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Stability and Convergence of a Nonlocal Explicit Finite-Difference Scheme for Numerical Solution of the Fractional Van der Pol–Airy Equation

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Abstract. This paper presents a study of a nonlocal explicit finite-difference scheme for the numerical solution of the Cauchy problem for the fractional Van der Pol–Airy oscillator, which describes forced oscillations with memory effects. The study focuses on the stability and convergence of the proposed numerical scheme. These properties are rigorously established using the properties of the scheme’s coefficients, the approximation of fractional derivative operators, and discrete Volterra inequalities. The conditional convergence and stability of the numerical scheme are proved. The scheme is shown to have first-order accuracy, provided that a condition on the time step of the computational grid is satisfied, which is also derived.

Keywords: fractional oscillator, Van der Pol–Airy equation, finite-difference scheme, stability, convergence, numerical modeling, fractional calculus, forced oscillations

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1. INTRODUCTION

Currently, there is increased interest in the development of mathematical models of oscillatory systems with memory (fractional oscillators). This interest is driven by the presence of relevant applied problems and their applications in various fields of knowledge: from physical and engineering sciences [1] to economics [2, 3]. Indeed, oscillations arise in various systems, including those possessing heredity, and they can carry important information about the properties of the oscillatory process under consideration. Studying these properties will allow us to establish important relationships between various parameters of the system and describe a particular oscillatory regime. Fractional oscillators are studied within the framework of fractional dynamics theory [4], and the tool for studying fractional oscillators is typically the apparatus of fractional calculus [5, 6, 7].

This article continues the study of the nonlinear Van der Pol–Airy oscillatory system with heredity effects. The peculiarity of this oscillatory system lies in the presence in the model equation of nonlinear friction, characteristic of relaxation oscillations of the Van der Pol oscillator [8], a linear dependence of the natural oscillation frequency on time, characteristic of oscillations with a slowly decaying amplitude in the Airy oscillator [9], as well as the consideration of heredity effects, which are typical for viscoelastic and plastic media [10]. Heredity effects are described using fractional derivatives, understood in the sense of Gerasimov–Caputo [11, 12]. Note that in the authors’ works [13, 14], free oscillations of the fractional Van der Pol–Airy oscillator were studied, numerical methods were investigated, as well as qualitative properties of the numerical solution, such as the stability of limit cycles and the computational order of the scheme according to the Runge rule. A software package was developed in the Python programming language that allows visualizing the modeling results: constructing oscillograms and phase trajectories [15]. In the authors’ work [16], forced oscillations of the fractional Van der Pol–Airy oscillator were investigated, and a nonlocal explicit finite-difference scheme was proposed for an algorithm to calculate amplitude-frequency and phase-frequency characteristics. It was shown that the orders of fractional derivatives are related to the Q-factor of the oscillatory system.

Despite previous research, a rigorous study of the stability and convergence of the nonlocal explicit finite-difference scheme has not been conducted. Therefore, the present work is devoted to filling this gap. The paper investigates the stability and convergence of the nonlocal explicit finite-difference scheme. Through rigorous proofs of the corresponding theorems, it is established that the proposed scheme is conditionally stable and converges with first-order accuracy. A condition on the computational grid step is also obtained.

2. PROBLEM STATEMENT

Consider the Cauchy problem for the fractional Van der Pol–Airy equation:

$$\frac{1}{\theta^{2-\alpha}} \partial_{0t}^\alpha x(t) + \frac{\lambda (ax(t)^2 - b)}{\theta^{1-\beta}} \partial_{0t}^\beta x(t) + ctx(t) = \delta \cos(\varphi t), \quad x(0) = x_0, \quad \dot{x}(0) = y_0. \quad (2.1)$$

Here $x(t) \in C^2[0, T]$ is the solution function, $T > 0$ is the simulation time, $t \in [0, T]$ is the process time, θ is the time scale coefficient [17], $\lambda, a, b, c, \delta, \phi, x_0, y_0$ are given constants. The fractional differentiation operators in the model equation (2.1) are understood in the sense of Gerasimov–Caputo with fractional orders $1 < \alpha < 2$, $0 < \beta < 1$ and are defined as follows [11, 12]:

$$\begin{aligned} \partial_{0t}^\alpha x(t) &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \ddot{x}(\tau) d\tau (t-\tau)^{\alpha-1}, \\ \partial_{0t}^\beta x(t) &= \frac{1}{\Gamma(1-\beta)} \int_0^t \dot{x}(\tau) d\tau (t-\tau)^\beta. \end{aligned} \quad (2.2)$$

Remark 2.1. The Van der Pol–Airy oscillator, which is described using fractional derivatives (2.2), will be called the fractional Van der Pol–Airy oscillator. The fractional derivatives here describe the heredity effect, where the current state of the oscillatory system depends on its previous states.

Remark 2.2. The Cauchy problem (2.1) describes forced oscillations with nonlinear friction and taking into account the increase in the natural frequency over time. Since nonlinear friction is characteristic of the Van der Pol oscillator, and the natural frequency increases linearly with time, which is characteristic of the Airy oscillator, we will call the Cauchy problem (2.1) the fractional Van der Pol–Airy oscillator.

Remark 2.3. Known special cases of the fractional Van der Pol–Airy oscillator (2.1) include:

- (1) For $c = \theta = 1$, $\delta = 0$, we obtain free oscillations of the Van der Pol–Airy oscillator, studied in [13, 14].
- (2) For $\alpha = 2$, $\beta = 1$, we arrive at the classical Van der Pol–Airy oscillator.
- (3) For $\lambda = 0$, we obtain the fractional Airy oscillator, studied in [18].

Remark 2.4. Note also that in the case where the natural oscillations in the model equation (2.1) are constant, we are dealing with the fractional Van der Pol oscillator, which has been fairly well studied [19].

For simplicity, we will henceforth use the notation $f(x) = \lambda(ax^2 - b)$. The function f satisfies the following Lipschitz condition.

Lemma 2.1. For any $M > 0$ the function $f(x) = \lambda(ax^2 - b)$ is Lipschitz continuous on $[-M, M]$ with constant $L_f = 2|\lambda a|M$. That is, for all $x, y \in [-M, M]$

$$|f(x) - f(y)| \leq L_f |x - y|. \quad (2.3)$$

3. APPROXIMATION OF FRACTIONAL DERIVATIVES

Because the model equation (2.1) is nonlinear, we solve it numerically. We assume that the exact solution $x(t)$ belongs to $C^3[0, T]$. Introduce a uniform grid $t_k = k\tau$, $k = 0, 1, \dots, N$, with step $\tau = T/N$, and let $x_k = x(t_k)$, $f_k = f(x_k)$.

We approximate the Gerasimov–Caputo derivatives using the well-known L1 discretization. For the second-order derivative inside the α -derivative we use a central difference, while for the first-order derivative inside the β -derivative we use a forward difference. This leads to

$$\begin{aligned} \partial_{0t}^\alpha x(t_k) &\approx A \sum_{j=0}^{k-1} \omega_j^\alpha \Delta^2 x_{k-j}, \\ \partial_{0t}^\beta x(t_k) &\approx B \sum_{j=1}^{k-1} \omega_j^\beta \Delta x_{k-j}, \end{aligned} \quad (3.1)$$

where $\Delta^2 x_m = x_{m+1} - 2x_m + x_{m-1}$, $\Delta x_m = x_{m+1} - x_m$,

$$A = \frac{\tau^{-\alpha}}{\theta^{2-\alpha}\Gamma(3-\alpha)}, \quad B = \frac{\tau^{-\beta}}{\theta^{1-\beta}\Gamma(2-\beta)},$$

and the weight coefficients are

$$\omega_j^\alpha = (j+1)^{2-\alpha} - j^{2-\alpha}, \quad \omega_j^\beta = (j+1)^{1-\beta} - j^{1-\beta}.$$

The coefficients possess the following elementary but important properties.

Lemma 3.1. *For all $1 < \alpha < 2$ and $0 < \beta < 1$ the following holds:*

- (1) $A > 0$, $B > 0$.
- (2) $\omega_j^\alpha > 0$, $\omega_j^\beta > 0$ for every $j \geq 0$ and the sequences are strictly decreasing: $\omega_{j+1}^\alpha < \omega_j^\alpha$, $\omega_{j+1}^\beta < \omega_j^\beta$.
- (3) $\sum_{j=0}^{\infty} |\Delta^2 \omega_j^\alpha| = \omega_1^\alpha$ and $\sum_{j=0}^{\infty} |\Delta \omega_j^\beta| = \omega_1^\beta$, where $\Delta \omega_j^\beta = \omega_{j+1}^\beta - \omega_j^\beta$ and $\Delta^2 \omega_j^\alpha = \omega_{j+1}^\alpha - 2\omega_j^\alpha + \omega_{j-1}^\alpha$ (with $\omega_{-1}^\alpha = 0$).

Proof: Property 1 follows directly from the positivity of the gamma function. For property 2, consider the function $f(x) = x^p$ with $p = 2 - \alpha \in (0, 1)$ or $p = 1 - \beta \in (0, 1)$. Then $\omega_j = f(j+1) - f(j)$. Since $f'(x) > 0$, the differences are positive. Concavity ($f''(x) < 0$) implies the differences decrease. Property 3 is verified by telescoping:

$$\sum_{j=0}^m \Delta^2 \omega_j^\alpha = \omega_{m+1}^\alpha - \omega_m^\alpha - \omega_1^\alpha \rightarrow -\omega_1^\alpha \quad (m \rightarrow \infty),$$

and because all $\Delta^2 \omega_j^\alpha$ are negative, their absolute values sum to ω_1^α . The proof for β is analogous. $\square$

Lemma 3.2. *Let $x \in C^3[0, T]$. Then the approximation errors satisfy*

$$\left| \partial_{0t}^\alpha x(t_k) - A \sum_{j=0}^{k-1} \omega_j^\alpha \Delta^2 x_{k-j} \right| \leq C_\alpha \tau, \quad \left| \partial_{0t}^\beta x(t_k) - B \sum_{j=1}^{k-1} \omega_j^\beta \Delta x_{k-j} \right| \leq C_\beta \tau,$$

with constants $C_\alpha = \frac{M_3 T^{2-\alpha}}{\theta^{2-\alpha}\Gamma(3-\alpha)}$, $C_\beta = \frac{M_2 T^{1-\beta}}{\theta^{1-\beta}\Gamma(2-\beta)}$, where $M_2 = \max_{[0, T]} |x''|$, $M_3 = \max_{[0, T]} |x'''|$.

Proof: We give the proof for the α -derivative; the β case is similar. Start from the definition

$$\partial_{0t}^\alpha x(t_k) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - s)^{1-\alpha} x''(s) ds.$$

On each subinterval $[t_j, t_{j+1}]$ we replace $x''(s)$ by its linear interpolant at t_j, t_{j+1} plus remainder:

$$x''(s) = \frac{t_{j+1} - s}{\tau} x''(t_j) + \frac{s - t_j}{\tau} x''(t_{j+1}) + R_j(s), \quad |R_j(s)| \leq \frac{M_3}{2} (s - t_j)(t_{j+1} - s).$$

Substituting and integrating, the principal part gives exactly the discrete formula with weights ω_j^α after summation. The remainder contributes at most

$$\frac{M_3}{2\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - s)^{1-\alpha} (s - t_j)(t_{j+1} - s) ds \leq \frac{M_3 T^{2-\alpha}}{\Gamma(3-\alpha)} \tau.$$

Dividing by $\theta^{2-\alpha}$ yields the stated constant. The estimate for the β -derivative follows the same lines using $x'(s)$ and forward differences. $\square$

4. NONLOCAL EXPLICIT FINITE-DIFFERENCE SCHEME

Substituting the approximations (3.1) into equation (2.1) and solving for x_{k+1} we obtain the following explicit scheme

$$x_{k+1} = \frac{2A + P_k B - c\tau k}{D_k} x_k - \frac{A}{D_k} x_{k-1} - \frac{A}{D_k} \sum_{j=1}^{k-1} \omega_j^\alpha \Delta^2 x_{k-j} - \frac{P_k B}{D_k} \sum_{j=1}^{k-1} \omega_j^\beta \Delta x_{k-j} + \frac{\delta \cos(k\tau\varphi)}{D_k}, \quad (4.1)$$

where $P_k = f(x_k)$ and $D_k = A + P_k B$.

Our aim is to write the scheme in the form of a linear combination of previous values with coefficients that can be estimated. Using the identities

$$\sum_{j=1}^{k-1} \omega_j^\alpha \Delta^2 x_{k-j} = \sum_{m=0}^k c_{km}^\alpha x_m, \quad \sum_{j=1}^{k-1} \omega_j^\beta \Delta x_{k-j} = \sum_{m=0}^k c_{km}^\beta x_m,$$

with the convention $\omega_0^\alpha = \omega_0^\beta = 0$, we can rewrite (4.1) as

$$x_{k+1} = \sum_{m=0}^k a_{km} x_m + g_k, \quad k = 0, 1, \dots, N-1, \quad (4.2)$$

where $g_k = \delta \cos(k\tau\varphi)/D_k$ and the coefficients a_{km} are given explicitly by

$$\begin{aligned} a_{k,k} &= \frac{2A + P_k B - c\tau k}{D_k} - \frac{A\omega_1^\alpha + P_k B\omega_1^\beta}{D_k}, \\ a_{k,k-1} &= -\frac{A}{D_k} + \frac{A(2\omega_1^\alpha - \omega_2^\alpha) + P_k B(\omega_1^\beta - \omega_2^\beta)}{D_k}, \\ a_{k,m} &= -\frac{A(\omega_{k-m+1}^\alpha - 2\omega_{k-m}^\alpha + \omega_{k-m-1}^\alpha) + P_k B(\omega_{k-m+1}^\beta - \omega_{k-m}^\beta)}{D_k}, \quad 0 \leq m \leq k-2, \end{aligned}$$

with the understanding that $\omega_{-1}^\alpha = 0$ and $\omega_0^\beta = 0$ wherever needed.

For the subsequent analysis the following estimate of the total variation of the coefficients is crucial.

Lemma 4.1. *Assume that there exist constants $M > 0$ and $R_0 \in (0, 1)$ such that $|x_k| \leq M$ and*

$$\tau \leq \tau_0 = \left[\frac{R_0}{\lambda(|a|M^2 + |b|)C_0} \right]^{1/(\alpha-\beta)}, \quad C_0 = \theta^{1+\beta-\alpha} \frac{\Gamma(3-\alpha)}{\Gamma(2-\beta)}.$$

Then for all k

$$\sum_{m=0}^k |a_{km}| \leq \Gamma_0,$$

where Γ_0 is a constant independent of τ and k , given by

$$\Gamma_0 = \frac{1}{1-R_0} \left(2 + 2R_0 + \omega_1^\alpha + R_0\omega_1^\beta + \max_k \frac{c\tau k}{A} \right).$$

In particular, Γ_0 is uniformly bounded as $\tau \rightarrow 0$.

Proof: First, from the definition of A, B and the bound on P_k we have

$$|P_k| \leq \lambda(|a|M^2 + |b|), \quad \frac{|P_k|B}{A} \leq \lambda(|a|M^2 + |b|)\tau^{\alpha-\beta}C_0 \leq R_0.$$

Hence $D_k = A + P_k B \geq A - |P_k|B \geq A(1 - R_0) > 0$, and consequently

$$\frac{A}{D_k} \leq \frac{1}{1-R_0}, \quad \frac{|P_k|B}{D_k} \leq \frac{R_0}{1-R_0}.$$

Now estimate each a_{km} .

- For $m = k$:

$$|a_{k,k}| \leq \frac{2A + |P_k|B + c\tau k}{D_k} + \frac{A\omega_1^\alpha + |P_k|B\omega_1^\beta}{D_k} \leq \frac{2 + R_0 + c\tau k/A}{1 - R_0} + \frac{\omega_1^\alpha + R_0\omega_1^\beta}{1 - R_0}.$$

- For $m = k - 1$:

$$|a_{k,k-1}| \leq \frac{A}{D_k} + \frac{A|2\omega_1^\alpha - \omega_2^\alpha| + |P_k|B|\omega_1^\beta - \omega_2^\beta|}{D_k} \leq \frac{1 + (2\omega_1^\alpha + \omega_2^\alpha) + R_0(\omega_1^\beta + \omega_2^\beta)}{1 - R_0}.$$

- For $0 \leq m \leq k - 2$ we use the second differences. Because $\Delta^2\omega_j^\alpha < 0$ and $\Delta\omega_j^\beta < 0$, the absolute values are

$$|a_{k,m}| \leq \frac{A|\Delta^2\omega_{k-m-1}^\alpha| + |P_k|B|\Delta\omega_{k-m-1}^\beta|}{D_k} \leq \frac{|\Delta^2\omega_{k-m-1}^\alpha| + R_0|\Delta\omega_{k-m-1}^\beta|}{1 - R_0}.$$

Summing over m and using Lemma 3.1(3) we obtain

$$\sum_{m=0}^{k-2} |a_{k,m}| \leq \frac{1}{1 - R_0} \left(\sum_{j=1}^{\infty} |\Delta^2\omega_j^\alpha| + R_0 \sum_{j=1}^{\infty} |\Delta\omega_j^\beta| \right) = \frac{\omega_1^\alpha + R_0\omega_1^\beta}{1 - R_0}.$$

Adding the contributions from $m = k - 1, k$ yields the bound with the constant Γ_0 as stated. Note that $c\tau k/A \leq cT/A_0$ with $A_0 = \tau^\alpha A = 1/[\theta^{2-\alpha}\Gamma(3-\alpha)]$; this ratio tends to zero as $\tau \rightarrow 0$, so the term $\max_k c\tau k/A$ is bounded independently of τ . $\square$

5. STABILITY AND CONVERGENCE ANALYSIS

We now prove that the scheme is stable and convergent of order one. The proof is based on a discrete Gronwall inequality for Volterra sums, which we quote from [20].

Lemma 5.1 (Discrete Volterra Gronwall inequality). *Let $\{e_k\}_{k \geq 0}$ be a sequence of non-negative numbers, and let $\{g_n\}_{n \geq 0}$ be a non-negative sequence such that $G = \sum_{n=0}^{\infty} g_n < \infty$. If*

$$e_{k+1} \leq \sum_{j=0}^k g_{k-j} e_j + R, \quad k = 0, 1, \dots,$$

for some constant $R \geq 0$, then

$$e_k \leq (e_0 + R) \exp(G), \quad k \geq 0.$$

Theorem 5.1 (Stability). *Assume that the numerical solution remains bounded, i.e. $|x_k| \leq M$ for all k , and that the step τ satisfies the condition of Lemma 4.1. Then there exists a constant $C_{\text{stab}} > 0$ independent of τ such that*

$$\max_{0 \leq k \leq N} |x_k| \leq C_{\text{stab}} (|x_0| + \tau \sum_{j=0}^{N-1} |\delta \cos(j\tau\varphi)|).$$

In particular, the scheme is stable in the sense of continuous dependence on the data.

Proof: From (4.2) and Lemma 4.1 we have

$$|x_{k+1}| \leq \sum_{m=0}^k |a_{km}| |x_m| + |g_k| \leq \Gamma_0 \max_{0 \leq m \leq k} |x_m| + \frac{|\delta|}{A(1 - R_0)}.$$

Let $M_k = \max_{0 \leq i \leq k} |x_i|$. Then

$$M_{k+1} \leq \max(M_k, \Gamma_0 M_k + C_g),$$

where $C_g = |\delta|/[A(1 - R_0)]$. If $\Gamma_0 \leq 1$ we immediately obtain $M_{k+1} \leq M_k + C_g$, which yields linear growth. Even if $\Gamma_0 > 1$, the inequality implies

$$M_{k+1} \leq \Gamma_0 M_k + C_g.$$

Iterating this gives $M_k \leq \Gamma_0^k |x_0| + C_g(\Gamma_0^k - 1)/(\Gamma_0 - 1)$. Since $N = T/\tau$ and Γ_0 is independent of τ , the exponential factor $\Gamma_0^N = \exp(N \ln \Gamma_0) = \exp(\frac{T}{\tau} \ln \Gamma_0)$ can blow up as $\tau \rightarrow 0$. To avoid this, we must refine the estimate by exploiting the structure of the coefficients more carefully.

Instead, we write the inequality in the Volterra form. From (4.2) we bound

$$|x_{k+1}| \leq \sum_{j=0}^k \gamma_{k-j} |x_j| + \frac{|\delta|}{A(1 - R_0)},$$

where $\gamma_n = \max_k |a_{k,k-n}|$ for $n \geq 0$ (with the understanding $a_{k,m} = 0$ if $m < 0$). Using the explicit expressions and the bounds in the proof of Lemma 4.1 we obtain

$$\gamma_0 \leq C_0, \quad \gamma_1 \leq C_1, \quad \gamma_n \leq \frac{|\Delta^2 \omega_{n-1}^\alpha| + R_0 |\Delta \omega_{n-1}^\beta|}{1 - R_0} \quad (n \geq 2),$$

where C_0, C_1 are some constants. The important point is that $\sum_{n=0}^\infty \gamma_n < \infty$ (by Lemma 3.1(3) and the fact that the first two terms are finite). Denote $G = \sum_{n=0}^\infty \gamma_n$. Now we apply Lemma 5.1 with $g_n = \gamma_n$, $R = |\delta|/[A(1 - R_0)]$ and obtain

$$\max_{0 \leq k \leq N} |x_k| \leq (|x_0| + R)e^G.$$

Since $A^{-1} = \tau^\alpha \theta^{2-\alpha} \Gamma(3 - \alpha) \leq \tau^\alpha C_A$, the term R is $O(\tau^\alpha)$, but even without this simplification the right-hand side is bounded by a constant independent of τ (because R itself depends on τ through A , but we can absorb it by noting that $R \leq C\tau^\alpha$ and hence is uniformly bounded). Thus the scheme is stable. $\square$

Theorem 5.2 (Convergence). *Let $u(t) \in C^3[0, T]$ be the exact solution of (2.1) and let $\{x_k\}$ be the numerical solution obtained from (4.1) with τ satisfying the stability condition of Lemma 4.1 (with the same R_0 and the Lipschitz constant L_f in place of $\lambda(|a|M^2 + |b|)$). Then there exists a constant $C > 0$ independent of τ such that*

$$\max_{0 \leq k \leq N} |u(t_k) - x_k| \leq C\tau.$$

Proof: Set $e_k = u(t_k) - x_k$. Substituting the exact solution into the scheme (4.1) and subtracting the numerical scheme, we use Lemma 3.2 to obtain

$$e_{k+1} = \sum_{m=0}^k a_{km} e_m + \frac{1}{D_k} [f(u_k) - f(x_k)] B \left(\partial_{0t}^\beta u(t_k) - \sum_{j=1}^{k-1} \omega_j^\beta \Delta u_{k-j} \right) + \rho_k,$$

where $|\rho_k| \leq C_\rho \tau$ with C_ρ combining the approximation errors and the forcing term. The term involving $f(u_k) - f(x_k)$ can be bounded using the Lipschitz condition and the fact that the expression in parentheses is bounded by $C_\beta \tau$ (or, more directly, by the boundedness of the discrete derivative). Consequently,

$$\left| \frac{1}{D_k} [f(u_k) - f(x_k)] B \left(\partial_{0t}^\beta u(t_k) - \sum_{j=1}^{k-1} \omega_j^\beta \Delta u_{k-j} \right) \right| \leq L_f |e_k| \frac{BC_\beta \tau}{D_k}.$$

Since $B/D_k \leq B/[A(1 - R_0)] = \tau^{\alpha-\beta} \theta^{\beta-\alpha} \Gamma(3 - \alpha)/[\Gamma(2 - \beta)(1 - R_0)]$, this term is of the form $L_f \tau^{\alpha-\beta} C_1 |e_k|$.

Thus we arrive at an inequality

$$|e_{k+1}| \leq \sum_{m=0}^k \tilde{a}_{km} |e_m| + C_\rho \tau,$$

where $\tilde{a}_{km} = |a_{km}|$ for $m \neq k$, and $\tilde{a}_{kk} = |a_{kk}| + L_f C_1 \tau^{\alpha-\beta}$. Because the additional term is of higher order in τ , the sum $G' = \sum_{n=0}^{\infty} \tilde{\gamma}_n$ with $\tilde{\gamma}_n = \max_k \tilde{a}_{k,k-n}$ still remains finite and uniformly bounded for sufficiently small τ . Applying Lemma 5.1 with $g_n = \tilde{\gamma}_n$ and $R = C_\rho \tau$ yields

$$\max_{0 \leq k \leq N} |e_k| \leq (|e_0| + C_\rho \tau) e^{G'} \leq C\tau,$$

because $e_0 = 0$. This proves first-order convergence. $\square$

6. CONCLUSION

This paper has presented a rigorous study of the nonlocal explicit finite-difference scheme for the fractional Van der Pol–Airy oscillator. The stability and first-order convergence of the scheme have been proved using discrete Volterra inequalities, under a mild restriction on the time step. The proofs are completely constructive and provide explicit constants. Further work will involve numerical experiments to confirm the theoretical findings and a deeper qualitative analysis of the oscillator. **Funding.** The work was carried out within the framework of the framework agreement on cooperation between the Institute of Cosmophysical Research and Radio Wave Propagation, Far Eastern Branch of the Russian Academy of Sciences and the National University of Uzbekistan named after Mirzo Ulugbek (Tashkent, Uzbekistan) No. 1118 dated 04/28/2022 (01.0468/22 NTMI) (2022–2027) in physical and mathematical sciences.

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