

Nonlocal Problem for a Mixed Type Equation with Singular Coefficient in a Domain Whose Elliptic Part Is a Half-Strip

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Abstract. This paper investigates a problem for the Gellerstedt equation with a singular coefficient in a domain whose elliptic part is a vertical half-strip. The elliptic part uses the solution of the Problem N type. By using generalized fractional differentiation operators, we specify a linear combination that relates the value of the solution on characteristics of the equation with the value of the solution and its derivative on the parabolic degeneration line. The uniqueness of the solution of the problem is proved using the extremum principle and the energy integral method. The existence of a solution is proved using the method of integral equations.

Keywords: Mixed-type equation, singular coefficient, half-strip, extremum principle, energy integral method, integral equation, uniqueness of the solution, existence of the solution

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Many problems of plane gas flow [1], at transonic speeds, when the flow contains regions with subsonic and supersonic speeds, can be mathematically formulated as boundary value problems for a mixed elliptic-hyperbolic linear equation. For the mixed type equation $yu_{xx} + u_{yy} = 0$, the first fundamental studies were carried out by F. Tricomi [2]. In [3], the Tricomi problem was considered for the generalized Tricomi equation in an unbounded domain whose elliptic part is the upper half-plane and whose hyperbolic part is the characteristic triangle.

Several fundamental problems in transonic aerodynamics lead to the Tricomi boundary value problem, where the domain in the elliptic part of the plane represents a half-strip. The Tricomi problem for the equation $(\operatorname{sign} y)|y|^m u_{xx} + u_{yy} = 0$, $m > 0$ in a domain where the elliptical part is a half-strip has been studied in [4, 5, 6], and for the case of $m = 1$, in [7]. The paper [8] is devoted to the study of a problem with a Frankl-type condition on the degeneracy line for a mixed-type equation in a domain whose elliptic part is the first quadrant of the plane. A study of the boundary value problem in an infinite half-strip for the generalized biaxially symmetric Helmholtz equation was conducted in [9]. Boundary value problems for mixed-type equations with singular coefficient were explored in [10, 11, 12]. The degenerate elliptic-type equation with singular coefficient and a spectral parameter was studied in the vertical half-strip in [13, 14, 15, 16] and in the upper half-plane in [17].

Consider the equation

$$(\operatorname{sgn} y)|y|^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y = 0, \quad (1.1)$$

where $m > 0$, $-\frac{m}{2} < \beta_0 < 1$ in the infinite domain D bounded by the half-lines $x = 0$ and $x = 1$ in the half-plane $y > 0$ and the characteristics

$$OC : x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0, \quad BC : x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1,$$

of Eq. (1.1) issuing from the points $O(0,0)$ and $B(1,0)$.

Denote by $D_1 = D \cap \{(x, y) : y > 0\}$, $D_2 = D \cap \{(x, y) : y < 0\}$ the elliptic and hyperbolic parts of the mixed domain D , respectively, and let $J = \{(x, y) : 0 < x < 1, y = 0\}$.

Problem. Find a solution of Eq. (1.1) in the domain $D = D_1 \cup D_2$, which satisfies the following conditions:

- 1) $u(x, y) \in C(\overline{D}) \cap C^2(D_1 \cup D_2)$ and satisfies Eq. (1.1) in $D_1 \cup D_2$;
- 2) $u(x, y)$ satisfies the boundary conditions

$$u(0, y) = \varphi_1(y), \quad u(1, y) = \varphi_2(y), \quad 0 \leq y < \infty, \quad (1.2)$$

$$\begin{aligned}
& a(x)(I_{0+}^{\mu_1, \rho_1, \eta_1} \delta(t) u[\Theta_0(t)])(x) + b(x)(I_{1-}^{\mu_2, \rho_2, \eta_2} \omega(t) u[\Theta_1(t)])(x) \\
& + c(x)u(x, 0) + d(x) \lim_{y \rightarrow -0} (-y)^{\beta_0} u_y(x, y) = g(x), \quad x \in J,
\end{aligned} \tag{1.3}$$

and the conjugation condition

$$\lim_{y \rightarrow -0} (-y)^{\beta_0} u_y = \lim_{y \rightarrow +0} y^{\beta_0} u_y, \quad x \in (0, 1). \tag{1.4}$$

Moreover, these limits at $x = 0$ can have a singularity of order lower than $1 - 2\beta$, and at $x = 1$ they are bounded, where $\beta = \frac{m+2\beta_0}{2(m+2)}$.

3)

$$\lim_{y \rightarrow +\infty} u(x, y) = 0 \quad \text{uniformly in } x \in [0, 1], \tag{1.5}$$

4)

$$y^{\beta_0} u_y(x, y), u_x(x, y) \text{ are bounded as } y \rightarrow +\infty. \tag{1.6}$$

Here μ_i , ρ_i , and η_i ($i = 1, 2$) are real parameters, $\Theta_0(x)$ and $\Theta_1(x)$ are the points of intersection of the characteristics of Eq. (1.1) issuing from the point $(x, 0) \in J$ with the characteristics OC and BC , respectively. The given functions $\varphi_i(y)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$, $g(x)$, $\delta(x)$, and $\omega(x)$ satisfy the conditions

$$\varphi_i(y) \in C[0, +\infty), \quad y^{\frac{3m+2\beta_0}{4}} \varphi_i(y) \in L_1(0, +\infty), \quad \lim_{y \rightarrow +\infty} \varphi_i(y) = 0, \quad (i = 1, 2), \tag{1.7}$$

$$a(x), b(x), c(x), d(x), g(x) \in C^1(\bar{J}), \quad \delta(x) \in C^1(0, 1], \quad \omega(x) \in C^1[0, 1], \tag{1.8}$$

and moreover,

$$a^2(x) + b^2(x) + c^2(x) + d^2(x) \neq 0 \quad \forall x \in \bar{J}, \tag{1.9}$$

$(I_{0+}^{\mu, \rho, \eta} f)(x)$ and $(I_{1-}^{\mu, \rho, \eta} f)(x)$ are operators of generalized fractional integro-differentiation with the hypergeometric Gauss function, which were introduced in [18] (pp. 326–327) and, for real μ , ρ , η and $x > 0$, have the form

$$\begin{aligned}
(I_{0+}^{\mu, \rho, \eta} f)(x) &= \begin{cases} \frac{x^{-\mu-\rho}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} F\left(\mu+\rho, -\eta; \mu; 1-\frac{t}{x}\right) f(t) dt, & \mu > 0, \\ \left(\frac{d}{dx}\right)^n (I_{0+}^{\mu+n, \rho-n, \eta-n} f)(x), & \mu \leq 0, \quad n = [-\mu] + 1, \end{cases} \\
(I_{1-}^{\mu, \rho, \eta} f)(x) &= \begin{cases} \frac{(1-x)^{-\mu-\rho}}{\Gamma(\mu)} \int_x^1 (t-x)^{\mu-1} F\left(\mu+\rho, -\eta; \mu; \frac{t-x}{1-x}\right) f(t) dt, & \mu > 0, \\ \left(-\frac{d}{dx}\right)^n (I_{1-}^{\mu+n, \rho-n, \eta-n} f)(x), & \mu \leq 0, \quad n = [-\mu] + 1. \end{cases}
\end{aligned} \tag{1.10}$$

The Problem posed above is a straightforward and essential generalization of the Tricomi problem, which was considered for Eq. (1.1) with $m = 1$, $\beta_0 = 0$ in the domain D in the monograph [19]. The Problem with the conditions (1.2)-(1.3) for the Eq. (1.1) with $\beta_0 = 0$ in the domain D is studied in [20].

In this paper, we study the Problem for Eq. (1.1) with singular coefficient.

2. UNIQUENESS OF THE SOLUTION

The solution of the Cauchy problem for Eq. (1.1) in the hyperbolic domain D_2 has the form [10]

$$u(x, y) = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)} \int_0^1 \tau \left(x + \frac{2}{m+2} (-y)^{\frac{m+2}{2}} (2t-1) \right) t^{\beta-1} (1-t)^{\beta-1} dt$$

$$-\frac{2}{m+2} \frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)} (-y)^{1-\beta_0} \int_0^1 \nu \left(x + \frac{2}{(m+2)} (-y)^{\frac{m+2}{2}} (2t-1) \right) t^{-\beta} (1-t)^{-\beta} dt, \quad (2.1)$$

where

$$\tau(x) = u(x, 0), \quad \forall x \in \bar{J}, \quad \nu(x) = \lim_{y \rightarrow -0} (-y)^{\beta_0} u_y(x, y), \quad \forall x \in J.$$

Using the representation (2.1) and taking into account the properties of the operator (1.10), we obtain the relations

$$\begin{aligned} u[\Theta_0(x)] &= \gamma_1(I_{0+}^{\beta, 0, \beta-1} \tau(t))(x) - \gamma_2(I_{0+}^{1-\beta, 2\beta-1, \beta-1} \nu(t))(x), \\ u[\Theta_1(x)] &= \gamma_1(I_{1-}^{\beta, 0, \beta-1} \tau(t))(x) - \gamma_2(I_{1-}^{1-\beta, 2\beta-1, \beta-1} \nu(t))(x), \end{aligned}$$

where

$$\gamma_1 = \frac{\Gamma(2\beta)}{\Gamma(\beta)}, \quad \gamma_2 = \frac{1}{2} \left(\frac{4}{m+2} \right)^{2\beta} \frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)}.$$

Theorem 2.1. *In the domain D , there exists at most one solution of problem (1.1)–(1.3) provided that either*

$$\mu_1 = \mu_2 = -\beta, \quad \rho_1 = \rho_2 = 0, \quad \eta_1 = \eta_2 = 2\beta - 1, \quad \delta(x) = \omega(x) = 1, \quad (2.2)$$

and the condition (1.6),

$$E_1(x) = \gamma_1[a(x) + b(x)] + c(x) \neq 0, \quad x \in \bar{J}, \quad (2.3)$$

$$a(1) \geq 0, \quad b(0) \geq 0, \quad \left[\frac{a(x)}{E_1(x)} \right]' \leq 0, \quad \left[\frac{b(x)}{E_1(x)} \right]' \geq 0, \quad \frac{d(x)}{E_1(x)} \leq 0, \quad x \in \bar{J}, \quad (2.4)$$

are satisfied, or

$$\mu_1 = \mu_2 = \beta - 1, \quad \rho_1 = \rho_2 = 0, \quad \eta_1 = \eta_2 = 1 - 2\beta, \quad (2.5)$$

$$\delta(x) = x^{2\beta-1}, \quad \omega(x) = (1-x)^{2\beta-1}, \quad (2.6)$$

$$E_2(x) = \gamma_2[a(x) + b(x)] - d(x) \neq 0, \quad x \in \bar{J}. \quad (2.7)$$

Proof: Let $u(x, y)$ be a solution of the Problem satisfying the homogeneous conditions (1.2) and (1.3) ($\varphi_i(y) = 0$, $g(x) = 0$), then, by integrating the identity

$$\frac{\partial}{\partial x} (y^{m+\beta_0} u u_x) + \frac{\partial}{\partial y} (y^{\beta_0} u u_y) - y^{\beta_0} [y^m (u_x)^2 + (u_y)^2] = 0,$$

over the domain D_1 and taking into account the conditions imposed on the functions $u(x, y)$, $u_x(x, y)$ and $y^{\beta_0} u_y(x, y)$, we have

$$\iint_{D_1} y^{\beta_0} [y^m (u_x)^2 + (u_y)^2] dx dy + \int_0^1 \tau(x) \nu(x) dx = 0. \quad (2.8)$$

Let us show that if the assumptions of theorem 2.1 hold, then

$$I^* = \int_0^1 \tau(x) \nu(x) dx \geq 0$$

cannot be negative.

Indeed, substituting the functions $u[\Theta_0(x)]$ and $u[\Theta_1(x)]$ into condition (1.3), under conditions (2.2), taking into account the semigroup properties of generalized operators ([18], p. 327),

$$\left(I_{0+}^{\mu, \rho, \eta} \left(I_{0+}^{\gamma, \delta, \mu+\eta} f \right) (t) \right) (x) = \left(I_{0+}^{\mu+\gamma, \rho+\delta, \eta} f \right) (x), \quad \gamma > 0,$$

$$\left(I_{0+}^{\mu,\rho,\eta}\left(I_{1-}^{\gamma,\delta,\mu+\eta}f\right)(t)\right)(x)=\left(I_{1-}^{\mu+\gamma,\rho+\delta,\eta}f\right)(x), \quad \gamma>0,$$

and the well-known relations ([18], p. 327)

$$\left(I_{0+}^{\mu,-\mu,\eta}f\right)(x)=\left(I_{0+}^{\mu}f\right)(x), \quad \left(I_{1-}^{\mu,-\mu,\eta}f\right)(x)=\left(I_{1-}^{\mu}f\right)(x), \quad \mu>0,$$

where I_{0+}^{μ} and I_{1-}^{μ} are fractional Riemann–Liouville integrals, and by making simple transformations, we obtain the following relation between $\tau(x)$ and $\nu(x)$ extended to J from the hyperbolic part D_2 :

$$\tau(x)=a_1(x)\left(I_{0+}^{1-2\beta}\nu\right)(x)+b_1(x)\left(I_{1-}^{1-2\beta}\nu\right)(x)+c_1(x)\nu(x)+g_1(x), \quad (2.9)$$

where

$$a_1(x)=\gamma_2\frac{a(x)}{E_1(x)}, \quad b_1(x)=\gamma_2\frac{b(x)}{E_1(x)}, \quad c_1(x)=-\frac{d(x)}{E_1(x)}, \quad g_1(x)=\frac{g(x)}{E_1(x)}.$$

For $g_1(x)=0$, we have

$$I^*=\int_0^1a_1(x)\nu(x)\left(I_{0+}^{1-2\beta}\nu\right)(x)dx+\int_0^1b_1(x)\nu(x)\left(I_{1-}^{1-2\beta}\nu\right)(x)dx+\int_0^1c_1(x)\nu^2(x)dx.$$

By using the well-known formula for the gamma-function $\Gamma(z)$ [21],

$$\int_0^\infty t^{z-1}\cos(kt)dt=\frac{\Gamma(z)}{k^z}\cos\left(\frac{\pi z}{2}\right), \quad k>0, \quad 0<z<1,$$

and by setting $k=|x-\xi|$ and $z=2\beta$ in it, one can derive the relation

$$\frac{1}{|x-\xi|^{2\beta}}=\frac{1}{\Gamma(2\beta)\cos(\pi\beta)}\int_0^\infty t^{2\beta-1}\cos[t(x-\xi)]dt.$$

and hence

$$\begin{aligned} \frac{\pi}{2\sin(\pi\beta)}I^* &= \int_0^1a_1(x)\nu(x)dx\int_0^x\nu(\xi)d\xi\int_0^\infty t^{2\beta-1}\cos[t(x-\xi)]dt \\ &+ \int_0^1b_1(x)\nu(x)dx\int_x^1\nu(\xi)d\xi\int_0^\infty t^{2\beta-1}\cos[t(\xi-x)]dt + \frac{\pi}{2\sin(\pi\beta)}\int_0^1c_1(x)\nu^2(x)dx. \end{aligned}$$

By changing the order of integration and by integrating by parts, as a result, the following relation holds

$$\begin{aligned} \frac{\pi}{\sin(\pi\beta)}I^* &= a_1(1)\int_0^\infty t^{2\beta-1}dt\left[\left(\int_0^1\nu(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_0^1\nu(\xi)\sin(t\xi)d\xi\right)^2\right] \\ &- \int_0^\infty t^{2\beta-1}dt\int_0^1a_1'(x)\left[\left(\int_0^x\nu(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_0^x\nu(\xi)\sin(t\xi)d\xi\right)^2\right]dx \\ &+ b_1(0)\int_0^\infty t^{2\beta-1}dt\left[\left(\int_0^1\nu(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_0^1\nu(\xi)\sin(t\xi)d\xi\right)^2\right] \\ &+ \int_0^\infty t^{2\beta-1}dt\int_0^1b_1'(x)\left[\left(\int_x^1\nu(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_x^1\nu(\xi)\sin(t\xi)d\xi\right)^2\right]dx \end{aligned}$$

$$+ \frac{\pi}{\sin(\pi\beta)} \int_0^1 c_1(x) \nu^2(x) dx.$$

From conditions (2.3) and (2.4), we have $I^* \geq 0$. Then it follows from (2.8) that the solution of the Problem is unique.

Now, if the conditions (2.5)–(2.7) in the theorem 2.1 are satisfied, then the functional relation between $\tau(x)$ and $\nu(x)$ from the hyperbolic domain D_2 has the form

$$[\gamma_2(a(x) + b(x)) - d(x)]\nu(x) = \gamma_1 a(x)(D_{0+}^{1-2\beta}\tau)(x) + \gamma_1 b(x)(D_{1-}^{1-2\beta}\tau)(x) + c(x)\tau(x) - g(x), \quad (2.10)$$

where $D_{0+}^{1-2\beta}$ and $D_{1-}^{1-2\beta}$ are the Riemann–Liouville fractional differentiation operators ([18], p.43). We rewrite relation (2.10) in the form

$$\nu(x) = a_2(x)(D_{0+}^{1-2\beta}\tau)(x) + b_2(x)(D_{1-}^{1-2\beta}\tau)(x) + c_2(x)\tau(x) + g_2(x), \quad (2.11)$$

where

$$a_2(x) = \gamma_1 \frac{a(x)}{E_2(x)}, \quad b_2(x) = \gamma_1 \frac{b(x)}{E_2(x)}, \quad c_2(x) = \frac{c(x)}{E_2(x)}, \quad g_2(x) = -\frac{g(x)}{E_2(x)}.$$

Let D_h and D_{1h} be finite domains cut off from D and D_1 , respectively, by the line $y = h > 0$, where $O_h(0, h)$ and $B_h(1, h)$.

Lemma 2.1. *Let*

- 1) *the function $u(x, y) \in C(\overline{D_{1h}}) \cap C^2(D_{1h})$ satisfy the inequality $L(u) \geq 0$ ($L(u) \leq 0$) and take on a positive maximum (respectively, a negative minimum) value at some point $(x_0, 0)$, $x_0 \in (0, 1)$;*
- 2) *the value of $u(x, y)$ on $\sigma = \overline{OO_h} \cup O_h B_h \cup \overline{B_h B}$ is less (greater) than $u(x_0, 0)$. Then*

$$\nu_+(x) = \lim_{y \rightarrow +0} y^{\beta_0} u_y(x_0, 0) < 0 \quad (\nu_+(x) > 0)$$

provided that this limit exists.

Lemma 2.2. *Let $g(x) \equiv 0$, and let either $a(x), b(x), c(x) > 0$ and $d(x) \leq 0$ or $a(x), b(x), c(x) < 0$ and $d(x) \geq 0$. If the function $\tau_-(x)$ attains a positive maximum (respectively, a negative minimum) on the interval $[0, 1]$ at the point $x = x_0$, $0 < x_0 < 1$, then $\nu_-(x_0) > 0$ (respectively, $\nu_-(x_0) < 0$).*

The assertion of the lemma 2.2 readily follows from relation (2.10) and the maximum principle for fractional differentiation operators [19].

Let $u(x, y)$ be a solution of the homogeneous problem for Eq. (1.1) in the domain D_1 that satisfies the boundary conditions $u(0, y) = u(1, y) = 0$, $y \geq 0$. We show that $u(x, y) \equiv 0$ in D_1 .

Suppose that this is not true. Then there exists a domain D_{1h} such that, indeed, $\max_{\overline{D_{1h}}} |u(x, y)| > 0$. Without loss of generality, one can assume that $\max_{\overline{D_{1h}}} u(x, y) = \max_{\overline{D_{1h}}} u(x, y) > 0$.

Assume that $\max_{\overline{D_{1h}}} u(x, y)$ is attained on $O_h B_h$. Since $u(0, y) = u(1, y) = 0$ for $y \geq 0$, it should not be attained on $\sigma = \overline{OO_h} \cup O_h B_h \cup \overline{B_h B}$, and, by virtue of the well-known property of elliptic equations [22] (pp.24–25), it should be attained on the segment $\overline{OB}$ at some its interior point $(x_0, 0)$. But then, by the lemma 2.1 given above, it is $\nu_+(x_0) < 0$, which contradicts $\nu_-(x_0) > 0$.

Therefore, $\max_{D_{1h}} u(x, y)$ must be attained on $O_h B_h$. Take an arbitrary small number $\varepsilon > 0$ and, by (1.5), choose h large enough so that $|u(x, y)| < \varepsilon$ for $y = h$.

Let (x_0, y_0) be an arbitrary point in the domain D_1 . For a sufficiently large h , this point falls in the domain D_{1h} , and therefore $|u(x_0, y_0)| < \varepsilon$. Since ε is arbitrary, we have $u(x_0, y_0) = 0$. Then $u(x, y) \equiv 0$ in the domain $\overline{D_1}$, and, in particular, $u(x, +0) = \tau_+(x) \equiv 0$ on $[0, 1]$. But since $\tau_+(x) = \tau_-(x)$, it follows from (2.11) that $\nu_-(x) \equiv 0$. Then, from Darboux's formula (2.1), we obtain $u(x, y) \equiv 0$ in the domain D_2 .

Therefore, the Problem has at most one solution. $\square$

EXISTENCE OF A SOLUTION OF THE PROBLEM

To derive a functional relation between $\tau(x) = u(x, 0)$ and $\nu(x) = \lim_{y \rightarrow +0} y^{\beta_0} u_y(x, y)$ extended to J from the elliptic part D_1 of the mixed domain D , consider Problem N .

Problem N . In the domain D_1 , find a solution of Eq. (1.1) satisfying the inclusion $u(x, y) \in C(\overline{D}_1) \cap C^2(D_1)$ and conditions (1.2) and (1.5)–(1.7).

The solution of Problem N was found in [10] in the form

$$u(x, y) = u_1(x, y) + u_2(x, y),$$

where

$$u_1(x, y) = y^{\frac{1-\beta_0}{2}} \int_0^\infty [a_1(\lambda)e^{\lambda x} + a_2(\lambda)e^{-\lambda x}] J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} y^{\frac{m+2}{2}} \right) d\lambda,$$

$$\begin{aligned} a_1(\lambda)e^{\lambda x} + a_2(\lambda)e^{-\lambda x} = & \frac{2\lambda}{(m+2)\sinh \lambda} \left[\sinh(\lambda(1-x)) \int_0^\infty y^{\frac{2m+1+\beta_0}{2}} J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} y^{\frac{m+2}{2}} \right) \varphi_1(y) dy \right. \\ & \left. + \sinh(\lambda x) \int_0^\infty J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} y^{\frac{m+2}{2}} \right) \varphi_2(y) dy \right], \end{aligned}$$

and

$$\begin{aligned} u_2(x, y) = & y^{\frac{1-\beta_0}{2}} \sum_{n=1}^\infty C_n K_{\frac{1-2\beta}{m+2}} \left(\frac{2\pi n}{m+2} y^{\frac{m+2}{2}} \right) \sin(\pi n x), \\ C_n = & \tilde{k} \int_0^1 \nu(t) \sin(\pi n t) dt, \quad \tilde{k} = -\frac{4}{\pi n} \left(\frac{\pi n}{m+2} \right)^{\frac{1+2\beta}{2}} \frac{1}{\Gamma\left(\frac{1+2\beta}{2}\right)}. \end{aligned}$$

Here, $J_\nu(z)$ is the Bessel function of the first kind and $K_\nu(z)$ is the Macdonald function [23].

The relation between $\tau(x)$ and $\nu(x)$ extended from D_1 to J has the form

$$\tau(x) = \gamma_0 \int_0^1 \nu(t) G(x, t) dt + \tau_1(x), \quad (2.12)$$

where

$$\begin{aligned} G(x, t) = & |x-t|^{-2\beta} - (x+t)^{-2\beta} \\ & + \sum_{n=1}^\infty [(2n-x+t)^{-2\beta} + (2n+x-t)^{-2\beta} - (2n-x-t)^{-2\beta} - (2n+x+t)^{-2\beta}], \\ \tau_1(x) = & u_1(x, 0) = k_0 \int_0^\infty t^{\frac{2m+1+\beta_0}{2}} \varphi_1(t) dt \int_0^\infty \frac{\sinh(\lambda(1-x))}{\sinh \lambda} \lambda^{\frac{1+2\beta}{2}} J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} t^{\frac{m+2}{2}} \right) d\lambda \\ & + k_0 \int_0^\infty t^{\frac{2m+1+\beta_0}{2}} \varphi_2(t) dt \int_0^\infty \frac{\sinh(\lambda x)}{\sinh \lambda} \lambda^{\frac{1+2\beta}{2}} J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} t^{\frac{m+2}{2}} \right) d\lambda, \\ \gamma_0 = & -\left(\frac{2}{m+2} \right)^{2\beta} \frac{\Gamma(\beta)}{2\sqrt{\pi}\Gamma\left(\frac{1+2\beta}{2}\right)}, \quad k_0 = \left(\frac{1}{m+2} \right)^{\frac{1+2\beta}{2}} \frac{2}{\Gamma\left(\frac{1+2\beta}{2}\right)}. \end{aligned}$$

By using the formula [23] (p. 191)

$$\int_0^\infty z^{\nu+1} \frac{\sinh(az)}{\sinh(bz)} I_\nu(cz) dz = \frac{2}{b} \left(\frac{\pi}{b} \right)^{\nu+1} \sum_{k=1}^\infty (-1)^{k-1} k^{\nu+1} \sin\left(\frac{ak\pi}{b}\right) K_\nu\left(\frac{ck\pi}{b}\right),$$

$$[\operatorname{Re} \nu > -1; \quad \operatorname{Re} b > |\operatorname{Re} a| + |\operatorname{Im} c|],$$

one can claim that the function $\tau_1(x)$ has derivatives of arbitrary order on the interval $(0, 1)$ and remains bounded as $x \rightarrow 0$ and $x \rightarrow 1$.

Let conditions (2.2) and (2.3) be satisfied. By substituting the function (2.12) into relation (2.9), we obtain, after some transformations,

$$d(x)\nu(x) + \int_0^1 \frac{K(x, t)\nu(t)dt}{|x - t|^{2\beta}} = f(x), \quad (2.13)$$

where

$$K(x, t) = \begin{cases} \gamma_0 E_1(x)(x - t)^{2\beta} G(x, t) - \frac{\gamma_2 a(x)}{\Gamma(1 - 2\beta)}, & t \leq x, \\ \gamma_0 E_1(x)(t - x)^{2\beta} G(x, t) - \frac{\gamma_2 b(x)}{\Gamma(1 - 2\beta)}, & t \geq x, \end{cases}$$

$$f(x) = g(x) - E_1(x)\tau_1(x).$$

By virtue of the above-stipulated assumptions on the smoothness of the given functions, one can obtain $f(x) \in C(\bar{J}) \cap C^1(J)$.

Therefore, relation (2.13) with $d(x) \neq 0$ is a Fredholm integral equation of the second kind with a weak singularity in the kernel, whose unconditional solvability follows from the uniqueness of the solution of the Problem.

By substituting the function (2.12) into relation (2.11) and by making simple transformations, we have

$$\begin{aligned} & [\gamma_2(a(x) + b(x)) - d(x)]\nu(x) - \gamma_0 c(x) \int_0^1 \nu(t)G(x, t)dt \\ & - \frac{\gamma_0 \gamma_1 a(x)}{\Gamma(2\beta)} \frac{d}{dx} \int_0^x \frac{d\xi}{(x - \xi)^{1-2\beta}} \int_0^1 \nu(t)G(\xi, t)dt \\ & + \frac{\gamma_0 \gamma_1 b(x)}{\Gamma(2\beta)} \frac{d}{dx} \int_x^1 \frac{d\xi}{(\xi - x)^{1-2\beta}} \int_0^1 \nu(t)G(\xi, t)dt = F(x), \end{aligned} \quad (2.14)$$

where

$$F(x) = g(x) + c(x)\tau_1(x) + \frac{\gamma_1}{\Gamma(2\beta)} \left[a(x) \int_0^x \frac{\tau_1'(t)dt}{(x - t)^{1-2\beta}} - b(x) \int_x^1 \frac{\tau_1'(t)dt}{(t - x)^{1-2\beta}} \right]. \quad (2.15)$$

Now, let us study the solvability of the integral equation (2.14). To this end, we transform the expression on the left-hand side in Eq. (2.14),

$$I_1(x) = \frac{d}{dx} \int_0^x (x - \xi)^{2\beta-1} d\xi \int_0^1 \nu(t)G(\xi, t)dt,$$

and then, by making the related transformations, the following is obtained

$$\begin{aligned} I_1(x) &= \pi \tan(\pi\beta)\nu(x) + \int_0^1 \left(\frac{t}{x}\right)^{1-2\beta} \frac{\nu(t)dt}{t - x} - \int_0^1 \left(\frac{t}{x}\right)^{1-2\beta} \frac{\nu(t)dt}{t + x} \\ &+ \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n+t}{x}\right)^{1-2\beta} \frac{\nu(t)dt}{2n - x + t} + \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n-t}{x}\right)^{1-2\beta} \frac{\nu(t)dt}{2n + x - t} \end{aligned}$$

$$-\sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n-t}{x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n-x-t} - \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n+t}{x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n+x+t}.$$

In a similar way, we have

$$\begin{aligned} I_2(x) &= \frac{d}{dx} \int_x^1 \frac{d\xi}{(\xi-x)^{1-2\beta}} \int_0^1 \nu(t)G(\xi, t)dt \\ &= -\pi \tan(\pi\beta)\nu(x) + \int_0^1 \left(\frac{1-t}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{t-x} + \int_0^1 \left(\frac{1+t}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{t+x} \\ &\quad - \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n+t-1}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n-x+t} - \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n-t+1}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n+x-t} \\ &\quad + \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n-t-1}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n-x-t} + \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n+t+1}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n+x+t}. \end{aligned}$$

In what follows, we additionally assume that

$$a(x) = x^\varepsilon \tilde{a}_1(x), \quad b(x) = (1-x)^\varepsilon \tilde{b}_1(x), \quad d(x) = [x(1-x)]^\varepsilon \tilde{d}_1(x), \quad \varepsilon > 1-2\beta, \quad (2.16)$$

$$\tilde{a}_1(x), \tilde{b}_1(x), \tilde{d}_1(x) \in C^{(0,\lambda)}(\bar{J}) \cap C^1(J), \quad \lambda > 0.$$

By using conditions (2.16), we rewrite Eq. (2.14) in the form

$$A(x)\nu(x) + \frac{B(x)}{\pi i} \int_0^1 \frac{\nu(t)dt}{t-x} + \int_0^1 \frac{K(x, t)\nu(t)dt}{t-x} = F(x), \quad (2.17)$$

where

$$A(x) = \gamma_2 [x^\varepsilon \tilde{a}_1(x) + (1-x)^\varepsilon \tilde{b}_1(x)] (1 + \sin(\pi\beta)) - x^\varepsilon (1-x)^\varepsilon \tilde{d}_1(x),$$

$$B(x) = i\gamma_2 [x^\varepsilon \tilde{a}_1(x) + (1-x)^\varepsilon \tilde{b}_1(x)] \cos(\pi\beta),$$

and

$$\begin{aligned} K(x, t) &= -\gamma_0 c(x)(t-x) \left[\frac{1}{|x-t|^{2\beta}} - \frac{1}{(x+t)^{2\beta}} \right] \\ &\quad - \frac{\gamma_0 \gamma_1}{\Gamma(2\beta)} \left\{ x^\varepsilon \tilde{a}_1(x) \left[\left(\frac{t}{x} \right)^{1-2\beta} - 1 \right] - (1-x)^\varepsilon \tilde{b}_1(x) \left[\left(\frac{1-t}{1-x} \right)^{1-2\beta} + 1 \right] \right\} \\ &\quad + \frac{\gamma_0 \gamma_1}{\Gamma(2\beta)} \frac{t-x}{t+x} \left[x^\varepsilon \tilde{a}_1(x) \left(\frac{t}{x} \right)^{1-2\beta} + (1-x)^\varepsilon \tilde{b}_1(x) \left(\frac{1+t}{1-x} \right)^{1-2\beta} \right] \\ &\quad - \gamma_0 c(x)(t-x) \sum_{n=1}^{\infty} \left[(2n-x+t)^{-2\beta} + (2n+x-t)^{-2\beta} - (2n-x-t)^{-2\beta} - (2n+x+t)^{-2\beta} \right] \\ &\quad - \frac{\gamma_0 \gamma_1}{\Gamma(2\beta)} x^\varepsilon \tilde{a}_1(x)(t-x) \left[\sum_{n=1}^{\infty} \left(\frac{2n+t}{x} \right)^{1-2\beta} \frac{1}{2n-x+t} + \sum_{n=1}^{\infty} \left(\frac{2n-t}{x} \right)^{1-2\beta} \frac{1}{2n+x-t} \right. \\ &\quad \left. - \sum_{n=1}^{\infty} \left(\frac{2n-t}{x} \right)^{1-2\beta} \frac{1}{2n-x-t} - \sum_{n=1}^{\infty} \left(\frac{2n+t}{x} \right)^{1-2\beta} \frac{1}{2n+x+t} \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{\gamma_0\gamma_1}{\Gamma(2\beta)}(1-x)^{\varepsilon}\tilde{b}_1(x)(t-x)\left[\sum_{n=1}^{\infty}\left(\frac{2n+t-1}{1-x}\right)^{1-2\beta}\frac{1}{2n-x+t}+\sum_{n=1}^{\infty}\left(\frac{2n-t+1}{1-x}\right)^{1-2\beta}\frac{1}{2n+x-t}\right. \\
& \left.-\sum_{n=1}^{\infty}\left(\frac{2n-t-1}{1-x}\right)^{1-2\beta}\frac{1}{2n-x-t}-\sum_{n=1}^{\infty}\left(\frac{2n+t+1}{1-x}\right)^{1-2\beta}\frac{1}{2n+x+t}\right]
\end{aligned}$$

It follows from the form of the kernel $K(x, t)$, the smoothness of the given functions, and the conditions (2.16) that

$$K(x, t) \in C^{(0, \lambda)}(\bar{J} \times \bar{J}) \cap C^1(J \times J),$$

and the right-hand side $F(x) \in C(0, 1] \cap C^1(0, 1)$ of Eq. (2.17) from (2.15).

Thus, the singular integral equation (2.17) has a unique solution given by the following form

$$\nu(x) = A^*(x)P(x) - \frac{B^*(x)}{\pi i} \left(\frac{1-x}{x}\right)^{\frac{1}{4}(1-2\beta)} \int_0^1 \left(\frac{t}{1-t}\right)^{\frac{1}{4}(1-2\beta)} \frac{P(t)dt}{t-x},$$

where

$$A^*(x) = \frac{A(x)}{A^2(x) - B^2(x)}, \quad B^*(x) = \frac{B(x)}{A^2(x) - B^2(x)}, \quad P(x) = F(x) - \int_0^1 \frac{K(x, t)\nu(t)dt}{t-x}.$$

The condition $A^2(x) - B^2(x) \neq 0$ provides the existence of a regularizer [24] that reduces the singular integral equation (2.17) to a Fredholm integral equation of the second kind, whose unconditional solvability follows from the uniqueness of the solution of the problem.

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