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Geometric and topological properties of the Minkowski operator on convex sets

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Abstract. This paper systematically establishes the fundamental geometrical and topological properties of the Minkowski operator on convex sets in an Euclidean space. The convexity, closedness, and boundary properties of the Minkowski sum and difference are analyzed. In addition, results concerning the Minkowski sum and difference of open and closed sets are presented. The paper also constructs illustrative examples to clarify each of the obtained results.

Keywords: Convex sets, Minkowski difference, Minkowski sum, homothetic set, boundary of a set, compact set
MSC (2020): 52A20, 54C10

1. Preliminaries

Definition 1.1. [1] . Let X and Y be non-empty sets in $\mathbb{R}^n$. The geometric (or Minkowski) sum of X and Y is defined as

$$X + Y = \{x + y : x \in X, y \in Y\}$$

and the geometric (or Minkowski) difference is defined as

$$X \dot{-} Y = \{z \in \mathbb{R}^n : z + Y \subset X\}.$$

For a set $X \subset \mathbb{R}^n$ and a scalar $\lambda \in \mathbb{R}$, the scalar multiplication of X by λ is defined as

$$\lambda X = \{\lambda x : x \in X\}.$$

For any vector $a \in \mathbb{R}^n$ and non-empty set $X \subset \mathbb{R}^n$, the Minkowski sum of the vector a and the set X is defined as the Minkowski sum of the singleton a and X:

$$a + X = \{a\} + X = \{a + x : x \in X\}.$$

Definition 1.2. [2] Let X be a nonempty set in $\mathbb{R}^n$ and let $\lambda > 0$ be a real number. The set $X_{(\lambda,t)}$ defined by $X_{(\lambda,t)} = \lambda X + t$ is called a homothetic set of X. Here $t \in \mathbb{R}^n$.

Let N be a compact set and let M a convex compact set in $\mathbb{R}^n$. Assume that a foliation $\mathcal{F} = \{L_\alpha : L_\alpha = \partial M_\alpha,\ \alpha \in A\}$ is given, consisting of the boundaries of homothetic subsets of M.

Definition 1.3. If there exist an index $\alpha \in A$ and an element $z \in \mathbb{R}^n$ such that the relation $z+N \subset M_\alpha$ holds, then the set N is said can embed into the foliation $\mathcal{F}$.

Definition 1.4. If for some index $\alpha_0 \in A$ the relation $z + N \subset M_{\alpha_0}$ holds, but for every index $\alpha \in A$ with $\alpha < \alpha_0$ the inclusion $z + N \subset M_\alpha$ does not hold then the set N is said can embed into the set M_{α_0}.

From this definition it is easy to see that if a compact set N can embed into the set M_{α_0}, then the dimension of the geometric difference $M_{\alpha_0} \dot{-} N$ is smaller than the dimension of the space $\mathbb{R}^n$.

Definition 1.5. If for some index $\alpha_0 \in A$ the Minkowski difference $M_{\alpha_0} \dot{-} N$ consists of a single point, that is, $M_{\alpha_0} \dot{-} N = \{a\}$ for some $a \in \mathbb{R}^n$, then the compact set N is said to be fixedly embedded into the set M_{α_0}.

2. Results on the Minkowski Difference of Convex Sets

Lemma 2.1. *For a convex set M in $\mathbb{R}^n$ and its homothetic set $M_\alpha = \alpha M$ (where $0 < \alpha \le 1$), equality $D = M \dot{-} M_\alpha$ implies equality $M = D + M_\alpha$.*

Proof: The equality $D = M \dot{-} M_\alpha$ and the definition of the Minkowski difference imply the inclusion

$$D + M_\alpha \subset M. \tag{2.1}$$

Let us show the inverse inclusion. For each vector $x \in M$ and number α $(0 < \alpha \le 1)$, denote the difference $x - \alpha x$ by d, i. e., $d = x - \alpha x = (1-\alpha)x$. Since M is a convex set, for any point $y \in M$, we have

$$d + \alpha y = (1-\alpha)x + \alpha y \in M.$$

Hence, $d + M_\alpha \subset M$, which means $d \in D$. Thus, for every point $x \in M$, there exists a point $d \in D$ such that the equality $x = \alpha x + d$ holds. This implies

$$M \subset D + M_\alpha \tag{2.2}$$

The relations (2.1) and (2.2) together complete the proof of the lemma. □

Theorem 2.2. *Let M and N be nonempty convex sets in $\mathbb{R}^n$, and suppose that $M_\alpha \dot{-} N \ne \varnothing$ for some $\alpha \in (0,1]$. Then the equality*

$$M \dot{-} N = (M \dot{-} M_\alpha) + (M_\alpha \dot{-} N) \tag{2.3}$$

holds.

Proof: Let $z \in M \dot{-} N$. It must be shown that there exist points $z_1 \in M \dot{-} M_\alpha$ and $z_2 \in M_\alpha \dot{-} N$ satisfying the equality $z = z_1 + z_2$.

For an element $z \in M \dot{-} N$, using the definition of the Minkowski difference, we can write the relation $z + N \subset M$. Hence, for every $c \in N$, there exists an element $a \in M$ such that the equality $z + c = a$ holds.

Since $M_\alpha = \alpha M$, by Lemma 2.1, for each $a \in M$ there exist points $z_1 \in M \dot{-} M_\alpha$ and $m_\alpha \in M_\alpha$ such that a can be expressed in the form $a = m_\alpha + z_1$. Taking into account the relation $z + N \subset M$, for every $c \in N$ we have $z + c = m_\alpha + z_1$. From this equality we can write $z - z_1 + c = m_\alpha$. This implies that $z - z_1 \in M_\alpha \dot{-} N$. Thus, for any element $z \in M \dot{-} N$, we can always find a $z_1 \in M \dot{-} M_\alpha$ such that $z_2 = z - z_1 \in M_\alpha \dot{-} N$.

Now, let $z \in (M \dot{-} M_\alpha) + (M_\alpha \dot{-} N)$. Then there exist elements $z_1 \in M \dot{-} M_\alpha$ and $z_2 \in M_\alpha \dot{-} N$ such that $z_1 + z_2 = z$. From the relation $z_1 \in M \dot{-} M_\alpha$, by the definition of the Minkowski difference, we can write $z_1 + M_\alpha \subset M$. Similarly, from $z_2 \in M_\alpha \dot{-} N$ we obtain $z_2 + N \subset M_\alpha$. From these two inclusions it follows that $z_1 + z_2 + N \subset M$, and hence $z_1 + z_2 \subset M \dot{-} N$. Theorem 2.2 is proved. □

Let us consider an ordered set A such that for every pair $\alpha, \beta \in A$, $\alpha \ne \beta$, there exists an element $\gamma \in A$ satisfying $\alpha < \gamma < \beta$. As an example, the interval $[0,1]$ can be taken as the set A.

Suppose a convex compact set M_1 in $\mathbb{R}^n$ and one of its interior points $a \in \operatorname{int} M_1$ are given. Denote its boundary by $\partial M_1 = L_1$. Convex subsets M_α of M_1 are chosen such that their boundaries $\partial M_\alpha = L_\alpha$, $\alpha \in A$, satisfy the following conditions:

1) for all $\alpha, \beta \in A$, $\alpha \ne \beta$, we have $L_\alpha \cap L_\beta = \varnothing$;
2) $\bigcup_{\alpha \in A} L_\alpha = M_1 \setminus \{a\}$;
3) for any $\alpha, \beta \in A$ with $\alpha \le \beta$, the condition $M_\beta \dot{-} M_\alpha \ne \varnothing$ holds.

The family $\mathcal{F} = \{L_\alpha \colon \alpha \in A\}$ is called a foliation, and each L_α is called a leaf, $\alpha \in A$.

Now consider an arbitrary convex compact set M in $\mathbb{R}^n$. Choose a point $a \in \operatorname{int} M$. For each number $\alpha \in (0,1]$, the set $\alpha M + (1-\alpha)a$ is a convex compact subset of M. Introducing the notation $L_\alpha = \partial(\alpha M + (1-\alpha)a)$, we form the family $\mathcal{F} = \{L_\alpha \colon 0 < \alpha \le 1\}$. This family consists of the boundaries of subsets of M that are homothetic to M. At $\alpha = 0$, a special case arises: $M_0 = \{a\}$. This set is generally not homothetic to the given set $M = M_1$.

For the constructed family $\mathcal{F} = \{L_\alpha \colon 0 < \alpha \le 1\}$, the following statement follows directly from the definition of the Minkowski difference.

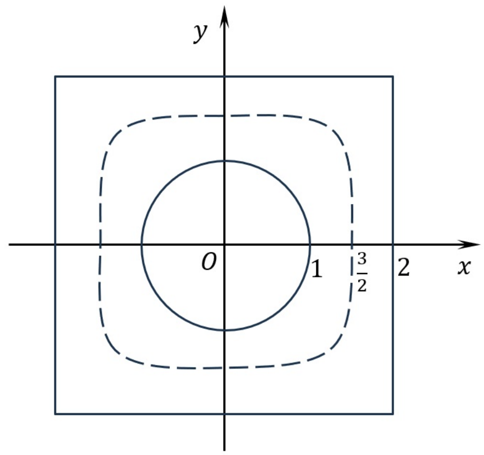

Figure 1.

Lemma 2.3. *The family $\mathcal{F}$ of boundaries of homothetic subsets of a convex compact set M in $\mathbb{R}^n$ satisfies the three conditions in the definition of a foliation, namely:*

1) for all $\alpha, \beta \in A$, $\alpha \neq \beta$, we have $L_\alpha \cap L_\beta = \varnothing$;

2) $\bigcup_{\alpha \in A} L_\alpha = M \setminus \{a\}$;

3) for any $\alpha, \beta \in A$ with $\alpha \leq \beta$, the condition $M_\beta \dot{-} M_\alpha \neq \varnothing$ holds.

It should be noted that for any family of subsets of M satisfying conditions 1), 2), and 3), Lemma 2.1 and Theorem 2.2 may not necessarily hold.

Example 2.4. Consider convex sets in the plane $\mathbb{R}^2$ with boundaries defined as following:

$$\partial M_1 = \{(x,\,y)\colon\ x = \pm 2 \ \text{ and } \ -2 \leq y \leq 2\} \cup \{(x,\,y)\colon\ -2 \leq x \leq 2 \ \text{ and } \ y = \pm 2\};$$

$$\partial M_{\frac{1}{2}} = \{(x,\,y)\colon\ x^2 + y^2 = 1\};$$

$$\partial M_t = \begin{cases} \left\{2(1-t)X + (2t-1)Y : X \in \partial M_{\frac{1}{2}}, Y \in \partial M_1\right\}, & \text{if } \frac{1}{2} \leq t \leq 1, \\ \left\{2tX\colon\ X \in \partial M_{\frac{1}{2}}\right\}, & \text{if } 0 < t \leq \frac{1}{2}. \end{cases}$$

Take the origin as the fixed interior point of M_1, i.e., let $a = (0,\,0)$ (see Fig. 1).

For the family of sets M_t, $t \in (0,1]$, the family $\{\partial M_t\colon\ t \in (0,1]\}$ satisfies conditions 1), 2), and 3) stated above. However, the conclusion of Lemma 2.1 does not hold.

Indeed, consider the square $M_1 = \{(x,\,y)\colon\ -2 \leq x \leq 2,\ -2 \leq y \leq 2\}$ and the disk $M_{\frac{1}{2}} = \{(x,\,y)\colon\ x^2 + y^2 \leq 1\}$. The Minkowski difference $D = M_1 \dot{-} M_{\frac{1}{2}}$ is the square with side length 1:

$$D = M_1 \dot{-} M_{\frac{1}{2}} = \{(x,\,y)\colon\ -1 \leq y \leq 1,\ -1 \leq x \leq 1\} \qquad \text{(see Fig. 2).}$$

But the Minkowski sum of the square D and the disk $M_{\frac{1}{2}}$ is not equal to the set M_1 (Fig. 3) [3].

According to [1], for these sets the strict inclusion

$$\left(M_1 \dot{-} M_{\frac{1}{2}}\right) + M_{\frac{1}{2}} \subset M_1$$

holds. Consequently, the conclusion of Theorem 2.2 is also not satisfied for this family.

The following result is established by directly verifying the definitions of the Minkowski difference and sum, as well as by applying Theorem 2.2.

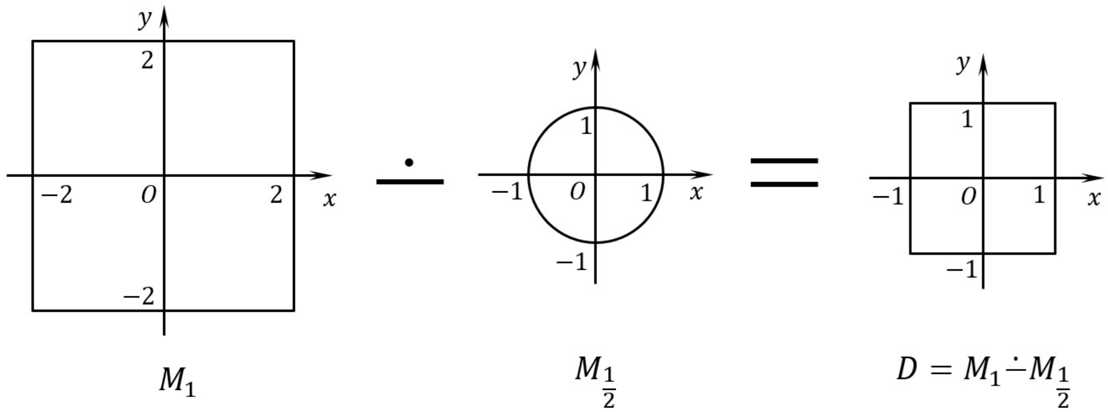

FIGURE 2.

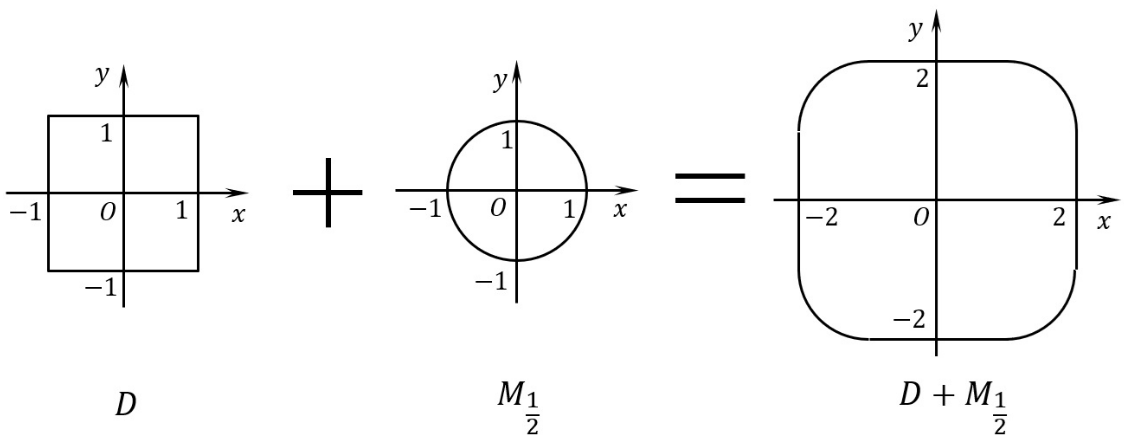

FIGURE 3.

Theorem 2.5. *If a compact set N is fixedly embedded in some set M_{α_0}, $\alpha \in (0, 1]$, then the equality*

$$M \dot{-} N = (M \dot{-} M_{\alpha_0}) + a \tag{2.4}$$

holds, where $M_{\alpha_0} \dot{-} N = \{a\}$.

The essence of this theorem is that in space $\mathbb{R}^n$, the problem of finding the Minkowski difference of an arbitrary compact set N from a convex compact set M can be reduced to the following, relatively simpler, problem: find the Minkowski difference from M of a set M_{α_0} that is homothetic to M and into which the compact set N is embedded (if it exists, fixedly embedded).

Problems of finding the Minkowski difference for mutually homothetic sets have been described in works [4], [5], [6].

The methods and examples presented above show that performing Minkowski operations in spaces of smaller dimension (for example, in $\mathbb{R}^2$) is easier and more efficient. However, in solving practical problems, one often has to work with sets given in higher-dimensional spaces. Under such conditions, it is convenient to decompose the given set defined by the Cartesian product into lower-dimensional sets and work with them.

Theorem 2.6. *For nonempty sets A, B, C and D in $\mathbb{R}^n$, the following equality holds:*

$$(A \times B) + (C \times D) = (A + C) \times (B + D). \tag{2.5}$$

Here $A \times B$ denotes the Cartesian product of sets.

Proof: For any point $x \in (A \times B) + (C \times D)$, by the definition of the Minkowski sum there exist points $x_1 \in A \times B$ and $x_2 \in C \times D$ such that

$$x = x_1 + x_2. \tag{2.6}$$

Since $x_1 \in A \times B$, by the definition of the Cartesian product it can be written as $x_1 = (a, b)$, where $a \in A$ and $b \in B$. Similarly, since $x_2 \in C \times D$, we have $x_2 = (c, d)$, where $c \in C$ and $d \in D$. Then equality (2.6) can be written in the form

$$x = x_1 + x_2 = (a, b) + (c, d).$$

By the definition of the Minkowski sum,

$$(a, b) + (c, d) = (a + c, b + d) \in (A + C) \times (B + D).$$

Now take an arbitrary element $z \in (A + C) \times (B + D)$. By the definition of the Cartesian product of sets, it can be written as an ordered pair

$$z = (z_1, z_2), \qquad z_1 \in (A + C), \; z_2 \in (B + D). \tag{2.7}$$

Since $z_1 \in (A + C)$, by the definition of the Minkowski sum there exist $a \in A$ and $c \in C$ such that $z_1 = a + c$. Similarly, from $z_2 \in (B + D)$ there exist $b \in B$ and $d \in D$ such that $z_2 = b + d$. Then expression (2.7) can be written as

$$z = (z_1, z_2) = (a + c, b + d) = (a, b) + (c, d). \tag{2.8}$$

Since $(a, b) \in A \times B$ and $(c, d) \in C \times D$, it follows that $z \in (A \times B) + (C \times D)$. Thus, Theorem 2.6 is proved. □

Lemma 2.7. *Let X and Y be nonempty sets, and let Z be a set homothetic to $X + Y$. If the condition $Z \subset X + Y$ is satisfied, then there exist nonempty sets X' and Y', homothetic to X and Y, respectively, such that $X' + Y' = Z$.*

Proof: Assume that $Z = \alpha(X + Y) + t$, where $t \in \mathbb{R}^n$ and $0 < \alpha \leq 1$. Then, for the sets $X' = \alpha X + t_X$ and $Y' = \alpha Y + t_Y$, the equality $X' + Y' = Z$ holds, where t_X and t_Y are any vectors with $t_X + t_Y = t$. Lemma 2.7 is proved. □

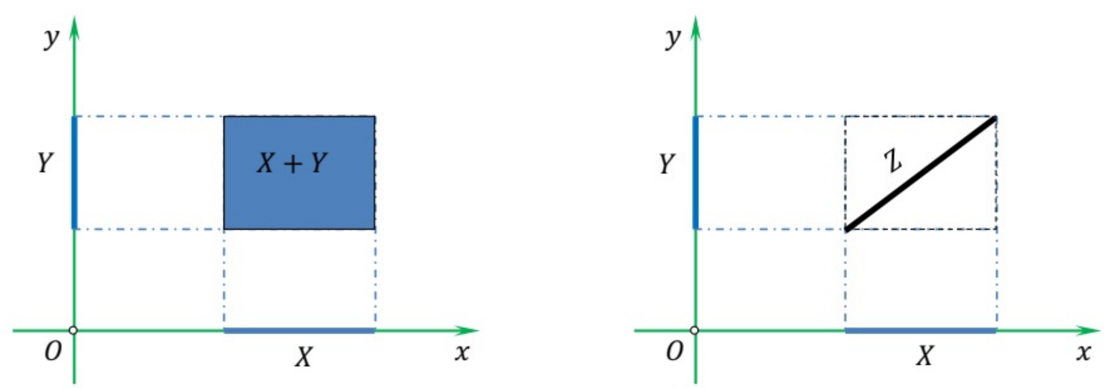

Figure 4.

The condition that the set Z be homothetic to the sum $X + Y$ is essential. Indeed, the set $Z = \left\{(x, y)\colon\; a \leq x \leq b;\; y = \frac{d-c}{b-a}(x - a) + c\right\}$ lies in the sum $X + Y = [a, b] \times [c, d]$ of the sets $X = [a, b] \times \{0\}$ and $Y = \{0\} \times [c, d]$. However, it is clear that the shape of the set Z is not similar to the shape of the sum $X + Y$; that is, Z is not a homothetic set. Moreover, there is no pair of sets $X' \subset X$ and $Y' \subset Y$ such that $X' + Y' = Z$ (see Fig. 4).

3. Topological Properties of Minkowski Operations

The density of topological spaces is one of the important concepts. The following statements can be established by direct verification.

Proposition 3.1. *If a set A is dense in a set $X \subset \mathbb{R}^n$, then for any vector $a \in \mathbb{R}^n$ the set $a + A$ is dense in $a + X$, and the set $A \dot{-} a$ is dense in $X \dot{-} a$.*

This statement can be generalized as follows.

Proposition 3.2. *If A is dense in X and B is dense in Y, then the set $A + B$ is dense in $X + Y$.*

To illustrate this statement consider the set $A = [2, 5]_{\mathbb{Q}}$ of all rational numbers in the interval $X = [2, 5]$, and the set $B = [8, 9]_{\mathbb{I}}$ of all irrational numbers in the interval $Y = [8, 9]$. Then the sum $A + B = [10, 14]_{\mathbb{I}}$ is the set of all irrational numbers in the interval $X + Y = [10, 14]$.

The following lemmas are used to represent a given set in the form of the Minkowski sum of two sets and make it possible to compute the Minkowski sum by decomposing it into parts.

Lemma 3.3. *Let X, Y and Z be nonempty sets such that $Z \cap (X + Y) \neq \varnothing$ and suppose that the set Z is homothetic to the sum $X + Y$. Then there exist sets U and V with $U \cap X \neq \varnothing$ and $V \cap Y \neq \varnothing$ such that $U + V = Z$.*

Proof: Denote the intersection $Z \cap (X + Y)$ by E, that is, $E = Z \cap (X + Y)$. Since for some $\alpha > 0$ and $t \in \mathbb{R}^n$ we have $E = \alpha(X + Y) + t$, by Lemma 2.7 there exist nonempty sets $X' = \alpha X + t_X$ and $Y' = \alpha Y + t_Y$ such that $X' + Y' = E$. Because $X' + Y' = E \subset Z$, by the definition of the Minkowski sum there exist sets $U \subset X'$ and $V \subset Y'$ satisfying $U + V = Z$. Consequently, $X' \cap X \neq \varnothing$ and $Y' \cap Y \neq \varnothing$. Thus, Lemma 3.3 is proved. □

From Figure 4 it can be seen that the condition that the set Z be homothetic to the sum $X + Y$ is essential.

Theorem 3.4. *For any nonempty sets X and Y, the inclusion*

$$\operatorname{int} X + \operatorname{int} Y \subset \operatorname{int}(X + Y)$$

holds. If, in addition, the sets X and Y are convex and $\dim X = \dim Y = \dim(X + Y)$, then the converse inclusion

$$\operatorname{int} X + \operatorname{int} Y \supset \operatorname{int}(X + Y)$$

also holds. Consequently, for such sets we have

$$\operatorname{int} X + \operatorname{int} Y = \operatorname{int}(X + Y).$$

Proof: Let $z \in \operatorname{int} X + \operatorname{int} Y$. Then there exist points $x \in \operatorname{int} X$ and $y \in \operatorname{int} Y$ such that $z = x + y$. Consider neighborhoods $B_x \subset X$ and $B_y \subset Y$. Then $B_x + B_y \subset X + Y$. Since B_x and B_y are open balls centered at x and y, respectively, their sum is an open set U. From the inclusion $U \subset X + Y$ it follows that $z \in \operatorname{int}(X + Y)$.

In general, the inclusion $\operatorname{int}(X + Y) \subset \operatorname{int} X + \operatorname{int} Y$ does not always hold. For example, if $\dim X < \dim(X + Y)$, then $\operatorname{int} X = \varnothing$, or if $\dim Y < \dim(X + Y)$, then $\operatorname{int} Y = \varnothing$.

Now assume that $\dim X = \dim Y = \dim(X + Y)$. Then

$$\operatorname{int} X + Y = X + \operatorname{int} Y = \operatorname{int} X + \operatorname{int} Y = \operatorname{int}(X + Y).$$

Theorem 3.4 is proved. □

We note that the convexity of the sets in the second part of Theorem 3.4 is essential.

Corollary 3.5. *a) for any convex sets X and Y in $\mathbb{R}^n$ with $\dim X = \dim Y = n$, the equality*

$$\operatorname{int} X + \operatorname{int} Y = \operatorname{int}(X + Y)$$

holds.

b) in the family $c(\mathbb{R}^n)$ of n-dimensional convex sets in $\mathbb{R}^n$, the Minkowski sum

$$+\colon\ c(\mathbb{R}^n) \times c(\mathbb{R}^n) \to c(\mathbb{R}^n)$$

is a continuous mapping.

Theorem 3.6. *Let X and Y be nonempty convex sets satisfying* $\dim X = \dim Y = \dim(X \dot{-} Y)$. *Then the inclusion* $\operatorname{int}(X \dot{-} Y) \subset \operatorname{int} X \dot{-} \operatorname{int} Y$ *holds.*

Proof: Take an arbitrary point $z \in \operatorname{int}(X \dot{-} Y)$. Then there exists an open neighborhood $B_r(z) = x \in \mathbb{R}^n \colon\ |z - x| < r$ such that $B_r(z) \subset X \dot{-} Y$. From here we obtain $B_r(z) + Y \subset X$. Since $\operatorname{int} Y \subset Y$, it follows that $B_r(z) + \operatorname{int} Y \subset B_r(z) + Y \subset X$. Because both $B_r(z)$ and $\operatorname{int} Y$ are open sets, the set $B_r(z) + \operatorname{int} Y$ is open. Hence, $B_r(z) + \operatorname{int} Y \subset \operatorname{int} X$. Therefore, $B_r(z) \subset \operatorname{int} X \dot{-} \operatorname{int} Y$, and consequently $z \in \operatorname{int} X \dot{-} \operatorname{int} Y$. Theorem 3.6 is proved. □

The existence of a point $z \in \operatorname{int} X \dot{-} \operatorname{int} Y$ that does not belong to $\operatorname{int}(X \dot{-} Y)$ can be seen from the following example. Let $X = [1, 5]$ and $Y = [3, 4]$. Then $\operatorname{int} X \dot{-} \operatorname{int} Y = [-2, 1]$, $\operatorname{int}(X \dot{-} Y) = (-2, 1)$. The points $z = -2$ and $z = 1$ have the stated property.

Lemma 3.7. *For arbitrary convex sets X and Y, the inclusion*

$$\mathbb{R}^n \setminus (X + Y) \subset (\mathbb{R}^n \setminus X) + (\mathbb{R}^n \setminus Y)$$

holds.

Proof: Let $z \in \mathbb{R}^n \setminus (X + Y)$. Then $z \notin X + Y$. This means that for any pair of vectors $x \in X$ and $y \in Y$ the equality $z = x + y$ does not hold. For each point $z \in \mathbb{R}^n \setminus (X + Y)$ and fixed points $x \in X$ and $y \in Y$ let us introduce the notations

$$Y_x(z) = \{y \in \mathbb{R}^n \colon\ z = x + y\}, \qquad X_y(z) = \{x \in \mathbb{R}^n \colon\ z = x + y\}.$$

Then, for every $z \in \mathbb{R}^n \setminus (X + Y)$, we obtain

$$\bigcup_{x \in X} Y_x(z) \subset (\mathbb{R}^n \setminus Y), \qquad \bigcup_{y \in Y} X_y(z) \subset (\mathbb{R}^n \setminus X).$$

On the other hand, by the definitions of the sets $Y_x(z)$ and $X_y(z)$ one has

$$z \in X_y(z) + Y_x(z) \subset \bigcup_{y \in Y} X_y(z) + \bigcup_{x \in X} Y_x(z) \subset (\mathbb{R}^n \setminus X) + (\mathbb{R}^n \setminus Y).$$

Thus, $\mathbb{R}^n \setminus (X + Y) \subset (\mathbb{R}^n \setminus X) + (\mathbb{R}^n \setminus Y)$.

Now we show that, in general, $\mathbb{R}^n \setminus (X + Y) \supset (\mathbb{R}^n \setminus X) + (\mathbb{R}^n \setminus Y)$ does not hold. For simplicity, it suffices to consider the case $n = 2$. Consider the following convex sets:

$$X = \{(x,\, y) \in \mathbb{R}^2 \colon\ 1 \le x \le 4,\ 3 \le y \le 6\},$$

$$Y = \{(x,\, y) \in \mathbb{R}^2 \colon\ 5 \le x \le 6,\ 1 \le y \le 2\}.$$

Then their Minkowski sum is the convex set

$$X + Y = \{(x,\, y) \in \mathbb{R}^2 \colon\ 6 \le x \le 10,\ 4 \le y \le 8\}.$$

Consider also the sets

$$T = \{(x,\, y) \in \mathbb{R}^2 \colon\ 8 \le x \le 10,\ 9 \le y \le 11\},$$

$$S = \{(x,\, y) \in \mathbb{R}^2 \colon\ -2 \le x \le 0,\ -5 \le y \le -3\}.$$

Clearly, $T + S = X + Y$. Moreover, $T \subset \mathbb{R}^2 \setminus X$ and $S \subset \mathbb{R}^2 \setminus Y$. Hence,

$$X + Y \subset (\mathbb{R}^2 \setminus X) + (\mathbb{R}^2 \setminus Y).$$

Above, the inclusion $\mathbb{R}^2 \setminus (X + Y) \subset (\mathbb{R}^2 \setminus X) + (\mathbb{R}^2 \setminus Y)$ was established. From the last two inclusions we obtain

$$\mathbb{R}^2 = (X + Y) \cup \big(\mathbb{R}^2 \setminus (X + Y)\big) \subset (\mathbb{R}^2 \setminus X) + (\mathbb{R}^2 \setminus Y),$$

that is, $\mathbb{R}^2 = (\mathbb{R}^2 \setminus X) + (\mathbb{R}^2 \setminus Y)$. Lemma 3.7 is proved. □

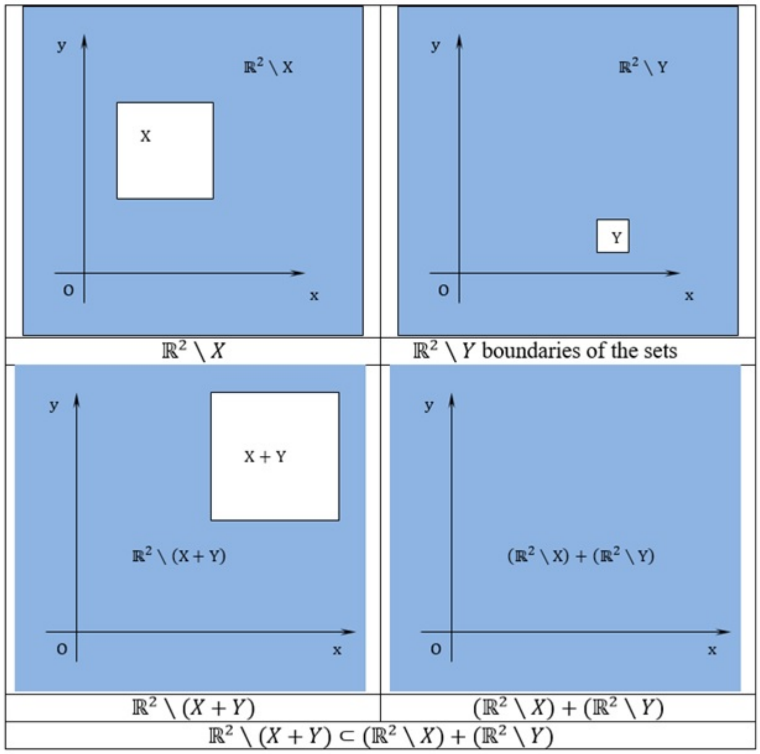

FIGURE 5.

An illustration of the proven lemma is presented in Figure 5.

Theorem 3.8. *For nonempty convex sets X, Y the following inclusions hold:*

$$\partial(X+Y) \subset \partial X + \partial Y \subset [X+Y].$$

Proof: Since the second inclusion in the conclusion of the theorem holds automatically, it suffices to show that any point $z \in \partial(X+Y)$ belongs to the set $\partial X + \partial Y$. By the definition of a boundary point, for any neighborhood $B_r(z)$ of the point z the relations $B_r(z) \cap (X+Y) \neq \varnothing$ and $B_r(z) \cap (\mathbb{R}^n \setminus (X+Y)) \neq \varnothing$ hold.

We can find x, $y \in \mathbb{R}^n$ such that for x and y we claim following

1) $z = x + y$,

2) $x \in \partial X$ and $y \in \partial Y$.

Suppose that for all x, $y \in \mathbb{R}^n$ satisfying condition 1), condition 2) does not hold. Then, for all x, $y \in \mathbb{R}^n$ such that $z = x + y$, it must be that either $x \notin \partial X$ and $y \in \partial Y$, or $x \in \partial X$ and $y \notin \partial Y$, or $x \notin \partial X$ and $y \notin \partial Y$. For clarity, let us assume $x \notin \partial X$ and $y \in \partial Y$. Then, either $x \in \operatorname{int} X$ or $x \in \mathbb{R}^n \setminus [X]$. In the first case, there exists an open ball $B_\varepsilon(x)$ such that $B_\varepsilon(x) \subset X$; in the second case, there exists an open ball $B_\varepsilon(x)$ such that $B_\varepsilon(x) \cap X = \emptyset$. In the first case,

$$B_\varepsilon(x+y) = B_\varepsilon(z) = B_\varepsilon(x) + y \subset X + Y,$$

which implies $z \in \operatorname{int}(X+Y)$. We get a contradiction. In the second case,

$$B_\varepsilon(x+y) = B_\varepsilon(z) = B_\varepsilon(x) + y \subset \mathbb{R}^n \setminus [X+Y].$$

In this case as well, we obtain $z \notin \partial(X+Y)$ which contradicts our assumption $z \in \partial(X+Y)$. Thus, $z \in \partial X + \partial Y$. Theorem 3.8 is proved. □

It should be noted that the relation $\partial(X+Y) \supset \partial X + \partial Y$ may not hold. For example, consider the squares

$$X = \{(x, y) \in \mathbb{R}^2 \colon\ 1 \leq x \leq 4,\ 3 \leq y \leq 6\},$$

$$Y = \{(x,\, y) \in \mathbb{R}^2 \colon\ 4 \le x \le 5,\ 1 \le y \le 2\}.$$

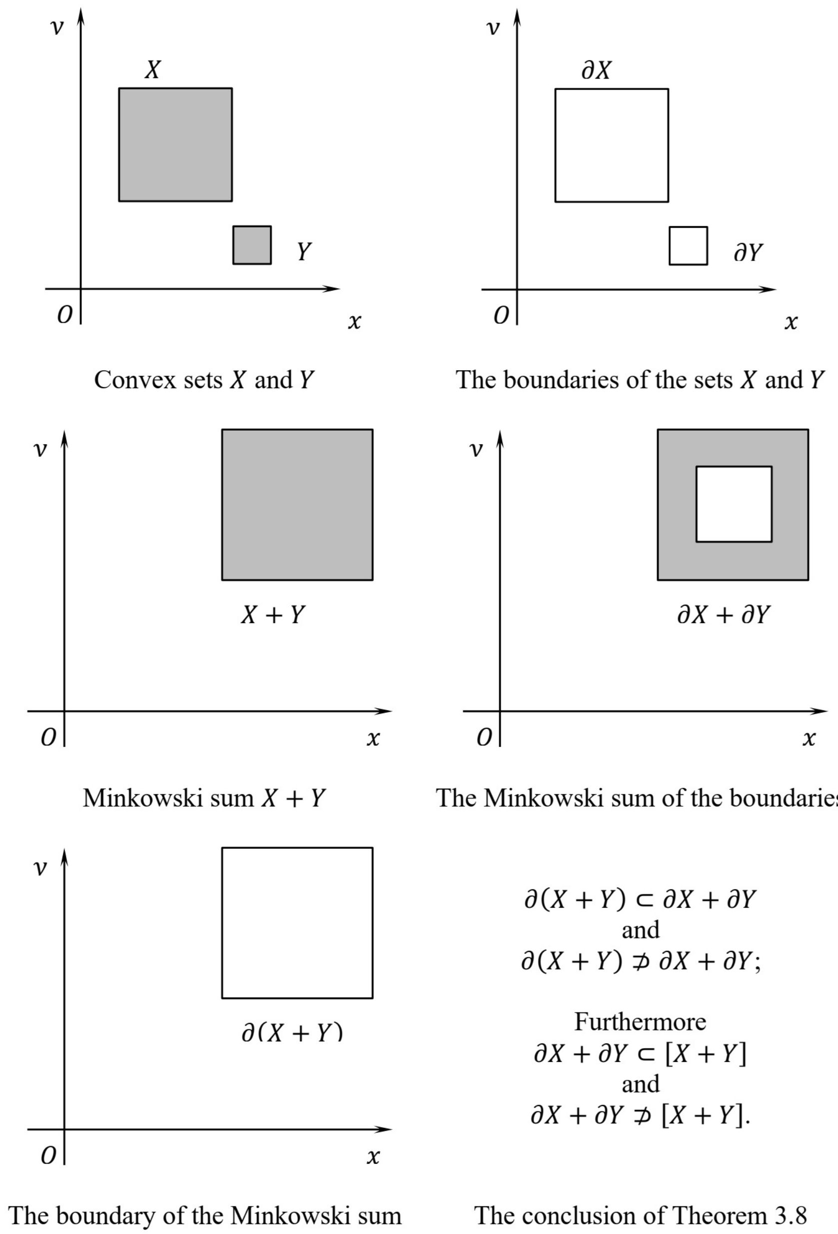

FIGURE 6.

It is clear that (see Fig. 6),

$$\partial X = \{(x,\, y) \in \mathbb{R}^2 \colon\ (x \in \{1,\, 4\},\, 3 \le y \le 6) \vee (1 \le x \le 4,\, y \in \{3,\, 6\})\},$$

$$\partial Y = \{(x,\, y) \in \mathbb{R}^2 \colon (x \in \{4,\, 5\},\, 1 \le y \le 2) \vee (4 \le x \le 5,\, y \in \{1,\, 2\})\},$$

$$\partial X + \partial Y = (X + Y) \setminus \{(x,\, y) \in \mathbb{R}^2 \colon\ 6 \le x \le 8,\, 5 \le y \le 7\},$$

$$X + Y = \{(x,\, y) \in \mathbb{R}^2\, 5 \le x \le 9,\, 4 \le y \le 8\},$$

$$\partial(X + Y) = \{(x,\, y) \in \mathbb{R}^2 \colon\ (x \in \{5,\, 9\},\ 4 \le y \le 8) \vee (5 \le x \le 9,\, y \in \{4\,8\})\}.$$

Thus,

$$\partial(X+Y)\subset\partial X+\partial Y$$

and

$$\partial(X+Y)\not\supset\partial X+\partial Y.$$

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