Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 28–33
DOI: 10.29229/uzmj.2026-1-3

On a trinomial analogue of Muirhead's inequaty

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Abstract. It is studied inequalities between three elementary symmetric polynomials being trinomial generalization of the well-known Muirhead's inequality. Some sufficient conditions have been derived. A sufficient and necessary condition is given in the case of two variables.

Keywords: Muirhead's inequality, symmetric polynomial, circle division polynomial, arithmetic function
MSC (2020): 26D05; 26E60

1. Introduction

The notable Muirhead's inequality [1]

$$\forall \mathbf{x} \in \mathbb{R}_+^n : \mu_{\boldsymbol{\alpha}}(\mathbf{x}) \geq \mu_{\boldsymbol{\beta}}(\mathbf{x}) \Leftrightarrow \boldsymbol{\alpha} \succeq \boldsymbol{\beta} \tag{1.1}$$

occupies an explicit position in both inequalities and symmetric polynomials theories.

In (1.1), the following denotations are used:

$\mathbb{R}_+^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \,|\, x_k > 0,\, k = 1, 2, \ldots, n\}$ is the positive ortant;

$\mu_{\boldsymbol{\alpha}}(\mathbf{x}) = \frac{1}{n!} \sum\limits_{\boldsymbol{\sigma} \in S_n} x_{\sigma(1)}^{\alpha_1} x_{\sigma(2)}^{\alpha_2} \ldots x_{\sigma(n)}^{\alpha_n}$ is an elementary (i.e., one-term) symmetric polynomial;

S_n is the symmetric group;

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an ordered set of integers satisfying the conditions

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_n = m, \tag{1.2}$$

where m is a fixed positive integer called a degree of the elementary symmetric polynomial $\mu_{\boldsymbol{\alpha}}(\mathbf{x})$. (Bold letters are used for vector quantities.)

The special order $\succeq$ in the set of collections satisfying (1.2) is determined by the following way: $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$ means

$$\alpha_1 \geq \beta_1,\ \alpha_1 + \alpha_2 \geq \beta_1 + \beta_2,\ \ldots,,$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \geq \beta_1 + \beta_2 + \cdots + \beta_{n-1}.$$

Several proofs of Muirhead's inequality are known ([2], Sec. 2.18 and 2.74; [3], Sec. 11.3; three proofs were given in [4]).

The subject was generalized to arbitrary positive quotients by R. Rado [3, 5] (for other generalizations, see [6, 7, 8, 9, 10]). Below, only the classical version of Muirhead's inequality with positive integer quotients will be considered. We are interested in the following question: when does the inequality

$$\forall \mathbf{x} \in \mathbb{R}_+^n :\ \mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \geq 2\mu_{\boldsymbol{\beta}}(\mathbf{x}) \tag{1.3}$$

hold?

If (1.3) is true, then it will be called the trinomial Muirhead inequality. Note if $\boldsymbol{\alpha} = \boldsymbol{\gamma}$, then (1.3) turns into the classical (binomial) Muirhead's inequality. Therefore, we will assume $\boldsymbol{\alpha} < \boldsymbol{\gamma}$ further. Besides, cases $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\boldsymbol{\gamma} = \boldsymbol{\beta}$ are trivial, therefore, it will be supposed $\boldsymbol{\alpha} > \boldsymbol{\beta} > \boldsymbol{\gamma}$.

Some trinomial generalization of Muirhead's inequality was given by I. Schur ([2], Sec. 2.81) for $n = 3$:

$$\mu_{(\alpha+2\beta,\,0,\,0)}(x,\, y,\, z) + \mu_{(\alpha,\,\beta,\,\beta)}(x,\, y,\, z) \geq 2\mu_{(\alpha+\beta,\,\beta,\,0)}(x,\, y,\, z)\,. \tag{1.4}$$

The simplest case of this looks

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + y^2x + x^2z + z^2x + y^2z + z^2y.$$

The present paper is devoted to the discussion of (1.3). It turned out that, unlike the classical binomial Muirhead's inequality (1.1), the trinomial case meets some surprises. In Section 2, a sufficient condition for (1.3) is given. In Section 3, a sufficient and necessary condition is derived for the case $n = 2$. Properties of two arithmetic functions connected with this condition are considered in Section 4. In Section 5, some inequalities of Muirhead-Schur type are given. Section 5 consists of final notes. Besides, two open problems are formulated.

2. A sufficient condition for the general case

Theorem 2.1. *If $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$, then $\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \geq 2\mu_{\boldsymbol{\beta}}(\mathbf{x})$.*

Proof. Let $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$. We set $\bar{\boldsymbol{\alpha}} = 2\boldsymbol{\beta} - \boldsymbol{\gamma}$, i.e. $\bar{\alpha}_k = 2\beta_k - \gamma_k$, $k = 1, 2, \ldots, n$. Then, on one hand

$$\mu_{\bar{\boldsymbol{\alpha}}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) = \tfrac{1}{n!}\sum_{\boldsymbol{\sigma}\in S_n}\left(x_{\sigma(1)}^{\bar{\alpha}_1}x_{\sigma(2)}^{\bar{\alpha}_2}\ldots x_{\sigma(n)}^{\bar{\alpha}_n} + x_{\sigma(1)}^{\gamma_1}x_{\sigma(2)}^{\gamma_2}\ldots x_{\sigma(n)}^{\gamma_n}\right) \geq$$
$$\geq 2\tfrac{1}{n!}\sum_{\boldsymbol{\sigma}\in S_n}\sqrt{x_{\sigma(1)}^{\bar{\alpha}_1+\gamma_1}x_{\sigma(2)}^{\bar{\alpha}_2+\gamma_2}\ldots x_{\sigma(n)}^{\bar{\alpha}_n+\gamma_n}} = 2\mu_{\boldsymbol{\beta}}(\mathbf{x}).$$

On the other hand, $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$ means

$$\begin{aligned}
&\alpha_1 + \gamma_1 \geq 2\beta_1,\\
&\alpha_1 + \gamma_1 + \alpha_2 + \gamma_2 \geq 2\beta_1 + 2\beta_2,\\
&\ldots,\\
&\alpha_1 + \gamma_1 + \alpha_2 + \gamma_2 + \ldots + \alpha_{n-1} + \gamma_{n-1} \geq 2\beta_1 + 2\beta_2 + \ldots + 2\beta_{n-1},\\
&\alpha_1 + \gamma_1 + \alpha_2 + \gamma_2 + \ldots + \alpha_n + \gamma_n = 2\beta_1 + 2\beta_2 + \ldots + 2\beta_n.
\end{aligned}$$

These relations imply

$$\begin{aligned}
&\alpha_1 \geq 2\beta_1 - \gamma_1 = \bar{\alpha}_1,\\
&\alpha_1 + \alpha_2 \geq 2\beta_1 - \gamma_1 + 2\beta_2 - \gamma_2 = \bar{\alpha}_1 + \bar{\alpha}_2,\ \ldots,
\end{aligned}$$

i.e. $\boldsymbol{\alpha} \succeq \bar{\boldsymbol{\alpha}}$. Therefore, due to (1.1)

$$\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \succeq \mu_{\bar{\boldsymbol{\alpha}}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \succeq 2\mu_{\boldsymbol{\beta}}(\mathbf{x}).$$

□

At first, we thought that, analogously to Muirhead's theorem, the condition $\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \succeq 2\mu_{\boldsymbol{\beta}}(\mathbf{x})$ might be not only sufficient but also necessary for (1.2). But (1.4) refutes this supposition. Here is another sample with two variables:

$$x^6 + y^6 + 2x^3y^3 \geq 2\left(x^5y + y^5x\right) \tag{2.1}$$

where $n = 2$, $\boldsymbol{\alpha} = (6, 0)$, $\boldsymbol{\gamma} = (3, 3)$, $\boldsymbol{\beta} = (5, 1)$, and the condition $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$ do not hold. Nevertheless

$$x^6 + y^6 + 2x^3y^3 - 2\left(x^5y + y^5x\right) = (x - y)^2\left(x^4 - x^2y^2 + y^4\right) \geq 0.$$

In this regard, it may be of interest

Problem 1. *Find a necessary and sufficient condition in terms of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ for (1.1) to hold.*

3. A necessary and sufficient condition in the case of two variables.

It is natural to begin investigating trinomial inequalities with the simplest case $n = 2$. Further, we will use the notations: $x_1 = x$, $x_2 = y$. A symmetric polynomial in this case contains only a couple of summands:

$$\frac{x^{k_1}y^{k_2} + y^{k_1}x^{k_2}}{2},\ k_1 + k_2 = m.$$

If all quotients in the trinomial Muirhead's inequality are positive, then it can be divided term by term to the greatest quotient of the product xy. Then, after at least one of the vector-quotients $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ will have a lower component equal to 0. If $\beta_2 = 0$, but $\alpha_2 > 0$ and $\gamma_2 > 0$, the trinomial Muirhead's inequality cannot hold, since β_1 would be greater than both α_1 and γ_1. Therefore, we can assume $\boldsymbol{\alpha} = (m, 0)$. Thus, we will deal with the pairs $(\beta, m - \beta)$ and $(\gamma, m - \gamma)$ instead of the vector-quotients $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ respectively.

Now, the trinomial Muirhead's inequality takes the form

$$x^m + y^m + x^\gamma y^{m-\gamma} + y^\gamma x^{m-\gamma} \geq 2\left(x^\beta y^{m-\beta} + y^\beta x^{m-\beta}\right) \tag{3.1}$$

in the considering case. Among all possible relations between the quotients m, β, γ, only the case $m > \beta > \gamma$ is nontrivial. Moreover, one may suppose $\gamma \geq m - \gamma$. In this regard, we will assume

$$m > \beta > \gamma \geq \frac{m}{2}. \tag{3.2}$$

This condition immediately excludes the case $\gamma = 1$, so that $\gamma \geq 2$, $\beta \geq 3$, and $m \geq 4$. Below, unless otherwise stated, condition (3.2) is assumed to be satisfied. In addition, $x \geq y$ can also be supposed due to the symmetry. Then (3.1) will be equivalent to the inequality

$$\left(x^\beta - 1\right)\left(x^{m-\beta} - 1\right) \geq x^{m-\beta}\left(x^{\beta-\gamma} - 1\right)\left(x^{\beta+\gamma-m} - 1\right). \tag{3.3}$$

Reduction by $(x-1)^2$ brings the last to the form

$$G_\beta(x)\, G_{m-\beta}(x) \geq x^{m-\beta} G_{\beta-\gamma}(x)\, G_{\beta+\gamma-m}(x) \tag{3.4}$$

for circle division polynomials $G_l(x) = 1 + x + x^2 + \cdots + x^{l-1}$ ([11].)

Setting $x = 1$, we come to the conclusion that the condition

$$\beta(m - \beta) \geq (\beta - \gamma)(\beta + \gamma - m) \tag{3.5}$$

is necessary in order (3.1) to be held.

It turns out that the quantity $\Delta = \beta(m-\beta) - (\beta-\gamma)(\beta+\gamma-m)$ plays a key role for trinomial Muirhead's inequality in the discussing case.

Theorem 3.1. $\Delta \geq 0$ *is necessary and sufficient for* (3.1).

Proof. Let $\Delta \geq 0$. It is required to establish the inequality

$$\varphi(x) \stackrel{def}{=} x^m + 1 + x^\gamma + x^{m-\gamma} - 2x^\beta - 2x^{m-\beta} \geq 0$$

for $x \geq 1$. We are going to show, by means of derivatives, that $\varphi(x)$ is increasing. It is more convenient to use the operator $x\frac{d}{dx}$ instead of usual derivation.

We have

$$x\varphi'(x) = mx^m + \gamma x^\gamma + (m-\gamma)\,x^{m-\gamma} - 2\beta x^\beta - 2(m-\beta)\,x^{m-\beta} = x^{m-\beta}\psi(x)$$

where $\psi(x) = mx^\beta + \gamma x^{\beta+\gamma-m} + (m-\gamma)\,x^{\beta-\gamma} - 2\beta x^{2\beta-m} - 2(m-\beta)$.

Further,

$$x\psi'(x) = m\beta x^\beta + \gamma(\beta+\gamma-m)\,x^{\beta+\gamma-m} + (m-\gamma)(\beta-\gamma)\,x^{\beta-\gamma} - 2\beta(2\beta-m)\,x^{2\beta-m} = x^{\beta-\gamma}\chi(x)$$

where $\chi(x) = m\beta x^{\gamma} + \gamma(\beta+\gamma-m)x^{2\gamma-m} + (m-\gamma)(\beta-\gamma) - 2\beta(2\beta-m)x^{\beta+\gamma-m}$.
Similarly

$$x\chi'(x) = m\beta\gamma x^{\gamma} + \gamma(\beta+\gamma-m)(2\gamma-m)x^{2\gamma-m} - 2\beta(2\beta-m)(\beta+\gamma-m)x^{\beta+\gamma-m} = x^{2\gamma-m}\xi(x),$$

where $\xi(x) = m\beta\gamma x^{m-\gamma} + \gamma(\beta+\gamma-m)(2\gamma-m) - 2\beta(2\beta-m)(\beta+\gamma-m)x^{\beta-\gamma}$, and finally,

$$x\xi'(x) = \beta\left[m\gamma(m-\gamma)x^{m-\gamma} - 2(2\beta-m)(2\beta-m)(\beta-\gamma)x^{\beta-\gamma}\right].$$

Since $x \geq 1$ and $m-\gamma > \beta-\gamma > 0$, the inequality $\xi'(x) \geq 0$ is equivalent to

$$m\gamma(m-\gamma) - 2(2\beta-m)(\beta+\gamma-m)(\beta-\gamma) \geq 0.$$

One may verify by direct calculation that the expression on the left side of the last inequality equals $(2\beta-m)\Delta + \gamma(m-\beta)(m-\gamma)$, that is non-negative. Thus $\xi'(x) \geq 0$. Further, it turns out

$$\xi(1) = m\beta\gamma + \gamma(\beta+\gamma-m)(2\gamma-m) - 2\beta(2\beta-m)(\beta+\gamma-m) = (2\beta+2\gamma-m)\Delta \geq 0.$$

(In order to verify, it is enough to open all brackets). Thus, $\xi(x) \geq 0$ implies $\chi'(x) \geq 0$. Moreover, the formula $\chi(1) = 2\Delta$ can be checked as well. Therefore $\chi(x) \geq 0$, that in turn implies $\psi'(x) \geq 0$. However, $\psi(1) = 0$, so $\psi(x) \geq 0$. Now, taking into account the fact that the signs of $\varphi'(x)$ and $\psi(x)$ coincide, as well the value $\varphi(1) = 0$, we obtain $\varphi(x) \geq 0$, which concludes the proof. □

Now we determine the values of m, β and γ for which the condition $\Delta \geq 0$ holds. It is easy to see that the last is equivalent to the relation

$$m \geq m_*(\beta,\gamma) \overset{def}{=} \left\lceil \frac{2\beta^2-\gamma^2}{2\beta-\gamma} \right\rceil = \beta + \left\lceil \frac{\beta\gamma-\gamma^2}{2\beta-\gamma} \right\rceil \tag{3.6}$$

Besides, obviously, $2\beta-\gamma \geq \left\lceil \frac{2\beta^2-\gamma^2}{2\beta-\gamma} \right\rceil$. (Here and below $\lceil s \rceil$ denotes the selling function and $\lfloor s \rfloor$ does the floor function).

Similarly, from $\Delta \geq 0$, we can derive the formula for the biggest value of β for given m and γ. But now we get a function containing quadratic irrationality:

$$\beta^*(m,\gamma) \overset{def}{=} \left\lfloor \frac{m + \sqrt{m^2 - 2m\gamma + \gamma^2}}{2} \right\rfloor$$

Thus, the condition $\Delta \geq 0$ is equivalent to $\beta \leq \beta^*(m,\gamma)$ as well.

Table 1 illustrates values of the function $m_*(\beta,\gamma)$ for $\beta = 3 \div 33$. If $\gamma \geq \beta$, then inequality (1.3) becomes trivial, and the condition $\Delta \geq 0$ is also evident. Cells corresponding to such pairs (γ, β) are left white in the table. Moreover, $\Delta < 0$ for pairs (γ, β) associated with gray cells.

Nontrivial pairs (γ, β), for those (3.1) is valid, are represented by colored cells. In particular, blue cells reflect Theorem 2.1. Other values for which inequality (1.3) holds are highlighted in red. Inequality (1.3) for values γ, β, m_* corresponding to the light-red cells follows from the inequality associated with the dark-red cell located to the left in the same row. Thus, each dark-red cell expresses a significant three-term Muirhead's inequality.

4. One more inequality of I. Schur's type.

Consider the case $n = 3$. We have mentioned that the inequality (1.4) is usually associated with the name of I. Schur [2]. (It should be noted that (1.4) differs from more essential Schur's inequality on the estimation of the polynomial norm called Schur's Lemma as well [12].) Here is a four-term generalization of (1.4).

Theorem 4.1.

$$\mu_{(\alpha+\beta+\gamma,0,0)}(x,y,z) + \mu_{(\alpha,\beta,\gamma)}(x,y,z) \geq \mu_{(\alpha+\beta,\gamma,0)}(x,y,z) + \mu_{(\alpha+\gamma,\beta,0)}(x,y,z) \tag{4.1}$$

Proof. One can assume $x \geq y \geq z$ without loss of generality. This condition implies

$$x^\alpha \left(x^\beta - y^\beta\right)\left(x^\gamma - z^\gamma\right) \geq y^\alpha \left(x^\beta - y^\beta\right)\left(y^\gamma - z^\gamma\right)$$

$$x^\alpha \left(x^\beta - z^\beta\right)\left(x^\gamma - y^\gamma\right) \geq y^\alpha \left(y^\beta - z^\beta\right)\left(x^\gamma - y^\gamma\right)$$

$$z^\alpha \left(z^\beta - x^\beta\right)\left(z^\gamma - y^\gamma\right) + z^\alpha \left(z^\beta - y^\beta\right)\left(z^\gamma - x^\gamma\right) \geq 0$$

(4.1) implies another trinomial Muirhead's inequality. □

Corollary.

$$\mu_{(\alpha+2\beta+2\gamma,\,0,\,0)}\left(x,\,y,\,z\right) + \mu_{(\alpha,\,2\beta,\,2\gamma)}\left(x,\,y,\,z\right) \geq \mu_{(\alpha+\beta+\gamma,\,\beta+\gamma,\,0)}\left(x,\,y,\,z\right). \tag{4.2}$$

One may note that inequalities (4.1) and (4.2) are not mutually comparable.

5. Final notes

$1^{\#}$. Theorems 2.1 and 4.2 remain valid for any non-negative quotients.

$2^{\#}$. It is easy to show that if inequality (1.1) holds then both sides are equal if and only if $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$ or $x_1 = x_2 = \ldots = x_n$.

$3^{\#}$. Each trinomial inequality (1.1) of n variables generates an inequality of $n+1$ variables of the same type by means of the transformation of elementary symmetric polynomials

$$\mu_{\boldsymbol{\alpha}}\left(x_1,\,x_2,\,\ldots,\,x_n\right) \to \mu_{(\boldsymbol{\alpha},\,0)}\left(x_1,\,x_2,\,\ldots,\,x_n,\,x_{n+1}\right) =$$

$$= \tfrac{1}{n+1}\left[\mu_{\boldsymbol{\alpha}}\left(x_1,\,x_2,\,\ldots,\,x_n\right) + \sum_{k=1}^{n} \mu_{\boldsymbol{\alpha}}\left(x_1,\,x_2,\,\ldots,\,x_n\right)|_{x_k = x_{n+1}}\right]$$

For example, the inequality $x^6 + y^6 + 2x^3y^3 \geq 2\left(x^5y + xy^5\right)$ with characteristics $\Delta = 1$ generates a sequence of trinomial Muirhead inequalities

$$x^6 + y^6 + z^6 + x^3y^3 + x^3z^3 + y^3z^3 \geq x^5y + y^5x + x^5z + z^5x + y^5z + z^5y,$$

$$\tfrac{1}{4}\sum_{k=1}^{4} x_k^6 + \tfrac{1}{6}\sum_{1\leq i \leq j \leq 4} x_i^3x_j^3 \geq 2\tfrac{1}{12}\sum_{i,j=1,\,i\neq j}^{4} x_i^5x_j,\,\ldots$$

$4^{\#}$. If $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$ and $\boldsymbol{\beta} + \boldsymbol{\delta} \succeq 2\boldsymbol{\gamma}$ then $\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \geq 2\mu_{\boldsymbol{\beta}}(\mathbf{x})$, $\mu_{\boldsymbol{\beta}}(\mathbf{x}) + \mu_{\boldsymbol{\delta}}$. Adding these inequalities one gets four-term Muirhead's inequality $\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\delta}}(\mathbf{x}) \geq \mu_{\boldsymbol{\beta}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x})$ differ from (4.1).

Thus, we can conclude that the theory of symmetric homogeneous inequalities, initiated by Muirhead more than a century ago, is fraught with many more mysteries. We hope that the present work will serve as a stimulus for further research on the fundamental problem of three-term Muirhead's inequalities: to determine the necessary and sufficient conditions that α, β, γ must satisfy in order for inequality (1.3) to hold.

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Table 1

	γ																														
β	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
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33																				39	39	39	39	39	38	38	38	37	37	36	35	34

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