

Symbolic Dynamics and Markov Structure for Tent-Type Maps

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Abstract. We study a parameter-dependent family of tent-type maps and focus on the distinguished parameter value $\omega_0 = \frac{1+\sqrt{5}}{2}$. After normalizing the dynamics to the unit interval, we show that the backward orbit of the unique turning point generates a Markov partition with a Fibonacci-type combinatorial structure. This yields an explicit symbolic model: the normalized map is (up to a standard countable exceptional set) topologically conjugate to the golden mean shift. We also determine an invariant probability measure that is absolutely continuous with respect to Lebesgue measure by solving the transfer-operator fixed point equation for the corresponding density. Finally, using the Markov/symbolic description, we compute escape rates for Markov holes.

Keywords: chaotic map; tent map; Markov property; symbolic dynamics; escape rate

MSC (2020): 37B10, 37E05

1. INTRODUCTION

Piecewise monotone interval maps provide a classical source of chaotic dynamical systems while also offering a tractable framework in which geometric dynamics can be translated into symbolic and combinatorial models. Among them, tent maps and their variants occupy a central position: they exhibit rich topological behavior, admit explicit inverse branches, and often allow the construction of Markov partitions and symbolic codings. Such codings are particularly useful for studying invariant measures, periodic points, entropy, and quantitative features of open systems such as escape rates. In this paper we consider the tent family $T_\omega : \mathbb{R} \rightarrow \mathbb{R}$, $\omega > 0$, defined by

$$T_\omega(x) = \begin{cases} \omega x, & x \leq \frac{1}{2}, \\ \omega(1-x), & x > \frac{1}{2}, \end{cases}$$

and focus on the chaotic regime $1 < \omega < 2$. Hasfura-Buenaga and Lynch [1] proved that in this range the map is chaotic in the sense of Devaney [2]: it is topologically transitive, periodic points are dense, and the system has sensitive dependence on initial conditions. For our purposes it is convenient to restrict to a forward invariant interval and then normalize the dynamics to the unit interval. Indeed, for $1 < \omega < 2$ the map leaves invariant

$$I_\omega = \left[\omega - \frac{\omega^2}{2}, \frac{\omega}{2} \right],$$

and via the affine rescaling

$$h_\omega : [0, 1] \rightarrow I_\omega, \quad h_\omega(x) = \left(\omega - \frac{\omega^2}{2} \right) (1-x) + \frac{\omega}{2} x,$$

we obtain the conjugate map

$$\widehat{T}_\omega = h_\omega^{-1} \circ T_\omega \circ h_\omega : [0, 1] \rightarrow [0, 1],$$

which is piecewise linear and expanding with a single turning point. This normalization fixes the phase space to $[0, 1]$ and makes Markov and symbolic constructions transparent.

A fundamental theme in one-dimensional dynamics is that a Markov partition provides a bridge from interval dynamics to symbolic dynamics. If a map admits a Markov partition, then its dynamics can be modeled by a shift space whose admissibility rules are encoded by a transition matrix. This viewpoint allows one to import tools from symbolic dynamics and Perron–Frobenius theory, including

Parry measures and spectral methods (see, e.g., [3, 4]). In particular, for expansive Markov maps the coding map intertwines the dynamics and, outside a small exceptional set, yields a conjugacy with a subshift of finite type.

The central object of the present paper is the parameter value

$$\omega_0 = \frac{1 + \sqrt{5}}{2},$$

for which the partition generated by the backward orbit of the turning point forms a Markov partition with a Fibonacci-type combinatorial structure. More precisely, the primary partition $\mathcal{P}_1 = \{[0, c], [c, 1]\}$, where $c = (3 - \sqrt{5})/2$, generates a refining sequence of partitions $\{\mathcal{P}_n\}_{n \geq 1}$ whose endpoints belong to the backward orbit $\{0, c, \bar{T}_{\omega_0}^{-1}c, \dots, \bar{T}_{\omega_0}^{-n}c, 1\}$. Each $\mathcal{P}_n$ consists of intervals of exactly two lengths (“small” and “big”), and the numbers of small and big intervals follow the Fibonacci recursion. This structure leads naturally to the golden mean shift, i.e. the subshift of finite type on $\{0, 1\}$ with transition matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

equivalently the shift obtained by forbidding the word 00. We construct an explicit coding and show that $\bar{T}_{\omega_0}$ is topologically conjugate (up to the standard countable exceptional set) to this shift.

Since $\bar{T}_{\omega}$ does not preserve Lebesgue measure in general, a further goal is to determine a natural invariant probability measure. For the piecewise linear expanding map $\bar{T}_{\omega_0}$ we compute an absolutely continuous invariant measure using the transfer (Perron–Frobenius) operator. On the symbolic side, irreducible subshifts of finite type carry a unique measure of maximal entropy, the Parry measure [4], and we relate the interval invariant measure to the symbolic Parry measure via the conjugacy.

Finally, we apply the symbolic Markov structure to open dynamical systems. Given a hole $H \subset [0, 1]$ (typically a union of elements of $\mathcal{P}_n$), one studies the survivor sets of points whose orbits avoid H up to time n and defines the exponential escape rate from the decay of their measures. For Markov holes, escape rates can be expressed in terms of Perron roots (or, equivalently, topological entropies) of subshifts obtained by adding the forbidden words corresponding to the hole (see, e.g., [5, 3]). We also discuss local escape rates for shrinking holes, which distinguish between periodic and non-periodic points and admit explicit formulas in the symbolic setting.

Tent and Generalized Luroth Series (GLS) maps possess a unique combination of ergodicity, strong mixing, positive Lyapunov exponent, and robust chaos, which makes them particularly effective in engineering applications [6]. The identity $\lambda_{\text{GLS}} = H_{\text{GLS}}$ provides a rigorous bridge between dynamical systems and information theory, forming the theoretical basis for Shannon-optimal lossless compression and arithmetic coding schemes. Their symbolic dynamics and finite-precision implementability enable secure cryptographic protocols, pseudo-random number generation, and joint compression-encryption mechanisms. Furthermore, GLS-based neurons exploiting topological transitivity and universal orbits form the mathematical foundation of Neurochaos Learning architectures for efficient classification tasks [6].

Outline of the paper. Section 2 introduces the family of tent-type maps and establishes the existence and uniqueness of an absolutely continuous invariant measure for $\hat{T}_{\omega_0}$. Section 3 proves that the backward orbit of the turning point generates a sequence of Markov partitions $\mathcal{P}_n$, and establishes their explicit Fibonacci-type structure (Theorems 3.9 and 3.10). Section 4 develops the symbolic dynamics associated with these partitions and proves (semi-)conjugacy with the golden mean shift, together with the description of the corresponding Parry measure. Section 5 studies periodic points and derives escape rates through Markov holes, including local escape rates for shrinking holes.

2. THE FAMILY OF TENT-TYPE MAPS

Let $T_{\omega} : \mathbb{R} \rightarrow \mathbb{R}$ be the family of tent maps defined by (see Figure 1)

$$T_{\omega}(x) = \begin{cases} \omega x, & x \leq \frac{1}{2}, \\ \omega(1 - x), & x > \frac{1}{2}, \end{cases} \quad \omega > 0.$$

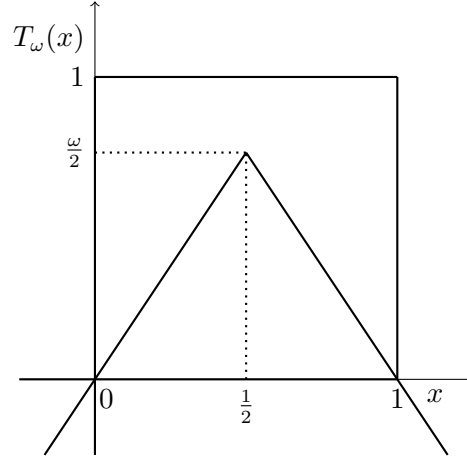


Figure 1

This family was studied by Hasfura–Buenaga and Lynch [1] in the parameter range $1 < \omega < 2$. They proved that for $1 < \omega < 2$ the map T_ω is chaotic in the sense of Devaney (see [2],[7]), namely:

- (1) T_ω is topologically transitive, i.e. for any nonempty open sets $U, V \subset \mathbb{R}$ there exists $n \geq 1$ such that $T_\omega^n(U) \cap V \neq \emptyset$;
- (2) the set of periodic points of T_ω is dense;
- (3) T_ω has sensitive dependence on initial conditions.

For $1 < \omega < 2$ the map T_ω leaves invariant the interval

$$I_\omega = \left[\omega - \frac{\omega^2}{2}, \frac{\omega}{2} \right] \subset [0, 1], \text{ i.e. } T_\omega(I_\omega) \subseteq I_\omega.$$

Hence, throughout this paper we study the restriction

$$T_\omega|_{I_\omega} : I_\omega \rightarrow I_\omega.$$

To normalize the dynamics to the unit interval, we introduce the affine homeomorphism

$$h_\omega : [0, 1] \rightarrow I_\omega, \quad h_\omega(x) = \left(\omega - \frac{\omega^2}{2} \right) (1 - x) + \frac{\omega}{2} x,$$

and define the conjugate map $\hat{T}_\omega : [0, 1] \rightarrow [0, 1]$ as

$$\hat{T}_\omega := h_\omega^{-1} \circ T_\omega \circ h_\omega.$$

A direct computation shows that $\hat{T}_\omega$ is a piecewise linear expanding map (see Figure 2)

$$\hat{T}_\omega(x) = \begin{cases} \omega x + 2 - \omega, & 0 \leq x \leq \frac{\omega - 1}{\omega}, \\ -\omega x + \omega, & \frac{\omega - 1}{\omega} < x \leq 1. \end{cases}$$

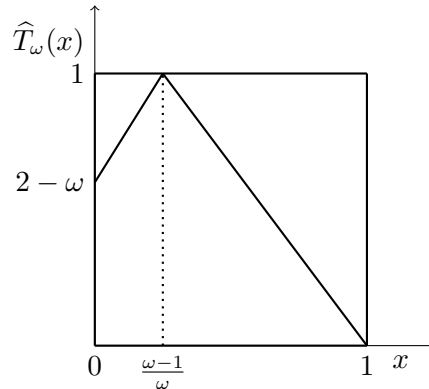


Figure 2

We determine the values of the parameter ω for which a Markov partition can be constructed on the interval $[0, 1]$ for the map $\hat{T}_\omega$. Let $f : [a, b] \rightarrow [a, b]$ be a map.

Definition 2.1. (see, for instance, [8]). Let $\mathcal{P} = \{I_0, I_1, \dots, I_n\}$ be a partition of the interval $[a, b]$ with endpoints $a = x_0 < x_1 < \dots < x_n = b$. The partition $\mathcal{P}$ is called a **Markov partition** if $f(x_i) = x_j$, for every i , that is, if the set of boundary points of the partition elements is invariant under the action of f .

We choose the initial partition

$$\mathcal{P}_0 = \left\{ I_0 = \left[0, \frac{\omega-1}{\omega} \right], I_1 = \left[\frac{\omega-1}{\omega}, 1 \right] \right\}.$$

The boundary points of $\mathcal{P}_0$ are given by

$$x_0 = 0, \quad x_1 = \frac{\omega-1}{\omega}, \quad x_2 = 1.$$

A direct computation shows that

$$\hat{T}_\omega(0) = 2 - \omega, \quad \hat{T}_\omega\left(\frac{\omega-1}{\omega}\right) = 1, \quad \hat{T}_\omega(1) = 0.$$

For the partition $\mathcal{P}_0$ to be a Markov partition for the map $\hat{T}_\omega$, the invariance condition for the boundary points must be satisfied. In particular, we require $\hat{T}_\omega(0) = \omega^{-1}(\omega-1)$. Solving the equation $2 - \omega = \frac{\omega-1}{\omega}$, we obtain the unique admissible value $\omega_0 = \frac{1+\sqrt{5}}{2}$.

In the present work we focus on the distinguished parameter value $\omega_0 := \frac{1+\sqrt{5}}{2}$, for which the partition generated by the backward orbit of the turning point forms a Markov partition. In this case the normalized map takes the form (see Figure 3)

$$\hat{T}_{\omega_0}(x) = \begin{cases} \frac{1+\sqrt{5}}{2}x + \frac{3-\sqrt{5}}{2}, & 0 \leq x \leq \frac{3-\sqrt{5}}{2}, \\ \frac{1+\sqrt{5}}{2}(1-x), & \frac{3-\sqrt{5}}{2} < x \leq 1. \end{cases}$$

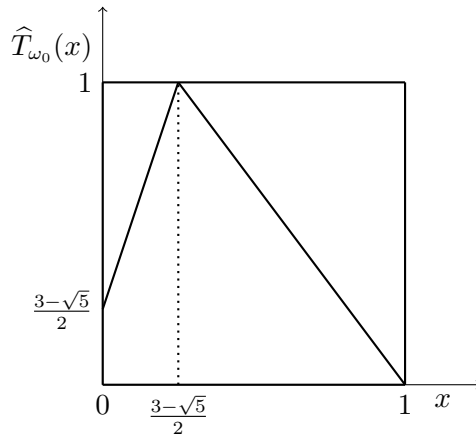


Figure 3

The map $\hat{T}_\omega$ does not preserve the Lebesgue measure λ . Indeed, for a general measurable set $A \subset [0, 1]$ one has

$$\lambda(A) \neq \lambda(\hat{T}_\omega^{-1}(A)).$$

Therefore, it is natural to seek an invariant probability measure that is absolutely continuous with respect to λ .

Lemma 2.1. *For the map $\hat{T}_{\omega_0}$ there exists a unique invariant probability measure μ that is absolutely continuous with respect to the Lebesgue measure λ .*

Proof: Since $\widehat{T}_{\omega_0}$ is a piecewise linear expanding map with constant slope φ on each monotonicity interval, the existence of an absolutely continuous invariant measure follows from the classical theory of Perron–Frobenius (transfer) operators for expanding interval maps (see [9],[10]).

Let L denote the Perron–Frobenius operator associated with $\widehat{T}_{\omega_0}$, acting on integrable functions h by

$$(Lh)(x) = \sum_{y \in \widehat{T}_{\omega_0}^{-1}(x)} \frac{h(y)}{|\widehat{T}'_{\omega_0}(y)|}.$$

Denoting $c := (3 - \sqrt{5})/2$ and $\omega_0 := (1 + \sqrt{5})/2$, the operator L satisfies

$$(Lh)(x) = \begin{cases} \omega_0^{-1} h\left(\frac{\omega_0 - x}{\omega_0}\right), & x \in [0, c), \\ \omega_0^{-1} \left[h\left(\frac{x - c}{\omega_0}\right) + h\left(\frac{\omega_0 - x}{\omega_0}\right) \right], & x \in [c, 1]. \end{cases}$$

Solving the fixed point equation $Lh = h$ under the normalization $\int_0^1 h(x) dx = 1$, we obtain the invariant density

$$h(x) = \begin{cases} \frac{5 + \sqrt{5}}{10}, & 0 \leq x \leq c, \\ \frac{3\sqrt{5} + 5}{10}, & c < x \leq 1. \end{cases}$$

Defining $\mu(A) = \int_A h(x) dx$ for measurable sets $A \subset [0, 1]$, it follows directly from $Lh = h$ that μ is invariant under $\widehat{T}_{\omega_0}$, i.e.

$$\mu(A) = \mu(\widehat{T}_{\omega_0}^{-1}(A)).$$

This completes the proof of Lemma 2.1. □

3. CONSTRUCTION OF A MARKOV PARTITION FOR $\widehat{T}_{\omega_0}$

In this section we construct a Markov partition for the normalized tent-type map $\widehat{T}_{\omega_0} : [0, 1] \rightarrow [0, 1]$, where

$$\omega_0 = \frac{1 + \sqrt{5}}{2}, \quad c := \frac{3 - \sqrt{5}}{2}.$$

The partition is generated by the backward orbit of the corner point c and yields a symbolic model conjugate to the golden mean shift.

3.1. Basic definitions. The **golden mean** is the irrational number $\omega_0 := \frac{1+\sqrt{5}}{2}$, which admits the purely periodic simple continued fraction expansion

$$\omega_0 = 1 + \frac{1}{1 + \frac{1}{1 + \dots}} = [1; 1, 1, 1, \dots].$$

Denoting by

$$\frac{p_n}{q_n} := [1; \underbrace{1, 1, \dots, 1}_{n \text{ times}}]$$

the n -th convergent of this expansion, the numerators and denominators satisfy the recursions

$$p_{-1} = 1, \quad p_0 = 1, \quad p_n = p_{n-1} + p_{n-2},$$

$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = q_{n-1} + q_{n-2}, \quad n \geq 1.$$

It is easy to see that $q_n = q_{n-1}$, $n \in \mathbb{N}$. Using the recursive relation for q_n we get the sequence the Fibonacci numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

Remark 3.1. An important arithmetic property of the Fibonacci sequence concerns the parity of its terms. In particular, the parity follows a periodic pattern of period three: for every integer $n \geq 0$, the Fibonacci number q_{3n} is even, while the two successive terms q_{3n+1} and q_{3n+2} are odd. As a result, the sequence alternates regularly according to the repeating pattern *odd, odd, even*.

Next we give necessary definitions from theory of Markov partitions, Expanding Markov maps their natural symbolic dynamics.

Definition 3.1. (see [11]) Let $I = [a, b] \subset \mathbb{R}$ be a closed interval. A finite collection of closed intervals $\{I_0, \dots, I_{N-1}\}$ is called a **partition** of I if

- (1) $I = \bigcup_{i=0}^{N-1} I_i$;
- (2) $\text{int}(I_i) \cap \text{int}(I_j) = \emptyset$ for $i \neq j$.

Definition 3.2. (see [11]) A map $f : I \rightarrow I$ is called a **Markov map** if there exists a partition $\{I_0, \dots, I_{N-1}\}$ of I such that for every pair (i, j) , $0 \leq i, j \leq N-1$,

- (1) either $f(\text{int } I_i) \cap \text{int } I_j = \emptyset$,
- (2) or $\text{int } I_j \subset f(\text{int } I_i)$.

Such a partition is called a **Markov partition** for f .

We note that the last definitions of Markov partitions for the class piecewise-expanding maps are equivalent.

Definition 3.3. (see [11]) A Markov map $f : I \rightarrow I$ is called **expanding** if f is C^1 on each $\text{int}(I_i)$ and there exists a constant $\lambda > 1$ such that

$$|f'(x)| \geq \lambda \quad \text{for all } x \in \text{int}(I_i), \quad i = 0, \dots, N-1.$$

Definition 3.4. (see [11]) Let $f : I \rightarrow I$ be a Markov map with Markov partition $\{I_0, \dots, I_{N-1}\}$. The **topological transition matrix** $A = (A_{ij})_{0 \leq i, j \leq N-1}$ is defined by

$$A_{ij} = \begin{cases} 1, & \text{int}(I_j) \subset f(\text{int}(I_i)), \\ 0, & f(\text{int}(I_i)) \cap \text{int}(I_j) = \emptyset. \end{cases}$$

Equivalently, $A_{ij} = 1$ if and only if the transition from I_i to I_j is allowed.

Definition 3.5. (see [11]) A word $x = (x_0 x_1 x_2 \dots) \in \{0, 1, \dots, N-1\}^{\mathbb{N}}$ is called **admissible** if $A_{x_n x_{n+1}} = 1$ for all $n \geq 0$. Let Σ_A denote the set of all admissible sequences and let $\sigma : \Sigma_A \rightarrow \Sigma_A$ be the left shift map

$$\sigma(x_0 x_1 x_2 \dots) = (x_1 x_2 x_3 \dots).$$

The following theorem shows a connection between symbolic representation of points from interval I and the basic intervals of the Markov partition.

Theorem 3.1. (see [11]) Let $f : I \rightarrow I$ be an expanding Markov map with Markov partition $\{I_0, \dots, I_{N-1}\}$ and transition matrix A . Then for every admissible sequence $x = (x_0 x_1 \dots) \in \Sigma_A$ there exists a unique point $\pi(x) \in I$ such that

$$f^n(\pi(x)) \in I_{x_n} \quad \text{for all } n \geq 0.$$

Moreover, the coding map $\pi : \Sigma_A \rightarrow I$ is onto and satisfies

$$\pi \circ \sigma = f \circ \pi.$$

3.2. The natural partition and its refinements. We start from the natural (two-interval) partition (see Figure 4)

$$\mathcal{P}_1 = \{I^s = [0, c], \quad I^b = [c, 1]\}, \quad c = \frac{3 - \sqrt{5}}{2}.$$

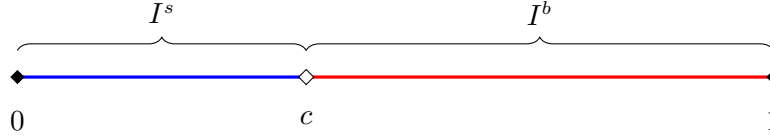


Figure 4

We refer to I^s as the small interval and I^b as the big interval. For $n \geq 1$, define the refinement

$$\mathcal{P}_{n+1} := \mathcal{P}_n \vee \widehat{T}_{\omega_0}^{-1}(\mathcal{P}_n),$$

i.e. $\mathcal{P}_{n+1}$ is obtained by subdividing each atom of $\mathcal{P}_n$ using the preimages of its endpoints under $\widehat{T}_{\omega_0}$. Equivalently, the set of endpoints of $\mathcal{P}_n$ is contained in

$$\{0, c, \widehat{T}_{\omega_0}^{-1}(c), \widehat{T}_{\omega_0}^{-2}(c), \dots, \widehat{T}_{\omega_0}^{-n}(c), 1\}.$$

Remark 3.2. Every interval of $\mathcal{P}_n$ has endpoints taken from the finite backward orbit of c up to depth n (together with 0 and 1).

3.3. Markov partitions and Fibonacci structure.

The following result describes the exact structure of the partitions $\mathcal{P}_n$ generated by the backward orbit of the critical point. It establishes their Markov property and the explicit refinement rule, which produces two interval types with Fibonacci-type growth. This structure forms the basis for the symbolic and matrix description of the system.

Theorem 3.2. 1. For every $n \geq 1$ the system of points $\{0, c, \widehat{T}_{\omega_0}^{-1}(c), \dots, \widehat{T}_{\omega_0}^{-n}(c), 1\}$ determine a Markov partition $\mathcal{P}_n$.

2. Each Markov partition $\mathcal{P}_n$ contain two type of intervals: big intervals $\{I_i^b, 1 \leq i \leq q_{n-1}\}$ with $|I_i^b| = \frac{\sqrt{5}-1}{2}\rho^{-(n-1)}$ and small intervals $\{I_i^s, 1 \leq i \leq q_{n-2}\}$ with $|I_i^s| = \frac{3-\sqrt{5}}{2}\rho^{-(n-1)}$

3. Under the refinement from the partition $\mathcal{P}_n$ to $\mathcal{P}_{n+1}$, each big interval I_i^b is subdivided into two subintervals, one of which becomes a big interval of the partition $\mathcal{P}_{n+1}$, while the other becomes a small interval. Moreover, every small interval of the partition $\mathcal{P}_n$ becomes a big interval in the partition $\mathcal{P}_{n+1}$.

Proof: (1) The partition $\mathcal{P}_n$ of the interval $[0, 1]$ is generated by the points

$$c, \widehat{T}^{-1}c, \dots, \widehat{T}^{-n}c, \quad c = \frac{3 - \sqrt{5}}{2}.$$

Then, the partition $\mathcal{P}_n$ can be written as

$$\mathcal{P}_n = \{I_i^n = [\widehat{T}^{-k}x, \widehat{T}^{-l}x] : 0 \leq k, l \leq n\}.$$

These points subdivide $[0, 1]$ into finitely many intervals with pairwise disjoint interiors, hence $\mathcal{P}_n$ is a partition in the sense of Definition 3.1. Moreover, for any $I_i^n \in \mathcal{P}_n$,

$$\widehat{T}(I_i^n) = [\widehat{T}^{-(k-1)}x, \widehat{T}^{-(l-1)}x],$$

and therefore the Markov property holds: for any $I_i^n, I_j^n \in \mathcal{P}_n$, either $f(\text{int } I_i^n) \cap \text{int } I_j^n \neq \emptyset$ or $\text{int } I_j^n \subseteq f(\text{int } I_i^n)$. Hence, by Definition 3.2, $\mathcal{P}_n$ is a Markov partition.

(2) We prove items (2) and (3) of the theorem by mathematical induction. For $n = 1$, the statements are readily verified. Assume that they hold for $n = k$.

The partition $\mathcal{P}^{n+1}$ is obtained from $\mathcal{P}^n$ via the refinement

$$\mathcal{P}^{n+1} = \mathcal{P}^n \vee \widehat{T}^{-1}\mathcal{P}^n,$$

that is, by taking the preimages $\widehat{T}^{-1}I_i^n$ of all intervals $I_i^n \in \mathcal{P}^n$. Since each inverse branch of $\widehat{T}$ contracts lengths by the factor $\rho = \frac{1+\sqrt{5}}{2}$, every interval is scaled by ϕ^{-1} and new intervals I_i^{n+1} are created.

More precisely, the lengths of the intervals in $\mathcal{P}^{n+1}$ take only two possible values:

$$\lambda(I^{n+1}) = \frac{3 - \sqrt{5}}{2} \cdot \rho^{-n} \quad \text{or} \quad \lambda(I^{n+1}) = \frac{\sqrt{5} - 1}{2} \cdot \rho^{-n}.$$

Hence, the partition $\mathcal{P}^{n+1}$ consists of exactly two types of intervals. Moreover, the length of a big interval at level $n + 1$ coincides with the length of a small interval at level n , that is,

$$\lambda(I_b^{n+1}) = \lambda(I_s^n).$$

$$\lambda(I_b^{n+1}) + \lambda(I_s^{n+1}) = \frac{3 - \sqrt{5}}{2} \cdot \rho^{-n} + \frac{\sqrt{5} - 1}{2} \cdot \rho^{-n} = \frac{\sqrt{5} - 1}{2} \cdot \rho^{-(n-1)} = \lambda(I_b^n).$$

Consequently, when passing from $\mathcal{P}^n$ to $\mathcal{P}^{n+1}$, each small interval I_s^n remains unchanged, while each big interval I_b^n is subdivided into one big and one small interval. As a result, the number of big intervals satisfies the recursion

$$q_n = q_{n-1} + q_{n-2},$$

while the number of small intervals equals q_{n-1} . This completes the inductive step. $\square$

Recall that the Fibonacci sequence of numbers q_n alternates regularly according to the repeating pattern *odd, odd, even*. Also, the intervals of each Markov partition $\{\mathcal{P}_n, n \geq 1\}$ enumerated in positive direction from 0 to 1.

Next theorem describes the Fibonacci structure and location of intervals Markov partitions $\{\mathcal{P}_n, n \geq 1\}$.

Theorem 3.3. *Let $\{\mathcal{P}_n, n \geq 1\}$ the sequence of Markov partitions associated by map $\hat{T}_{\omega_0}$. Then the small and big intervals are arranged according to the following three distinct cases:*

- (1) *If, for the n -th partition, the total number q_n of intervals is even, then the first interval is small, while the last interval is big.*
- (2) *For the $(n + 1)$ -st partition, the first two intervals are big, whereas the last interval is small.*
- (3) *For the $(n + 2)$ -nd partition, both the first two and the last two intervals are big.*

In all three cases, every remaining small interval appears exclusively in consecutive pairs, while the big intervals occur only in blocks consisting of either two or four consecutive intervals.

Proof: For the partitions $\mathcal{P}_n$ with $n = 3, 4, 5$, the validity of the stated properties is illustrated by the diagrams above.

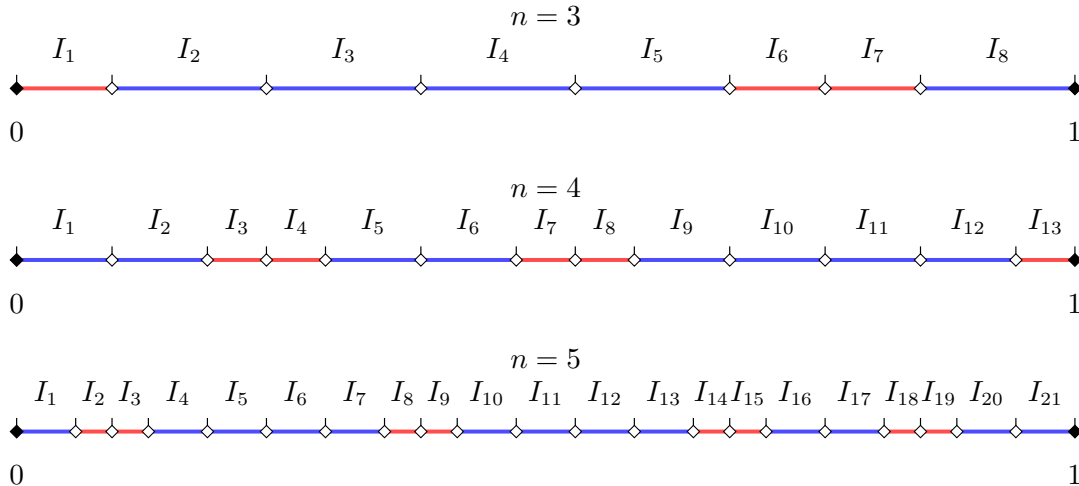


Figure 5

We now proceed by induction. Assume that the statements hold for $\mathcal{P}_n$ with $n = k$, where q_k is even. We prove that they also hold for $n = k + 1$.

Consider the partition $\mathcal{P}_{k+1}$. The subcollection of intervals contained in the subinterval $[\frac{3-\sqrt{5}}{2}, 1]$ is obtained from the intervals of $\mathcal{P}_k$ by reversing their order. Consequently, the last interval is small, while the first interval starting at $\frac{3-\sqrt{5}}{2}$ is big. On the other hand, the subcollection of intervals contained in $[0, \frac{3-\sqrt{5}}{2}]$ inherits the order of the intervals of $\mathcal{P}_k$ lying in $[\frac{3-\sqrt{5}}{2}, 1]$. In particular, the first two intervals and the interval immediately to the left of $\frac{3-\sqrt{5}}{2}$ are big. Since the relative order of the remaining intervals is preserved and big intervals appear on both sides of the point $\frac{3-\sqrt{5}}{2}$, the configuration of small and big intervals is maintained. This corresponds precisely to case (b) of the theorem.

Next, assume that the statements hold for $\mathcal{P}_n$ with $n = k + 1$, where q_k is even, and consider $\mathcal{P}_{k+2}$. The intervals of $\mathcal{P}_{k+2}$ contained in $[\frac{3-\sqrt{5}}{2}, 1]$ are obtained by reversing the order of the intervals of $\mathcal{P}_{k+1}$. Hence, the last interval is big, while the first interval starting at $\frac{3-\sqrt{5}}{2}$ is small. For the subinterval $[0, \frac{3-\sqrt{5}}{2}]$, the intervals inherit the order of those intervals of $\mathcal{P}_{k+1}$ lying in $[\frac{3-\sqrt{5}}{2}, 1]$. Thus, the first interval and the interval immediately to the left of $\frac{3-\sqrt{5}}{2}$ are small. Since the relative positions of the remaining intervals are unchanged and small intervals occur on both sides of $\frac{3-\sqrt{5}}{2}$, the pattern of small and big intervals is preserved. This corresponds to case (c) of the theorem.

Finally, assume that the statements hold for $\mathcal{P}_n$ with $n = k + 2$, where q_k is even, and consider $\mathcal{P}_{k+3}$. In this case, the intervals of $\mathcal{P}_{k+3}$ contained in $[\frac{3-\sqrt{5}}{2}, 1]$ are obtained by reversing the order of the intervals of $\mathcal{P}_{k+2}$. As a result, the last interval and the first two intervals starting at $\frac{3-\sqrt{5}}{2}$ are big. For the subinterval $[0, \frac{3-\sqrt{5}}{2}]$, the order of the corresponding intervals of $\mathcal{P}_{k+2}$ is preserved, and the first interval together with the two intervals immediately to the left of $\frac{3-\sqrt{5}}{2}$ are big. Since the mutual arrangement of the remaining intervals is unchanged and pairs of big intervals appear on both sides of $\frac{3-\sqrt{5}}{2}$, the configuration of small and big intervals is preserved. This is exactly case (a) of the theorem.

In conclusion, the arrangement of small and big intervals in the partition $\mathcal{P}_n$ repeats cyclically according to the three cases stated in item (4) of Theorem 3.3. $\square$

3.4. Transition matrix for partition $\mathcal{P}_1$. Label the intervals of $\mathcal{P}_1$ by

$$I^s \leftrightarrow 0, \quad I^b \leftrightarrow 1.$$

A direct inspection of $\widehat{T}_{\omega_0}$ shows that $\widehat{T}_{\omega_0}(I^s)$ intersects only I^b (so the transition $0 \rightarrow 0$ is forbidden), whereas $\widehat{T}_{\omega_0}(I^b)$ intersects both I^s and I^b . Hence the transition matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

4. SYMBOLIC CODING AND CONJUGACY

Let (X, T) be a dynamical system, where X is a measurable space and $T : X \rightarrow X$ is a measurable map. Assume that X is partitioned into finitely many pairwise disjoint measurable sets

$$X = P_0 \cup \dots \cup P_{m-1}, \quad P_i \cap P_j = \emptyset \text{ for } i \neq j.$$

For $x \in X$ we define its *itinerary* (symbolic name) by

$$\omega(x) = (\omega_0, \omega_1, \dots) \in \{0, 1, \dots, m-1\}^{\mathbb{N}}, \quad \omega_n = i \iff \widehat{T}^n(x) \in P_i, \quad n \geq 0.$$

This gives the coding map

$$\pi : X \rightarrow \{0, 1, \dots, m-1\}^{\mathbb{N}}, \quad \pi(x) = \omega(x).$$

Let σ denote the left shift on $\{0, 1, \dots, m-1\}^{\mathbb{N}}$,

$$\sigma(\omega_0, \omega_1, \omega_2, \dots) = (\omega_1, \omega_2, \omega_3, \dots).$$

By construction,

$$\pi \circ \widehat{T} = \sigma \circ \pi.$$

The image $\Sigma := \pi(X)$ is the *symbolic dynamical system* associated to (X, T) .

Let $\mathcal{A} = \{0, 1, \dots, m-1\}$. The *full shift* over $\mathcal{A}$ is $\mathcal{A}^{\mathbb{N}}$ with the shift map σ . A *subshift* is a closed, shift-invariant subset $\Sigma \subseteq \mathcal{A}^{\mathbb{N}}$, i.e. $\sigma(\Sigma) = \Sigma$. Equivalently, there exists a (possibly infinite) family $\mathcal{F}$ of finite words (forbidden blocks) such that Σ consists of all sequences in $\mathcal{A}^{\mathbb{N}}$ which do not contain any word from $\mathcal{F}$ as a contiguous block.

Lemma 4.1 (Conjugacy to the golden mean shift). *Let $\omega_0 = \frac{1+\sqrt{5}}{2}$ and let $\widehat{T} = \widehat{T}_{\omega_0} : [0, 1] \rightarrow [0, 1]$. Consider the Markov partition $\mathcal{P}_1 = \{I^s = [0, c], I^b = [c, 1]\}$, where $c = \frac{3-\sqrt{5}}{2}$, and label $I^s \leftrightarrow 0$, $I^b \leftrightarrow 1$. Then the associated symbolic space is the one-sided golden mean shift*

$$\Sigma_A = \{x \in \{0, 1\}^{\mathbb{N}} : \text{the block } 00 \text{ does not occur}\}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and the systems $([0, 1], \widehat{T})$ and (Σ_A, σ) are topologically conjugate.

Proof: Since $\widehat{T}$ is an expanding Markov map with Markov partition $\mathcal{P}_1$ and transition matrix A , Theorem 3.1 (coding for expanding Markov maps) provides a continuous surjection $\pi : \Sigma_A \rightarrow [0, 1]$ such that $\pi \circ \sigma = \widehat{T} \circ \pi$. Moreover, expansivity implies that for each admissible sequence $x = (x_0 x_1 \dots) \in \Sigma_A$ there is a unique point $\pi(x) \in [0, 1]$ whose itinerary equals x . Hence π is bijective and therefore a homeomorphism, proving topological conjugacy.

In particular, the transition $0 \rightarrow 0$ is forbidden (equivalently, the block 00 cannot occur), while all other transitions are allowed, which yields the golden mean shift. $\square$

4.1. Parry measure and the invariant measure. Let A be the transition matrix of an irreducible one-step shift of finite type Σ_A and let $\theta > 0$ be its Perron eigenvalue. Let v and u be positive Perron eigenvectors:

$$Av = \theta v, \quad u^T A = \theta u^T, \quad u^T v = 1.$$

Define the stochastic matrix $P = (P_{ij})$ by

$$P_{ij} = \frac{A_{ij}v_j}{\theta v_i}, \quad 0 \leq i, j \leq N-1,$$

and the stationary distribution $\pi = (\pi_i)$ by $\pi_i = u_i v_i$. Then the *Parry measure* μ on Σ_A is the Markov measure given by

$$\mu([x_0 x_1 \dots x_{n-1}]) = \pi_{x_0} P_{x_0 x_1} \dots P_{x_{n-2} x_{n-1}}, \quad n \geq 1,$$

where $[x_0 \dots x_{n-1}]$ denotes the cylinder set.

In our case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \theta = \frac{1+\sqrt{5}}{2}.$$

A convenient choice of normalized Perron eigenvectors yields

$$\pi_0 = \mu([0]) = \frac{5-\sqrt{5}}{10}, \quad \pi_1 = \mu([1]) = \frac{5+\sqrt{5}}{10},$$

and the corresponding transition matrix is

$$P = \begin{pmatrix} 0 & 1 \\ \frac{3-\sqrt{5}}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$

Finally, the $\widehat{T}$ -invariant probability measure on $[0, 1]$ is obtained as the push-forward of μ under the conjugacy map π^{-1} from Lemma 4.1.

4.2. Periodic points. In this section we study periodic points of $\widehat{T} = \widehat{T}_{\omega_0}$ on $[0, 1]$. Since $\widehat{T}$ is conjugate to the one-sided subshift of finite type (Σ_A, σ) (Lemma 4.1), periodic orbits of $\widehat{T}$ correspond to periodic sequences in Σ_A . More precisely, if $\pi : \Sigma_A \rightarrow [0, 1]$ denotes the conjugacy, then

$$\widehat{T}^m(\pi(s)) = \pi(\sigma^m(s)) \quad \text{for all } s \in \Sigma_A, m \geq 1.$$

Hence, if s is σ -periodic with (prime) period m , then $\pi(s)$ is $\widehat{T}$ -periodic with (prime) period m .

4.3. An explicit example: period 3.

Example 4.1 (Period-3 orbits). For the golden mean shift $\Sigma_A \subset \{0, 1\}^{\mathbb{N}}$ (no block 00), the σ -periodic sequences of period 3 are obtained from admissible words of length 3 whose infinite repetition is admissible. Up to cyclic rotation, the admissible period-3 sequences are

$$s_1 = (011)^\infty, \quad s_2 = (110)^\infty, \quad s_3 = (101)^\infty,$$

and, in addition, the fixed point

$$s_4 = 1^\infty$$

(which is periodic for every period, but has prime period 1).

Let $\rho = \frac{1+\sqrt{5}}{2}$ and $c = \frac{3-\sqrt{5}}{2}$. Recall that the conjugacy π is determined by the inverse branches:

$$\pi(s) = \begin{cases} \frac{\pi(\sigma(s)) - c}{\rho}, & s_0 = 0, \\ \frac{\rho - \pi(\sigma(s))}{\rho}, & s_0 = 1. \end{cases} \quad (*)$$

(Here we use the explicit linear formula for $\widehat{T}$ on each Markov interval.)

To compute the points corresponding to $s_1 = (011)^\infty$, $s_2 = (110)^\infty$, $s_3 = (101)^\infty$, set

$$x = \pi((011)^\infty), \quad y = \pi((110)^\infty), \quad z = \pi((101)^\infty).$$

Using $(*)$ and the relations $\sigma((011)^\infty) = (110)^\infty$, $\sigma((110)^\infty) = (101)^\infty$, $\sigma((101)^\infty) = (011)^\infty$, we obtain the system

$$\begin{cases} x = \frac{y - c}{\rho}, \\ y = \frac{\rho - z}{\rho}, \\ z = \frac{\rho - x}{\rho}. \end{cases}$$

Solving yields

$$x = 0, \quad y = c = \frac{3 - \sqrt{5}}{2}, \quad z = 1.$$

Finally, for the fixed point $s_4 = 1^\infty$ we have

$$t := \pi(1^\infty) = \frac{\rho - t}{\rho},$$

hence $t = \frac{\sqrt{5}-1}{2}$. Therefore, $\widehat{T}$ has periodic points

$$0, \quad \frac{3 - \sqrt{5}}{2}, \quad 1, \quad \frac{\sqrt{5} - 1}{2},$$

where the first three belong to a period-3 orbit, while $\frac{\sqrt{5}-1}{2}$ is a fixed point.

4.4. Uniqueness of short-period points in refined cylinders. Let $\mathcal{P}_n$ be the n -th Markov refinement and let $\{I_n^1, \dots, I_n^{q_n}\}$ denote the collection of its interval elements. Each I_n^i corresponds to a cylinder of length n in Σ_A .

Lemma 4.2. *For each $n \in \mathbb{N}$ and each element I_n^i of $\mathcal{P}_n$, the interval I_n^i contains at most one periodic point whose (prime) period is $\leq n$.*

Proof: The map $\hat{T}$ is strictly monotone and expanding on the interior of every $J \in \mathcal{P}_n$. Hence $\hat{T}^k$ is strictly monotone on $\text{int}(J)$ for every $1 \leq k \leq n$. If $x, y \in J$ were two distinct points with $\hat{T}^k(x) = x$ and $\hat{T}^k(y) = y$ for some $k \leq n$, then by monotonicity $\hat{T}^k - \text{Id}$ would have at least two zeros in $\text{int}(J)$, forcing it to change sign, which is impossible because $\hat{T}^k$ is strictly monotone on $\text{int}(J)$. Therefore, J contains at most one such periodic point. $\square$

Lemma 4.3. *The set of periodic points of $\hat{T}$ is dense in $[0, 1]$.*

Proof: Let $(a, b) \subset [0, 1]$ be a nonempty open interval. Choose n large enough so that $\mathcal{P}_n$ has an element $I_n^i \subset (a, b)$ (the mesh of $\mathcal{P}_n$ tends to 0 since $\hat{T}$ is expanding). The interval I_n^i corresponds to a cylinder $[w]$ of length n in Σ_A . Pick an admissible periodic sequence in $[w]$, e.g. by repeating w and, if necessary, modifying finitely many symbols to respect admissibility (no 00 occurs). Its image under the conjugacy π is a periodic point of $\hat{T}$ lying in $I_n^i \subset (a, b)$. Hence periodic points are dense in $[0, 1]$. $\square$

5. ESCAPE RATES THROUGH MARKOV HOLES

5.1. Survivor sets and exponential escape rate.

Definition 5.1 (see [12]). Let $(\hat{M}, T)$ be a dynamical system and let $A \subset \hat{M}$ be a measurable *hole*. For $n \geq 1$ the n -step survivor set is

$$\Omega_n(A) := \{x \in \hat{M} : T^k(x) \notin A \text{ for all } 0 \leq k < n\}.$$

Definition 5.2 (see [12]). Assume λ is a reference measure (e.g. Lebesgue measure on $[0, 1]$ or an invariant probability measure). The (exponential) escape rate into A is

$$\rho(A) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \lambda(\Omega_n(A)),$$

provided the limit exists.

For later use, define the set of points that fall into the hole exactly at time n :

$$\Theta_n(A) := \Omega_n(A) \setminus \Omega_{n+1}(A) = \{x : T^k(x) \notin A \text{ } (0 \leq k < n), \text{ and } T^n(x) \in A\}.$$

Lemma 5.1 (see [12]). *Assume $\sum_{n \geq 0} \lambda(\Theta_n(A)) = 1$ and that $\rho(A)$ exists. Then*

$$\rho(A) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \lambda(\Theta_n(A)).$$

Proof: Since $\Omega_n(A) = \bigsqcup_{k \geq n} \Theta_k(A)$, we have $\lambda(\Omega_n(A)) = \sum_{k \geq n} \lambda(\Theta_k(A))$. The assumption $\sum_{k \geq 0} \lambda(\Theta_k(A)) = 1$ ensures that the tail sums have the same exponential decay rate as $\lambda(\Theta_n(A))$, yielding the claim. $\square$

5.2. Escape rate via entropy of subshifts. We consider Markov holes defined by cylinders in the symbolic space. Let Σ_F be an irreducible subshift of finite type with transition matrix A_F and Perron eigenvalue $\theta > 1$ (so that $h_{\text{top}}(\Sigma_F) = \log \theta$). Let G be an additional finite family of forbidden words and write $\Sigma_{F \cup G}$ for the resulting subshift with Perron eigenvalue Λ .

Theorem 5.1 (see [5]). *Let H_G be the hole corresponding to forbidding the family G (equivalently, the set of sequences that enter the forbidden cylinder). Then the escape rate exists and satisfies*

$$\rho(H_G) = \log\left(\frac{\theta}{\Lambda}\right) > 0,$$

where $\log \theta = h_{\text{top}}(\Sigma_F)$ and $\log \Lambda = h_{\text{top}}(\Sigma_{F \cup G})$.

5.3. Examples of Markov holes. We illustrate Theorem 5.1 for the golden mean shift

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \theta = \frac{1 + \sqrt{5}}{2}.$$

Fix $N = 3$. The admissible cylinders of length 3 are

$$[010], [011], [101], [110], [111].$$

Each cylinder corresponds (under conjugacy) to one element of the Markov refinement $\mathcal{P}_3$.

Example 5.1. Consider the hole $H = [010]$ (i.e. we forbid the word 010). To describe the perturbed shift as a one-step SFT, we use states given by admissible words of length 2:

$$01, 10, 11.$$

Appending a symbol produces a transition; we keep the transition if the resulting length-3 word is admissible and not forbidden. This yields the 3×3 transition matrix

$$A_{010} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let Λ be the Perron eigenvalue of A_{010} . Then

$$\rho([010]) = \log\left(\frac{\theta}{\Lambda}\right).$$

(One may compute numerically $\Lambda \approx 1.4656$, hence $\rho([010]) \approx 0.102$.)

5.4. Local escape rate. Let $\alpha = \alpha_0 \alpha_1 \alpha_2 \cdots \in \Sigma_F$. For $n \geq 1$ set $w^n = \alpha_0 \cdots \alpha_{n-1}$ and $H_n = [w^n]$. The local escape rate around α is

$$\rho(\alpha) = \lim_{n \rightarrow \infty} \frac{\rho(H_n)}{\mu(H_n)},$$

provided the limit exists.

Theorem 5.2 (see [13]). *Let Σ_F be an irreducible subshift of finite type with Perron eigenvalue $\theta > 1$ and Parry measure μ . Then*

$$\rho(\alpha) = \begin{cases} 1, & \text{if } \alpha \text{ is non-periodic,} \\ 1 - \theta^{-m}, & \text{if } \alpha \text{ has prime period } m. \end{cases}$$

Lemma 5.2 (Local escape for shrinking Markov holes). *Let $x \in [0, 1]$ and let $\{A_n(x)\}_{n \geq 1}$ be a nested sequence of Markov intervals such that $\bigcap_{n \geq 1} A_n(x) = \{x\}$. Assume the corresponding symbolic point is $\alpha = \pi^{-1}(x) \in \Sigma_F$. Then:*

(1) *If x is periodic with prime period m , then*

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\mu(A_n(x))} = 1 - \theta^{-m}.$$

(2) *If x is non-periodic, then*

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\mu(A_n(x))} = 1.$$

Proof: Under the conjugacy, each Markov interval $A_n(x)$ corresponds to a cylinder $H_n = [w^n]$ shrinking to α . Therefore

$$\frac{\rho(A_n(x))}{\mu(A_n(x))} = \frac{\rho(H_n)}{\mu(H_n)}.$$

The conclusion follows immediately from Theorem 5.2. $\square$

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