) - \int_{\tau_p}^{t} u_{p1}(s) ds \\ &= y_1(\tau_p) - x_{p1}(\tau_p) + \int_{\tau_p}^{t} (V_{p1}(s) - u_{p1}(s))\, ds \\ &\ge \int_{\tau_p}^{t} (\alpha + |u_{p1}(s)| - u_{p1}(s))\, ds \ge \alpha(t - \tau_p) = h_1(t), \ \tau_p \le t \le \tau_p'. \end{aligned} \tag{3.8}$$

In the second way, we have

$$|z_p(t) - x_p(t)| \ge |z_p(\tau_p) - x_p(\tau_p)| - \int_{\tau_p}^{t} |V_p(s) - u_p(s)|\, ds = a_p - \int_{\tau_p}^{t} |V_p(s) - u_p(s)|\, ds, \ \tau_p \le t \le \tau_p'. \tag{3.9}$$

Since

$$\begin{aligned} \int_{\tau_p}^{t} |V_p(s) - u_p(s)|\, ds &\le \left(\int_{\tau_p}^{t} 1^2 ds \right)^{1/2} \left(\int_{\tau_p}^{t} (V_p(s) - u_p(s))^2 ds \right)^{1/2} \\ &\le \sqrt{t - \tau_p} \cdot \left(\int_{\tau_p}^{t} 2\left(|V_p(s)|^2 + |u_p(s)|^2\right) ds \right)^{1/2}, \end{aligned}$$

therefore using the inequalities (recall $t \le \tau_p' < T$)

$$\int_{\tau_p}^{t} |V_p(s)|^2 ds \le \sigma^2 T, \quad \int_{\tau_p}^{t} |u_p(s)|^2 ds \le \rho^2 T < \sigma^2 T$$

we obtain

$$\int_{\tau_p}^{t} |V_p(s) - u_p(s)| ds \le \sqrt{t - \tau_p} \cdot \sqrt{2 \cdot 2\sigma^2 T} = 2\sigma \sqrt{T(t - \tau_p)}.$$

Using this and (3.9) we obtain the following

$$|z_p(t) - x_p(t)| \ge a_p - 2\sigma \sqrt{T(t - \tau_p)} = h_2(t), \ \tau_p \le t \le \tau_p'. \tag{3.10}$$

Combining (3.8) and (3.10) we obtain

$$|z_p(t) - x_p(t)| \ge h(t) = \max\{h_1(t), h_2(t)\}, \quad t \ge \tau_p. \tag{3.11}$$

We observe that the function $h_1(t) = \alpha(t - \tau_p)$, $t \ge \tau_p$, increases and the function $h_2(t) = a_p - 2\sigma\sqrt{T(t - \tau_p)}$, $t \ge \tau_p$, decreases. It is not difficult to see that the function $h(t)$, $t \ge \tau_p$, takes its minimum at the point $t = t_*$ where the graphs of the functions $h_1(t)$ and $h_2(t)$ intersect. To find this point, we solve the equation $h_1(t) = h_2(t)$:

$$\alpha(t - \tau_p) = a_p - 2\sigma \sqrt{T(t - \tau_p)}.$$

Denote $\sqrt{t - \tau_p} = d$ to obtain the equation

$$\alpha d^2 + 2\sigma \sqrt{T} d - a_p = 0. \tag{3.12}$$

Using the inequality $\alpha a_p < \alpha a_1 \le \sigma^2 T$, we estimate the positive root of equation (3.12):

$$d = \frac{-\sigma\sqrt{T} + \sqrt{\sigma^2 T + \alpha a_p}}{\alpha} = \frac{a_p}{\sigma\sqrt{T} + \sqrt{\sigma^2 T + \alpha a_p}} \ge \frac{a_p}{\sigma\sqrt{T} + \sqrt{\sigma^2 T + \sigma^2 T}} = \frac{a_p}{\sigma\sqrt{T}(1 + \sqrt{2})}.$$

Hence,

$$h(t_*) = h_1(t_*) = \alpha d^2 = \frac{\alpha a_p^2}{\left(\sigma\sqrt{T}(1 + \sqrt{2})\right)^2} \ge \frac{\alpha a_p^2}{6\sigma^2 T}. \tag{3.13}$$

Since the FE moves only on the time interval $[\tau_p, \tau_p']$, therefore, we obtain from (3.11) and (3.13) that

$$|z_p(t) - x_p(t)| \ge h(t_*) > \frac{\alpha a_p^2}{6\sigma^2 T}, \quad \tau_p \le t \le \tau_p'. \tag{3.14}$$

C. We estimate the distance between the FE $z_p(t)$ and the evader $y(t)$ on $\tau_p \le t \le \tau_p'$.
Using equation $z_p(\tau_p) = y(\tau_p)$ we have the following

$$|y(t) - z_p(t)| = \left| \int_{\tau_p}^{t} (v(s) - V_p(s))\, ds \right|, \quad \tau_p \le t \le \tau_p', \tag{3.15}$$

where $V_p(t) = (V_{p1}(t), \alpha + U(t))$, $\tau_p \le t < \tau_p'$.
To estimate (3.15), we consider the following two cases: (i) $\tau_{p+1} \ge \tau_p'$ (ii) $\tau_{p+1} < \tau_p'$.
If (i) $\tau_{p+1} \ge \tau_p'$ is the case, then by the property of the function $r(t)$ we have $r = r(t) = p$ for $\tau_p \le t < \tau_p'$, and hence, by (3.3) the maneurer $v(t) = V_r(t) = (V_{r1}(t), \alpha + U(t))$ has the form $v(t) = V_p(t)$, $\tau_p \le t < \tau_p'$. Then, clearly, (3.15) implies that

$$|y(t) - z_p(t)| = 0.$$

Let now (ii) $\tau_{p+1} \le \tau'_p$ is the case. Then, $r = r(t) = p$ for $\tau_p \le t < \tau_{p+1}$, and, hence,

$$v(t) = V_p(t), \quad \tau_p \le t < \tau_{p+1}.$$

Consequently, $v(t)$ coincides with $w_p(t)$ on $\tau_p \le t < \tau_{p+1}$, implying that $y(t) = z_p(t)$, $t \in [\tau_p, \tau_{p+1}]$. In particular, we have $y(\tau_{p+1}) = z_p(\tau_{p+1})$. Using this we obtain the following, for the distance $|y(t)-z_p(t)|$ on $[\tau_{p+1}, \tau'_p]$,

$$\begin{aligned} |y(t) - z_p(t)| &= \left| \int_{\tau_{p+1}}^{t} (v(s) - V_p(s))ds \right| \le \int_{\tau_{p+1}}^{t} |v(s) - V_p(s)|ds \\ &\le \int_{[\tau_{p+1},t]\setminus J_{p+1}} |v(s) - V_p(s)|ds + \int_{[\tau_{p+1},t]\cap J_{p+1}} |v(s) - V_p(s)|ds. \end{aligned} \tag{3.16}$$

Since $r(t) = p$ for $t \in [\tau_p, \tau'_p) \setminus J_{p+1}$ and $[\tau_{p+1}, t] \setminus J_{p+1} \subseteq [\tau_p, \tau'_p) \setminus J_{p+1}$, therefore $v(t) = V_p(t)$ for $t \in [\tau_{p+1}, t] \setminus J_{p+1}$. This implies that the first integral in (3.16) is equal to 0. Thus, (3.16) takes the form

$$|y(t) - z_p(t)| \le \int_{[\tau_{p+1},t]\cap J_{p+1}} |v(s) - V_p(s)|\, ds.$$

From this in view of

$$|v(s) - V_p(s)| = |V_{r1}(s) - V_{p1}(s)| \le |V_{r1}(s)| + |V_{p1}(s)| = 2\alpha + |u_{r1}(s)| + |u_{p1}(s)|$$

we obtain

$$|y(t) - z_p(t)| \le \int_{J_{p+1}} (2\alpha + |u_{r1}(s)| + |u_{p1}(s)|)\, ds \quad = \int_{J_{p+1}} 2\alpha\, ds + \int_{J_{p+1}} |u_{r1}(s)|\, ds + \int_{J_{p+1}} |u_{p1}(s)|\, ds. \tag{3.17}$$

To estimate the first integral in (3.17), we have the following

$$\int_{J_{p+1}} 2\alpha\, ds \le 2\alpha\, \mathrm{mes}(J_{p+1}) \le 2\alpha \sum_{j=p+1}^{m_0} (\tau'_j - \tau_j) \le 2\alpha \sum_{j=p+1}^{m_0} \frac{2a_j}{\alpha} \le 4 \sum_{j=p+1}^{\infty} a_j \le 8a_{p+1}. \tag{3.18}$$

To estimate the second and third integrals in (3.17), we use the Cauchy-Schwarz inequality and (3.18) to obtain

$$\int_{J_{p+1}} |u_{r1}(s)|ds \le \left(\int_{J_{p+1}} 1^2 ds \right)^{1/2} \left(\int_{Jp+1} |u_{r1}(s)|^2(s)ds \right)^{1/2} \le \sqrt{8a_{p+1}}\, \rho\sqrt{T} \tag{3.19}$$

and

$$\int_{J_{p+1}} |u_{p1}(s)|ds \le \left(\int_{J_{p+1}} 1^2 ds \right)^{1/2} \left(\int_{Jp+1} |u_{p1}(s)|^2(s)ds \right)^{1/2} \le \sqrt{8a_{p+1}}\, \rho\sqrt{T}. \tag{3.20}$$

Combining equations (3.17), (3.18), (3.19) and (3.20) we obtain the following estimate for $|y(t)-z_p(t)|$:

$$|y(t) - z_p(t)| \le 8a_{p+1} + 4\rho\sqrt{T}\sqrt{2a_{p+1}} \le 5\rho\sqrt{2T}\sqrt{a_{p+1}}. \tag{3.21}$$

Here we used the inequality $a_{p+1} < a_1 < \frac{\rho^2 T}{32}$ following from (3.1).

D. We estimate the distance between the pursuer $x_p(t)$ and the evader $y(t)$ on the time interval $[\tau_p, \tau_p']$. By (3.14) and (3.21) $a_{p+1} \le \frac{\alpha^2}{7200\sigma^6 T^3} a_p^4$ we have

$$|y(t) - x_p(t)| \ge |x_p(t) - z_p(t)| - |z_p(t) - y(t)| \ge \frac{\alpha a_p^2}{6\sigma^2 T} - 5\rho\sqrt{2T}\sqrt{a_{p+1}} \ge \frac{\alpha a_p^2}{12\sigma^2 T}. \tag{3.22}$$

Next, we estimate $y_2(t) - x_{p2}(t)$ for $t > \tau_p'$. We observe

$$y_2(\tau_p) - x_{p2}(\tau_p) \ge -|y(\tau_p) - z_p(\tau_p)| = -a_p.$$

Since $z_{p2}(\tau_p) = y_2(\tau_p)$, we have

$$z_{p2}(\tau_p') - x_{p2}(\tau_p') = y_2(\tau_p) - x_{p2}(\tau_p) + \int_{\tau_p}^{\tau_p'} (\alpha + U(s) - u_{p2}(s))\, ds \ge -a_p + \int_{\tau_p}^{\tau_p'} \left(\alpha + \left(\sum_{i=1}^{M} u_{i2}^2(s) \right)^{1/2} - u_{p2}(s) \right) ds \ge -a_p + \alpha \int_{\tau_p}^{\tau_p'} ds = -a_p + \alpha(\tau_p' - \tau_p) = a_p. \tag{3.23}$$

Here we used the definition $\tau_p' = \tau_p + \frac{2a_p}{\alpha}$.

By definition of control of FE and evader $v_2(t) = V_{p2}(t) = \alpha + U(t)$, $t \in [\tau_p, \tau_p')$ and therefore $y_2(t) = z_{p2}(t)$, $t \in [\tau_p, \tau_p']$. In particular, $y_2(\tau_p') = z_{p2}(\tau_p')$. In addition, since $v_2(t) = \alpha + U(t)$, $t \in [0, T]$ and $v_2(t) = U(t)$, $t > T$, therefore, we have $v_2(t) \ge U(t) \ge |u_{p2}(t)|$ for all $t \ge 0$. Hence, by (3.23), for $t > \tau_p'$,

$$y_2(t) - x_{p2}(t) = y_2(\tau_p') + \int_{\tau_p'}^{t} v_2(s) ds - \left(x_{p2}(\tau_p') + \int_{\tau_p'}^{t} u_{p2}(s) ds \right) = z_{p2}(\tau_p') - x_{p2}(\tau_p') + \int_{\tau_p'}^{t} (v_2(s) - u_{p2}(s))\, ds \ge z_{p2}(\tau_p') - x_{p2}(\tau_p') \ge a_p. \tag{3.24}$$

Consequently,

$$y_2(t) - x_{p2}(t) \ge a_p, \quad t > \tau_p'. \tag{3.25}$$

In summary,

1) $|y(t) - x_p(t)| > a_p$, $0 \le t < \tau_p$, by the definition of time τ_p,
2) $|y(t) - x_p(t)| \ge \frac{\alpha a_p^2}{12\sigma^2 T} > a_{p+1}$, $\tau_p \le t \le \tau_p'$, by (3.22) and inequalities
$$a_{p+1} \le \frac{\alpha}{12\sigma^2 T} a_p^4 < \frac{\alpha}{12\sigma^2 T} a_p^2,$$
3) $|y(t) - x_p(t)| \ge a_p$, $t > \tau_p'$, by (3.25).

These inequalities allow us to draw the conclusion that $x_p(t) \ne y(t)$ for all $t \ge 0$, which means that evasion is possible.

The conclusions 1)-3) derived above imply that $|y(t) - x_p(t)| \ge a_{p+1}$ for the pursuer x_p, for which the a_p-approach time τ_p is occurred. This means that for the pursuer x_p another a_k-approach time with $k \ge p+1$ further will not occur. Therefore, for each pursuer x_i, $i \in \{1, 2, ..., m\}$, either an a_i-approach time τ_i will not occur or it occurs only once. Hence, the number m_0 of all approach times τ_i is finite. More precisely, $m_0 \le m$. The proof of the theorem is complete.

4. Conclusions

We have studied a differential game of evasion with multiple pursuers and a single evader in which all players' control resources vary over time. The evader's available control resource exceeds the aggregate resource of all pursuers. To the best of our knowledge, evasion problems under integral constraints of this type have not been addressed in the literature.

Our main contributions are (i) an evasion theorem establishing sufficient conditions for successful evasion and (ii) an explicit evasion strategy for the evader that guarantees capture avoidance under those conditions.

As future work, it would be natural to investigate analogous games under alternative constraint classes for the evader and/or to extend the analysis to higher-order dynamics.

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Ibragimov G.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
University of Public Safety of the Republic of Uzbekistan,
Tashkent, Uzbekistan
e-mail: ibragimov.math@gmail.com

Inomiddinov S.N.,
Namangan State University, Namangan, Uzbekistan.
e-mail: sn.inomiddinov@gmail.com

Ibrokhimjonov I.I.,
National University of Uzbekistan, Toshkent, Uzbekistan
e-mail: islombekibrokhimjonov03@gmail.com