

On $(O-C)$ -compactness of the space of idempotent probability measures

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Abstract. We establish a connection between connected sets and max-plus connected sets. Then this connection is applied between spaces of ordinary probability measures and idempotent (max-plus) probability measures on a given Tychonoff space. We show that for every Tychonoff space X the space $I_\beta(X)$ is max-plus convex and therefore, path max-plus-connected. Combining these properties with topological arguments we prove that for a Tychonoff space X the space $I_\beta(X)$ is $(O-C)$ -compact. Auxiliary propositions relating path connectedness and max-plus connectedness are established and used in the derivation of the main results.

Keywords: idempotent probability measures, max-plus probability measures, Tychonoff spaces, max-plus convexity, path connectedness, $(O-C)$ -compactness

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1. INTRODUCTION

Beginning in the last decade of the past century, idempotent mathematics has been extensively developed (see, e.g., [1, 2, 3, 4, 5]). In [6], the systematic study of the space of idempotent probability measures on the algebra of continuous functions on a given compact Hausdorff space was initiated and many results similar to those of [7, 8, 9] for usual probability measures were obtained. Further progress was made in [10, 11, 12, 13, 14]. In the paper [12], a modified method proposed in [15] was applied. More advanced results on idempotent probability measures were obtained in subsequent works.

By a *space* we mean a topological T_1 -space, and by a *map* a continuous map. Recall that a Tychonoff space is a space X such that every closed subset $F \subset X$ and any point $x \in X \setminus F$ are functionally separated, i. e., there exists a continuous function $\varphi: X \rightarrow \mathbb{R}$ for which $\varphi(x) = 0$ and $\varphi(y) = 1$ for $y \in F$.

A finite sequence of subsets $M_0, \dots, M_s$ of a set X is [14] a *chain* connecting sets M_0 and M_s , if $M_{i-1} \cap M_i \neq \emptyset$ for all $i = 1, \dots, s$. A collection ω of subsets of a set X is said to be *connected* if for any pair of sets $M, M' \subset X$ there exists a chain ω connecting the sets M and M' . The maximal connected sub-collections of ω are called *components* of ω . A star-finite open cover of a space X is said to be a *finite-component cover* if the number of elements of each component is finite.

1.1. Preliminaries. Let βX be the Stone-Čech compactification of a Tychonoff space X (i.e., the maximal compact extension).

The author of [6] observed the functor $I: \mathcal{Comp} \rightarrow \mathcal{Comp}$ and showed that it is normal. Then in [12] using the construction suggested by A.Ch. Chigogidze [15], it was obtained an extension $I_\beta: \mathcal{Tych} \rightarrow \mathcal{Tych}$. Here, the sign $\mathcal{Comp}$ means the category of compact Hausdorff spaces and their continuous maps, and $\mathcal{Tych}$ the category of Tychonoff spaces and their continuous maps.

Consider the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with two algebraic operations, addition $\oplus$ and multiplication $\odot$, defined by $u \oplus v = \max\{u, v\}$, $u \odot v = u + v$, where $u, v \in \mathbb{R}$.

Let X be a compact Hausdorff space, and let $C(X)$ be the algebra of continuous functions on X with ordinary algebraic operations. We also define operations $\oplus$ and $\odot$ on $C(X)$ by

$$\varphi \oplus \psi = \max\{\varphi, \psi\}, \quad \text{and} \quad \varphi \odot \psi = \varphi + \psi, \quad \text{where} \quad \varphi, \psi \in C(X).$$

Definition 1.1. [6] A functional $\mu: C(X) \rightarrow \mathbb{R}$ is called an *idempotent probability measure* on X if it has the following properties:

- 1I. $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$, where λ_X is the constant function taking the value λ ;

2I. $\mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$;

3I. $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

For a compact space X , we denote by $I(X)$ the set of all idempotent probability measures on X . We consider $I(X)$ as a subspace of $\mathbb{R}^{C(X)}$.

It can be shown that, for any compact spaces X and Y and any continuous map $f: X \rightarrow Y$, the map $I(f): I(X) \rightarrow I(Y)$ defined by

$$I(f)(\mu)(\psi) = \mu(\psi \circ f)$$

is continuous. Moreover, the construction I is a normal functor. Given any idempotent probability measure $\mu \in I(X)$, we define its support as

$$\text{supp } \mu = \cap \{A \subset X : A = \overline{A}, \mu \in I(A)\},$$

thus, the support of an idempotent probability measure μ is the least (by inclusion) closed subset of X on which μ is concentrated.

For each positive integer n , we set

$$I_n(X) = \{\mu \in I(X) : |\text{supp } \mu| \leq n\} \quad \text{and} \quad I_\omega(X) = \bigcup_{n=1}^{\infty} I_n(X).$$

The set $I_\omega(X)$ is dense in $I(X)$ [6]. An idempotent probability measure $\mu \in I_\omega(X)$ is said to be *finitely supported*. Note that if μ is an idempotent probability measure with finite support $\text{supp } \mu = \{x_1, x_2, \dots, x_k\}$, then μ admits a unique representation in the form

$$\mu = \lambda_1 \odot \delta_{x_1} \oplus \lambda_2 \odot \delta_{x_2} \oplus \dots \oplus \lambda_k \odot \delta_{x_k},$$

where $-\infty < \lambda_i \leq 0$ for $i = 1, 2, \dots, k$ and $\lambda_1 \oplus \lambda_2 \oplus \dots \oplus \lambda_k = 0$. Here, as usual, given $x \in X$, by δ_x we denote the functional on $C(X)$ defined by $\delta_x(\varphi) = \varphi(x)$ for $\varphi \in C(X)$ and called the *Dirac measure*. This measure is concentrated at x .

For a Tychonoff space X , let βX be its the maximal compact (i. e. the Stone-Čech) extension. Following [12], consider the set

$$I_\beta(X) = \{\mu \in I(\beta X) : \text{supp } \mu \subset X\}.$$

Since the support $\text{supp } \mu$ of $\mu \in I_\beta(X)$ is compact, μ is called an idempotent measure with compact support. $I_\beta(X)$ is considered as a subspace of $I(\beta X)$.

Note that a functional $\mu: C(X) \rightarrow \mathbb{R}$ is a *probability measure* on X if it satisfies the following properties:

1P. $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$;

2P. $\mu(\lambda\varphi) = \lambda\mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$;

3P. $\mu(\varphi + \psi) = \mu(\varphi) + \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

The sets $P(X)$ of all probability measures and $I(X)$ of all idempotent probability measures are endowed with the topology of pointwise convergence. For a compact Hausdorff space X the topological spaces $P(X)$ and $I(X)$ are compact Hausdorff spaces.

In the case when X is a Tychonoff space we put

$$P_\beta(X) = \{\mu \in P(\beta X) : \text{supp } \mu \subset X\}.$$

Although constructions P and I are not isomorphic functors, in [16] it was shown that the spaces $P(X)$ and $I(X)$ are homeomorphic for any compact metrizable space X .

2. FUNCTOR OF IDEMPOTENT PROBABILITY MEASURES AND $(O-C)$ -COMPACTNESS

In the current work we establish that the above mentioned result in [16] admits extension for wider classes of topological spaces, exactly, for the class of Tychonoff spaces. In [17] the authors shown that the functor I_β transforms Π -complete spaces into Π -complete spaces. We will show that acting at a Tychonoff space X by the functor I_β one gets a $(O-C)$ -compact space $I_\beta(X)$.

Theorem 2.1. *For any Tychonoff space X , the spaces $P_\beta(X)$ and $I_\beta(X)$ are homeomorphic.*

Proof: Let $\mu \in P_\beta(X)$. Then $\text{supp } \mu$ is a compact set. Consider the map

$$z_I^P : P_\omega(\text{supp } \mu) \rightarrow I_\omega(\text{supp } \mu)$$

given [16] by

$$z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) = \bigoplus_{i=1}^n \left(\left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right)$$

and the map

$$z_P^I : I_\omega(\text{supp } \mu) \rightarrow P_\omega(\text{supp } \mu)$$

given by

$$z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = \sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \cdot \delta_{x_i}.$$

It is easy to see that

$$\text{supp} \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) = \text{supp} \left(z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) \right)$$

and

$$\text{supp} \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = \text{supp} \left(z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) \right).$$

The maps z_I^P and z_P^I are continuous and mutually inverse maps [16]. Therefore, $z_P^I z_I^P : P_\omega(\text{supp } \mu) \rightarrow P_\omega(\text{supp } \mu)$ and $z_I^P z_P^I : I_\omega(\text{supp } \mu) \rightarrow I_\omega(\text{supp } \mu)$ are the identity maps. Really, for each probability measure

$$\sum_{i=1}^n \alpha_i \delta_{x_i} \in P_\omega(\text{supp } \mu),$$

we have

$$\begin{aligned} z_P^I z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) &= z_P^I \left(\bigoplus_{i=1}^n \left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right) = \sum_{i=1}^n \frac{\exp(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j)}{\sum_{l=1}^n \exp(\ln \alpha_l - \bigoplus_{j=1}^n \ln \alpha_j)} \delta_{x_i} \\ &= \sum_{i=1}^n \frac{\exp(\ln \alpha_i)}{\exp(\bigoplus_{j=1}^n \ln \alpha_j)} \cdot \frac{\sum_{l=1}^n \exp(\ln \alpha_l)}{\exp(\bigoplus_{j=1}^n \ln \alpha_j)} \delta_{x_i} = \sum_{i=1}^n \frac{\alpha_i}{\sum_{l=1}^n \alpha_l} \delta_{x_i} = \sum_{i=1}^n \alpha_i \delta_{x_i}. \end{aligned}$$

For each idempotent probability measure

$$\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I_\omega(\text{supp } \mu),$$

we have

$$z_I^P z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = z_I^P \left(\sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \delta_{x_i} \right) = \bigoplus_{i=1}^n \left(\ln \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} - \bigoplus_{l=1}^n \ln \frac{e^{\lambda_l}}{\sum_{j=1}^n e^{\lambda_j}} \right) \odot \delta_{x_i}$$

$$\begin{aligned}
&= \bigoplus_{i=1}^n \left(\ln e^{\lambda_i} - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \left(\ln e^{\lambda_l} - \ln \sum_{j=1}^n e^{\lambda_j} \right) \right) \odot \delta_{x_i} \\
&= \bigoplus_{i=1}^n \left(\lambda_i - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \lambda_l + \ln \sum_{j=1}^n e^{\lambda_j} \right) \odot \delta_{x_i} = \bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i}.
\end{aligned}$$

Theorem 2.1 is proved. $\square$

A nonempty set L is said to be a *max-plus linear space* over a field $\mathbb{K}$ if it is linear with respect to operations $\oplus$ and $\odot$. Here $\oplus$ is the new addition, and $\odot$ the multiplication by scalar. A subset A of a max-plus linear space L over a field $\mathbb{K}$ is said to be a *max-plus linear subspace* of L if

- (1) $c_X \in A$ for each $c \in \mathbb{K}$;
- (2) $\lambda \odot \varphi \in A$ for any $\lambda \in \mathbb{K}$ and $\varphi \in A$;
- (3) $\varphi \oplus \psi \in A$ for any $\varphi, \psi \in A$.

We note that the first condition can be replaced by the requirement $0_X \in L$. Then condition 2 yields condition 1.

A subset $A \subset L$ is called *max-plus-convex* if for every pair $x_1, x_2 \in A$ we have $\lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2 \in A$ for all $\lambda_1, \lambda_2 \in \mathbb{R}_{\max}$ with $\lambda_1 \oplus \lambda_2 = 0$. In other words,

$$[x_1, x_2]_{\oplus} = \{\lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2 : \lambda_1, \lambda_2 \in \mathbb{R}_{\max} \text{ with } \lambda_1 \oplus \lambda_2 = 0\} \subset A.$$

A subset A of a topological space X is *path max-plus-connected* when, for every pair of points $a, b \in A$, there exists a continuous function $f: [-\infty, 0]_{\oplus} \rightarrow A$ such that $f(-\infty) = a$ and $f(0) = b$.

It is easy to verify that the following statement is faithful.

Proposition 2.1. *Every max-plus-convex set is a path max-plus-connected set.*

Theorem 2.2. *Let X be a set in a linear space L , and Y a set in a max-plus linear space M . Moreover, let X and Y be homeomorphic. Then Y is max-plus connected if and only if X is path connected.*

Proof: Let $f: X \rightarrow Y$ is a homeomorphism, X a path connected set in a linear space, and $y_1, y_2 \in Y$ an arbitrary couple of points. Let $x_1, x_2 \in X$ be corresponding preimages of y_1, y_2 , i. e. $f(x_i) = y_i$, $i = 1, 2$.

Consider any continuous map $g: [0, 1] \rightarrow X$ with $g(0) = x_1, g(1) = x_2$. The exponential function $e: [-\infty, 0] \rightarrow [0, 1]$, $e(t) = e^t$, is a homeomorphism and satisfies equalities $e(-\infty) = 0$ and $e(0) = 1$. It reminds to remark that $h = f \circ g \circ e: [-\infty, 0]_{\oplus} \rightarrow Y$ is a max-plus path such that $h(-\infty) = y_1$ and $h(0) = y_2$.

Let Y be a max-plus connected set. Take a max-plus path $H: [-\infty, 0]_{\oplus} \rightarrow Y$ with $H(-\infty) = y_1, H(0) = y_2$, and any pair of points $x_1, x_2 \in X$. Suppose that $f(x_i) = y_i, i = 1, 2$. Then the composition $f^{-1} \circ H \circ \ln: [0, 1] \rightarrow X$ is the searching path. Here $\ln: [0, 1] \rightarrow [-\infty, 0]_{\oplus}$ is the logarithmic function.

Theorem 2.2 is proved. $\square$

It is well known [6] for a compact Hausdorff space X the set $I(X)$ is a max-plus convex. We affirm the following statement.

Proposition 2.2. *For a Tychonoff space X the set $I_{\beta}(X)$ is max-plus convex.*

Proof: The fact that a finite union of compact sets is also compact, completes the proof. $\square$

Definition 2.1. [14]. A topological space X is called (O-C)-compact if every clopen cover of X admits a finite subcover.

Since each connected space is (O-C)-compact [14] according to Theorem 2.2 we have the following result.

Proposition 2.3. *A max-plus connected space is (O-C)-compact.*

Theorem 2.2, Propositions 2.2, 2.3 imply the following main result of the paper.

Theorem 2.3. *For any Tychonoff space X the space $I_{\beta}(X)$ is (O-C)-compact.*

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