

Direct and inverse problems for Barenblatt-Zhel'tov-Kochina type fractional equations

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Abstract. In the present work, we consider a Cauchy problem for a nonhomogeneous fractional-order equation of the Barenblatt-Zhel'tov-Kochina type with the Caputo derivative. A solution to the problem is constructed by means of the Fourier method. The continuity of the obtained solution is justified using the convergence properties of functional series. In addition, the uniqueness of the solution is proved. Throughout the analysis, essential use is made of the properties of the Mittag-Leffler function. Moreover, the inverse problem of determining the source function is also investigated, and the existence and uniqueness of its solution are rigorously established.

Keywords: Cauchy problem, Caputo derivative, Mittag-Leffler function, direct problem, inverse problem

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1. INTRODUCTION AND PROBLEM STATEMENT

Let $A : H \rightarrow H$ be a positive self-adjoint unbounded operator on the separable Hilbert space H , possessing a complete set of orthonormal eigenfunctions $\{v_k\}$ with corresponding positive eigenvalues $\{\lambda_k\}$. By reindexing as needed, we may assume that the eigenvalues satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty.$$

Let $C((a, b); H)$ denote the set of continuous functions $u(t)$ on the interval $t \in (a, b)$ with values in H . Now, recall the concept of the degree of an operator A in a Hilbert space H . Let τ be an arbitrary real number. We introduce the degree of the operator A in H as follows:

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

where $h_k = (h, v_k)$ are the Fourier coefficients of the element $h \in H$.

Obviously, the domain of this operator has the form:

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

For elements $h, g \in D(A^\tau)$ we introduce the inner product and the norm as follows:

$$(h, g)_\tau = \sum_{k=1}^{\infty} \lambda_k^{2\tau} (h_k, g_k), \quad h, g \in H,$$

$$\|h\|_\tau^2 = \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 = \|A^\tau h\|^2.$$

Together with this inner product and norm $D(A^\tau)$ turns into a Hilbert space.

The Caputo fractional derivative of order $0 < \rho < 1$ is defined as follows (see, [1]):

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{h'(\xi)}{(t-\xi)^\rho} d\xi, \quad t > 0.$$

Consider the following problems:

Direct problem. Find a function $u(t)$ that satisfies the Cauchy problem for fractional equations of Barenblatt-Zhel'tov-Kochina type:

$$\begin{cases} D_t^\rho u(t) + A^a(D_t^\rho u(t)) + A^b u(t) = f(t), & 0 < t \leq T; \\ u(+0) = \varphi, \end{cases} \quad (1.1)$$

where $\varphi \in H$, $f(t) \in C([0; T], H)$, $0 < \rho < 1$ and $a, b > 0$.

Let $AC[0, T]$ be the set of absolutely continuous functions defined on $[0, T]$ and let $AC([0, T]; H)$ stand for the space of absolutely continuous functions $u(t)$ with values in H .

Definition 1.1. A function $u(t) \in AC([0, T]; H)$ with the properties $D_t^\rho u(t)$, $A^a(D_t^\rho u(t))$, $A^b u(t) \in C((0, T]; H)$ and satisfying conditions (1.1) is called the solution of the Cauchy problem (1.1).

In this paper, we study not only the direct problem but also the inverse problem. In particular, let us assume that in problem (1.1) the function $f(t)$ is unknown and independent of t . Under this assumption, we investigate the inverse problem of determining the unknown f , and for this purpose, we employ the following additional condition:

$$\int_0^T u(\tau) \omega(\tau) d\tau = \psi, \quad 0 < \tau \leq T, \quad (1.2)$$

where $\omega(t)$ and ψ are given functions, and their specific conditions will be provided later.

Inverse problem. Determine the pair of functions $\{u(t), f\}$ that satisfies problem (1.1) subject to the additional condition (1.2).

Definition 1.2. If the functions $u(t) \in AC([0, T]; H)$ and $f \in H$ have the properties $D_t^\rho u(t)$, $A^a(D_t^\rho u(t))$, $A^b u(t) \in C((0, T]; H)$ and satisfy conditions (1.1) and (1.2), then the pair of functions $\{u(t), f\}$ is called the solution of the inverse problem (1.1) and (1.2).

Let us consider the following equation:

$$\frac{\partial^k u}{\partial t^k} + \Delta \left(\frac{\partial^k u}{\partial t^k} \right) - \nu^2 \Delta u = 0, \quad (1.3)$$

where $\Delta = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator.

If $k = 1$, then this equation is called a partial differential equation of the Barenblatt-Zhel'tov-Kochina type; if $k = 2$, then it is called a partial differential equation of the Boussinesq type. Equations of the form (1.3) are encountered in modeling various processes. For example, equations of the Barenblatt-Zhel'tov-Kochina type are the basic equations for the filtration of homogeneous liquids. This was first noticed in the work of G. I. Barenblatt, Yu. P. Zhel'tov and I. N. Kochina [2]. The article [2] is devoted to the stabilization of solutions to the stochastic Barenblatt-Zhel'tov-Kochina equation. Later, such equations were studied in the papers [3, 4, 5, 6, 7]. Numerous other related works are discussed in the paper, and their results are compared with the corresponding theorems.

For example, in [7], the magneto-hydrodynamic equilibrium of magnetic stars with a poloidal-toroidal mixed magnetic field is analyzed using non-classical mathematical physics equations. The model is constructed and its consistency with observational data is examined. As a result, a general solution describing the equilibrium in the magnetosphere is obtained.

The present paper consists of four sections: in Section (2), the main properties of the Mittag-Leffler function required for the forward and inverse problems are presented. In Section (3), a complete proof of the existence and uniqueness of the solution to the forward problem is provided. Moreover, this section discusses research on problems similar to those studied in this work. Section (4) is devoted to the inverse problem, where the existence and uniqueness of its solution are established.

2. PRELIMINARIES

The Mittag-Leffler function $E_{\rho, \mu}(t)$ has the form

$$E_{\rho, \mu}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\rho n + \mu)},$$

where $\rho > 0$ and μ is a complex number.

We recall the following properties of the Mittag-Leffler function:

Lemma 2.1. *Let $0 < \rho < 1$, $\lambda > 0$. Then the following equality holds:*

$$D_t^\rho [t^\rho E_{\rho, \rho+1}(-\lambda t^\rho)] = E_{\rho, 1}(-\lambda t^\rho).$$

Lemma (2.1) has been proved in [8].

Lemma 2.2. *Let $0 < \rho < 1$, $\lambda > 0$. Then the following equality holds:*

$$D_t^\rho [E_{\rho, 1}(-\lambda t^\rho)] = -\lambda E_{\rho, 1}(-\lambda t^\rho).$$

Lemma (2.2) has been proved in [8].

Lemma 2.3. *Let $0 < \rho < 1$, $\lambda > 0$ and $\mu > 0$. Then the following equality holds:*

$$\int_0^t s^{\mu-1} E_{\rho, \mu}(-\lambda s^\rho) ds = t^\mu E_{\rho, \mu+1}(-\lambda t^\rho).$$

Lemma (2.3) has been proved in [9], p. 100.

We use the following estimates of the Mittag-Leffler function (see, [10] p. 134-136):

$$0 < E_{\rho, \mu}(-t) < 1, \quad t > 0, \quad (2.1)$$

$$0 < E_{\rho, \mu}(-t) \leq \frac{C}{1+t}, \quad t > 0, \quad (2.2)$$

$$E_{\rho, \mu}(-t) = \frac{1}{t} \left(1 + O\left(\frac{1}{t^2}\right) \right), \quad t > 1, \quad (2.3)$$

where $\rho > 0$, and μ is a complex number.

We also use an estimate with a positive number λ and $0 < \varepsilon < 1$ (see, e.g. [11]):

$$|t^{\rho-1} E_{\rho, \rho}(-\lambda t^\rho)| \leq \frac{C t^{\rho-1}}{1 + (\lambda t^\rho)^2} \leq C \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}, \quad t > 0. \quad (2.4)$$

3. CAUCHY PROBLEM

Theorem 3.1. *Let $0 < \rho < 1$. If 1) $b \leq a$, $\varphi \in D(A^b)$, $f \in C([0, T]; H)$ or 2) $b > a$, $\varphi \in D(A^a)$, $f \in C([0, T]; D(A^\varepsilon))$, then the problem (1.1) has a unique solution and this solution has the form:*

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k E_{\rho, 1}(-\mu_k t^\rho) + \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho, \rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right] v_k, \quad (3.1)$$

where φ_k are the Fourier coefficients of the function φ and $\mu_k = \frac{\lambda_k^b}{1 + \lambda_k^a}$.

We note that problems similar to (1.1), in the case $a = 1, b = 1$ were studied in [8]. In this work, if $\varphi \in D(A)$, $f \in H$, the existence and uniqueness of the solution are proved. When $a = 2, b = 1$, problem (1.1) is referred to as a Benney-Luke type problem. A problem of this type has been studied in the work [12]. In this work, the existence and uniqueness of the solution of the direct problem are proved for $\varphi \in D(A)$ and $f \in H$, while the existence and uniqueness of the inverse problem are shown for $\varphi, \psi \in D(A^2)$. In the case $a = 0$ and $b = 1$, the equation presented in problem (1.1) was investigated in [13] as a nonlocal problem. When $f = 0$, the problem (1.1) was studied in [14]. In the problem of the paper [14], if $c = \min(a, b)$ and $\varphi \in D(A^c)$, then the problem admits a unique solution.

We focus in particular on paper [15], which was carried out by M. Ruzhansky, D. Serikbaev, N. Tokmagambetov, and B. T. Torebek, as it is closely related to our work. In the paper [15], the following pseudo-parabolic equation is studied:

$$D_t^\rho [u(t) + \mathcal{L}u(t)] + \mathcal{M}u(t) = f(t), \quad 0 < t < T < \infty, \quad (3.2)$$

with the Cauchy condition

$$u(0) = \varphi \in H.$$

Here, $\mathcal{L}$ and $\mathcal{M}$ are self-adjoint operators on a separable Hilbert space H . In this work, the authors investigate the existence and uniqueness of solutions to the direct problem and show that the solution belongs to the following space:

$$u \in L_2([0, T]; D(\mathcal{M})) \cap W^\rho([0, T]; D(\mathcal{L})), \quad 0 < \rho < 1,$$

when $\varphi \in D(\mathcal{L}) \cap D(\mathcal{M})$, $f \in W^1([0, T]; H)$.

For $0 < \rho \leq 1$, the space $W^\rho([0, T]; X)$ consists of all L_2 -integrable functions $g : [0, T] \rightarrow X$ whose fractional derivative $D_t^\rho g$ is also L_2 -integrable.

Now we will compare the results obtained in [15] with the findings of our study, highlighting the similarities and differences between the two works. The operators $\mathcal{L}$ and $\mathcal{M}$ are used instead of the operators A^a and A^b appearing in equation (1.1). The difference between problem (1.1) and (3.2) is that the problem (1.1) is studied depending on the power of the operator A . In [15], the problem is studied in two cases. In the first case, for $1/2 < \rho < 1$, it is assumed that $\varphi \in D(\mathcal{L}) \cap D(\mathcal{M})$ and $f \in L_2([0, T]; H) \cap L_2([0, T]; H_{L, M}^{-1, 1})$. In the second case, for $0 < \rho < 1$, they require more conditions on the source term and $f \in W^1([0, T]; H)$. However, in our work, the problem is studied for $0 < \rho < 1$ without imposing any additional conditions.

Another distinction is that in [15], the variable t is taken into account only almost everywhere, which implies that $f(t) \in W^1([0, T]; H)$. In contrast, in the present work, t is considered at every point, ensuring the continuity of the function $f(t)$.

Next, we compare the inverse problem considered in our work with the inverse problem studied in [15]. In both works, the inverse problem is studied in order to determine the pair $\{u(t), f\}$. In [15], the additional condition $u(T) = \psi$ is imposed, whereas in the present paper the overdetermination condition is given in the form of (1.2). However, in both works it is assumed that $f \in H$ and $\varphi, \psi \in D(A^c)$, where $c = \max(a, b)$.

Now, we present the proof of Theorem (3.1):

Proof: Due to the completeness of the system $\{v_k\}$ in H , an arbitrary solution of (1.1) can be written in the form:

$$u(t) = \sum_{k=1}^{\infty} T_k(t) v_k. \quad (3.3)$$

Substituting into the first equation of (1.1) we have the following equation:

$$D_t^\rho T_k(t) (1 + \lambda_k^a) + T_k(t) \lambda_k^b = f_k(t),$$

Therefore,

$$D_t^\rho T_k(t) + \mu_k T_k(t) = \frac{f_k(t)}{1 + \lambda_k^a}, \quad \mu_k = \frac{\lambda_k^b}{1 + \lambda_k^a}.$$

Using the second condition of (1.1), we obtain the following problem:

$$\begin{cases} D_t^\rho T_k(t) + \mu_k T_k(t) = \frac{f_k(t)}{1 + \lambda_k^a}, \\ T_k(+0) = \varphi_k. \end{cases} \quad (3.4)$$

This problem has a unique solution (see, [16] p. 231):

$$T_k(t) = \varphi_k E_{\rho, 1}(-\mu_k t^\rho) + \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho, \rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta. \quad (3.5)$$

We transform (3.5) into equality (3.3). Then, we obtain the formal solution (3.1).

To prove uniqueness, it suffices to consider the homogeneous equation (1.1) with $f_k(t) = 0$ and $\varphi_k = 0$ for all k . Then from (3.5), it immediately follows that $T_k(t) \equiv 0$ for all $k \geq 1$. Taking into

account the definition of $T_k(t)$ and the completeness of the system $\{v_k\}$, we conclude that $u(t) \equiv 0$. This establishes the uniqueness of the solution.

It remains to prove that the constructed formal solution satisfies all the requirements of Definition (1.1). We will analyze the proof of the theorem separately for the cases $b \leq a$ and $b > a$.

1) Let $a \geq b$.

Let $S_n(t)$ denote the partial sums of (3.1):

$$S_n(t) = \sum_{k=1}^n \left[\varphi_k E_{\rho,1}(-\mu_k t^\rho) + \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right] v_k,$$

By applying the estimate (2.1) for the Mittag-Leffler function together with the Parseval equality, we obtain the estimate below:

$$\begin{aligned} \|S_n(t)\|^2 &= \sum_{k=1}^n \left| \varphi_k E_{\rho,1}(-\mu_k t^\rho) + \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right|^2 \\ &\leq 2 \sum_{k=1}^n |\varphi_k E_{\rho,1}(-\mu_k t^\rho)|^2 + 2 \sum_{k=1}^n \left| \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right|^2. \end{aligned}$$

Using the estimate (2.4), we have:

$$\begin{aligned} \|S_n(t)\|^2 &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + 2C \sum_{k=1}^n \left(\frac{1}{1 + \lambda_k^a} \int_0^t \mu_k^{\varepsilon-1} \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2 \\ &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + 2C \sum_{k=1}^n \frac{\lambda_k^{2b(\varepsilon-1)}}{(1 + \lambda_k^a)^{2\varepsilon}} \left(\int_0^t \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2 \\ &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + 2C \sum_{k=1}^n \lambda_k^{-2(a\varepsilon+b(1-\varepsilon))} \left(\int_0^t \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2. \end{aligned}$$

Applying the generalized Minkowski inequality, we obtain the following estimate:

$$\begin{aligned} \|S_n(t)\|^2 &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + \frac{2C}{\lambda_1^{2(a\varepsilon+b(1-\varepsilon))}} \left(\int_0^t \eta^{\varepsilon\rho-1} \left(\sum_{k=1}^n |f_k(t - \eta)|^2 \right)^{\frac{1}{2}} d\eta \right)^2 \\ &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + C_1 \max_{t \in [0, T]} \|f(t)\|^2. \end{aligned}$$

According to the above expression, if $\varphi \in H$, $f(t) \in C([0, T]; H)$ then $u(t) \in AC([0, T]; H)$.

Now let us estimate $A^b u(t)$. Using the Parseval equality, the estimate is as follows:

$$\begin{aligned} \|A^b S_n(t)\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k E_{\rho,1}(-\mu_k t^\rho)|^2 \\ &\quad + 2 \sum_{k=1}^n \lambda_k^{2b} \left| \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right|^2. \end{aligned}$$

Next, similarly to the above, using the properties of the Mittag-Leffler function and the generalized Minkowski inequality, we obtain the estimate below:

$$\|A^b S_n(t)\|^2 \leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + \frac{2C}{\lambda_1^{2\varepsilon(a-b)}} \left(\int_0^t \eta^{\varepsilon\rho-1} \left(\sum_{k=1}^n |f_k(t - \eta)|^2 \right)^{1/2} d\eta \right)^2$$

$$\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + C_1 \max_{t \in [0, T]} \|f(t)\|^2.$$

Thus, if $\varphi \in D(A^b)$ and $f(t) \in C([0, T]; H)$, then $A^b u(t) \in C([0, T]; H)$.

Next, we estimate the term $A^a(D_t^\rho u(t))$. To show this, we first carry out the necessary calculations:

$$\begin{aligned} G(t) &= \int_0^t \eta^{\rho-1} E_{\rho, \rho}(-\mu_k \eta^\rho) f_k(t-\eta) d\eta = \int_0^t (t-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k (t-\eta)^\rho) f_k(\eta) d\eta \\ &= -f_k(\eta) (t-\eta)^\rho E_{\rho, \rho+1}(-\mu_k (t-\eta)^\rho) \Big|_0^t + \int_0^t (t-\eta)^\rho E_{\rho, \rho+1}(-\mu_k (t-\eta)^\rho) f'_k(\eta) d\eta \\ &= f_k(0) t^\rho E_{\rho, \rho+1}(-\mu_k t^\rho) + \int_0^t (t-\eta)^\rho E_{\rho, \rho+1}(-\mu_k (t-\eta)^\rho) f'_k(\eta) d\eta = g_1(t) + g_2(t). \end{aligned}$$

By using Lemma (2.1), we obtain the following expression:

$$D_t^\rho g_1(t) = D_t^\rho (f_k(0) t^\rho E_{\rho, \rho+1}(-\mu_k t^\rho)) = f_k(0) E_{\rho, 1}(-\mu_k t^\rho),$$

and

$$\begin{aligned} D_t^\rho g_2(t) &= \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{1}{(t-s)^\rho} \int_0^s (s-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k (s-\eta)^\rho) f'_k(\eta) d\eta ds \\ &= \frac{1}{\Gamma(1-\rho)} \int_0^t f'_k(\eta) d\eta \int_\eta^t \frac{(s-\eta)^{\rho-1}}{(t-s)^\rho} E_{\rho, \rho}(-\mu_k (s-\eta)^\rho) ds \\ &= \frac{1}{\Gamma(1-\rho)} \int_0^t f'_k(\eta) \left(\sum_{k=0}^{\infty} \frac{(-\mu_k)^k}{\Gamma(\rho k + \rho)} \int_\eta^t \frac{(s-\eta)^{\rho k + \rho - 1}}{(t-s)^\rho} ds \right) d\eta \\ &= \frac{1}{\Gamma(1-\rho)} \int_0^t f'_k(\eta) \left(\sum_{k=0}^{\infty} \frac{(-\mu_k)^k}{\Gamma(\rho k + \rho)} (t-\eta)^{\rho k} B(\rho k + \rho, 1-\rho) \right) d\eta \\ &= \int_0^t f'_k(\eta) E_{\rho, 1}(-\mu_k (t-\eta)^\rho) d\eta = f_k(\eta) E_{\rho, 1}(-\mu_k (t-\eta)^\rho) \Big|_0^t - \int_0^t f_k(\eta) (t-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k (t-\eta)^\rho) d\eta \\ &= f_k(t) - f_k(0) E_{\rho, 1}(-\mu_k t^\rho) - \int_0^t f_k(\eta) (t-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k (t-\eta)^\rho) d\eta. \end{aligned}$$

As a result, we obtain the resulting equality:

$$D_t^\rho G(t) = f_k(t) - \int_0^t f_k(\eta) (t-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k (t-\eta)^\rho) d\eta.$$

To this end, we apply Lemma (2.2), which allows us to derive the required bound:

$$\begin{aligned} \|A^a(D_t^\rho S_n(t))\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + 2 \sum_{k=1}^n \left| D_t^\rho \left(\int_0^t \eta^{\rho-1} E_{\rho, \rho}(-\mu_k \eta^\rho) f_k(t-\eta) d\eta \right) \right|^2 \\ &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + 4 \sum_{k=1}^n |f_k(t)|^2 + C \max_{t \in [0, T]} \|f(t)\|^2 \\ &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + 4 \|f(t)\|^2 + C \max_{t \in [0, T]} \|f(t)\|^2. \end{aligned}$$

From this, if $\varphi \in D(A^b)$ and $f(t) \in C([0, T]; H)$, we obtain $A^a(D_t^\rho u(t)) \in C([0, T]; H)$.

Estimating $D_t^\rho u(t)$ in the same way as above, we obtain the estimate below:

$$\|D_t^\rho S_n(t)\|^2 \leq 2 \sum_{k=1}^n |\varphi_k|^2 + C_1 \|f(t)\|^2 + C_2 \max_{t \in [0, T]} \|f(t)\|^2.$$

From the above expression, for $\varphi \in H$ and $f(t) \in C([0, T]; H)$, $D_t^\rho u(t)$ belongs to $C([0, T]; H)$.

2) Let $a < b$.

Next, an estimate of $A^b u(t)$ is obtained by applying the Parseval equality:

$$\begin{aligned} \|A^b S_n(t)\|^2 &= 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k E_{\rho,1}(-\mu_k t^\rho)|^2 \\ &+ 2 \sum_{k=1}^n \lambda_k^{2b} \left| \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right|^2. \end{aligned}$$

Then, using (2.2), (2.4), and the generalized Minkowski inequality, we obtain the following estimate:

$$\begin{aligned} \|A^b S_n(t)\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2b} \left| \frac{C}{1 + \mu_k t^\rho} \right|^2 |\varphi_k|^2 + 2 \sum_{k=1}^n \lambda_k^{2b} \left(\frac{1}{1 + \lambda_k^a} \int_0^t C \mu_k^{\varepsilon-1} \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2 \\ &\leq \frac{C}{t^{2\rho}} \sum_{k=1}^n \lambda_k^{2a} |\varphi_k|^2 + 2C \sum_{k=1}^n \lambda_k^{2\varepsilon(b-a)} \left(\int_0^t \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2 \\ &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + 2C_1 \left(\int_0^t \eta^{\varepsilon\rho-1} \left(\sum_{k=1}^n \lambda_k^{2\varepsilon(b-a)} |f_k(t - \eta)|^2 \right)^{1/2} d\eta \right)^2 \\ &\leq \frac{C}{t^{2\rho}} \sum_{k=1}^n \lambda_k^{2a} |\varphi_k|^2 + C_2 \max_{t \in [0, T]} \|f(t)\|_{\varepsilon(b-a)}^2, \quad t > 0. \end{aligned}$$

It follows that, if $\varphi \in D(A^a)$, $f(t) \in C([0, T]; D(A^{\varepsilon(b-a)}))$, the term $A^b u(t)$ belongs to the space $C((0, T]; H)$.

The term $A^a(D_t^\rho u(t))$ is next estimated using Lemma (2.2) and the Parseval equality:

$$\begin{aligned} \|A^a(D_t^\rho S_n(t))\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2a} |-\mu_k \varphi_k E_{\rho,1}(-\mu_k t^\rho)|^2 + 2 \sum_{k=1}^n \left| D_t^\rho \left(\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right) \right|^2. \end{aligned}$$

Using the above properties of the Mittag-Leffler function and the generalized Minkowski inequality, we obtain an estimate of the form:

$$\begin{aligned} \|A^a(D_t^\rho S_n(t))\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2a} \left| \mu_k \varphi_k \frac{C_1}{1 + \mu_k t^\rho} \right|^2 + 4 \sum_{k=1}^n |f_k(t)|^2 + 4C \max_{t \in [0, T]} \|f(t)\|^2 \\ &\leq \frac{C_1}{t^{2\rho}} \sum_{k=1}^n \lambda_k^{2a} |\varphi_k|^2 + 4\|f(t)\|^2 + C_2 \max_{t \in [0, T]} \|f(t)\|^2, \quad t > 0. \end{aligned}$$

Based on the above result, assuming that $\varphi \in D(A^a)$ and $f(t) \in C([0, T]; H)$, the term $A^a(D_t^\rho u(t))$ belongs to the space $C((0, T]; H)$.

Estimating $D_t^\rho u(t)$ in the same way as above, we obtain the following estimate:

$$\|D_t^\rho S_n(t)\|^2 \leq \frac{C_1}{t^{2\rho}} \sum_{k=1}^n |\varphi_k|^2 + C_2 \|f(t)\|^2 + C_3 \max_{t \in [0, T]} \|f(t)\|^2, \quad t > 0.$$

Assuming that $\varphi \in H$, $f(t) \in C([0, T]; H)$, it follows that $D_t^\rho u(t)$ is an element of the space $C((0, T]; H)$.

Thus Theorem (3.1) is fully proved. $\square$

4. INVERSE PROBLEM

Theorem 4.1. *Let $0 < \rho < 1$. If $c = \max(a, b)$, $\varphi, \psi \in D(A^c)$, and the function $\omega(t)$ is sign-preserving on the interval $(0, T]$, then the inverse problem (1.1)–(1.2) admits a unique solution $\{u(t), f\}$, and this solution has the form:*

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k E_{\rho,1}(-\mu_k t^\rho) + \frac{f_k}{1 + \lambda_k^a} t^\rho E_{\rho,\rho+1}(-\mu_k t^\rho) \right] v_k \quad (4.1)$$

and

$$f = \sum_{k=1}^{\infty} f_k v_k, \quad (4.2)$$

where

$$f_k = \left(\frac{\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau}{\int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau} \right) (1 + \lambda_k^a). \quad (4.3)$$

Proof: Existence of the solution.

By (1.2) we have the following equality:

$$\sum_{k=1}^{\infty} \left[\int_0^T \varphi_k \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau + \frac{f_k}{1 + \lambda_k^a} \int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau \right] v_k = \psi$$

If the ψ function is expanded into a series with respect to $\{v_k\}$, then we obtain the following equality:

$$\begin{aligned} \int_0^T \varphi_k \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau + \frac{f_k}{1 + \lambda_k^a} \int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau &= \psi_k, \\ \sigma_k(T) \cdot f_k &= \left(\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau \right) (1 + \lambda_k^a), \end{aligned}$$

where

$$\sigma_k(T) = \int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau.$$

According to the theorem, if the function $\omega(t)$ is sign-preserving, then $\sigma_k(T) \neq 0$ for all $k \in \mathbb{N}$. To prove this, we derive a lower bound for the function $\sigma_k(T)$ by using lemma (2.3):

$$\begin{aligned} |\sigma_k(T)| &= \left| \int_0^T \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) \omega(\tau) d\tau \right| \geq \min_{\tau \in (0,T]} |\omega(\tau)| \int_0^T \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau \\ &= \min_{\tau \in (0,T]} |\omega(\tau)| \int_0^T \frac{1}{\mu_k} (1 - E_{\rho,1}(-\mu_k \tau^\rho)) d\tau = \min_{\tau \in (0,T]} |\omega(\tau)| \cdot \frac{T}{\mu_k} (1 - E_{\rho,2}(-\mu_k T^\rho)). \end{aligned}$$

We consider the following two cases:

1) Let $a \geq b$. In this case, $\mu_k \rightarrow 0$ and we use the following estimate for the Mittag-Leffler function:

$$E_{\rho,2}(-\mu_k T^\rho) = \sum_{k=0}^{\infty} \frac{(-\mu_k T^\rho)^k}{\Gamma(\rho k + 2)} = 1 - \frac{\mu_k T^\rho}{\Gamma(\rho + 2)} + O(\mu_k^2 T^{2\rho}).$$

Then we obtain the following expression:

$$|\sigma_k(T)| \geq \min_{\tau \in (0,T]} |\omega(\tau)| \cdot \frac{T}{\mu_k} \cdot \frac{\mu_k T^\rho}{\Gamma(\rho + 2)} = C T^{\rho+1}. \quad (4.4)$$

2) Let $a < b$. In this case, $\mu_k \rightarrow \infty$ and we obtain the following lower bound for $\sigma_k(T)$ by applying (2.3):

$$|\sigma_k(T)| \geq \frac{C_0}{\mu_k} \quad (4.5)$$

It should be emphasized that if $\omega_k(t) = 0$ or changes its sign on the interval $(0, T)$, then $\sigma_k(T)$ may be equal to zero. In this case, the problem (1.1)–(1.2) may have infinitely many solutions or may not have a solution.

It remains to prove that the constructed formal solution satisfies all the requirements of Definition 1.2. We will analyze the proof of the theorem separately for the cases $a \geq b$ and $a < b$.

1) Let $a \geq b$.

Let F_n denote the partial sums of (4.2):

$$F_n = \sum_{k=1}^n \left[\left(\frac{\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau}{\int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau} \right) (1 + \lambda_k^a) \right] v_k. \quad (4.6)$$

Then by applying the Parseval equality and (4.4), we derive the following estimate:

$$\begin{aligned} \|F_n\|^2 &= \sum_{k=1}^n \left| \frac{\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau}{\sigma_k(T)} \right|^2 |1 + \lambda_k^a|^2 \\ &\leq 2 \sum_{k=1}^n \left| \frac{\psi_k}{CT^{\rho+1}} \right|^2 |1 + \lambda_k^a|^2 + 2 \sum_{k=1}^n \left| \frac{\varphi_k T \max_{\tau \in (0,T]} \omega(\tau)}{CT^{\rho+1}} \right|^2 |1 + \lambda_k^a|^2 \\ &= \frac{C_1}{T^{2\rho+2}} \sum_{k=1}^n |\psi_k|^2 |1 + \lambda_k^a|^2 + \frac{C_2}{T^{2\rho}} \sum_{k=1}^n |\varphi_k|^2 |1 + \lambda_k^a|^2. \end{aligned}$$

Thus if $\varphi, \psi \in D(A^a)$, series (4.2) is convergent.

2) Let $a < b$. Then the following estimate of (4.6) is obtained by applying (4.5) together with the Parseval equality:

$$\begin{aligned} \|F_n\|^2 &= \sum_{k=1}^n \left| \frac{\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau}{\frac{C_0}{\mu_k}} \right|^2 |1 + \lambda_k^a|^2 \\ &\leq 2 \sum_{k=1}^n \left| \frac{\mu_k \psi_k}{C_0} \right|^2 |1 + \lambda_k^a|^2 + 2 \sum_{k=1}^n \left| \frac{\mu_k \varphi_k T \max_{\tau \in (0,T]} \omega(\tau)}{C_0} \right|^2 |1 + \lambda_k^a|^2 \\ &= C_1 \sum_{k=1}^n \lambda_k^{2b} |\psi_k|^2 + C_2 T \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2. \end{aligned}$$

Thus if $\varphi, \psi \in D(A^b)$, series (4.2) is convergent.

Uniqueness. Suppose by contradiction that we have two solutions: $\{u_1(t), f_1\}$ and $\{u_2(t), f_2\}$. It is required to prove $u(t) = u_1(t) - u_2(t) = 0$ and $f = f_1 - f_2 = 0$. Since the problem is linear, to determine $u(t)$ and f we have the problem:

$$D_t^\rho u(t) + A^a(D_t^\rho u(t)) + A^b u(t) = f, \quad (4.7)$$

$$u(+0) = 0,$$

$$\int_0^T u(\tau) \omega(\tau) d\tau = 0 \quad (4.8)$$

Let $u(t)$ be a solution to this problem and $u_k(t) = (u(t), v_k)$. From equation (4.7), taking into account that the operator A is self-adjoint and $\{v_k\}$ is a system of eigenfunctions, we obtain

$$\sum_{k=1}^{\infty} \left[D_t^\rho u_k(t) + \lambda_k^a D_t^\rho u_k(t) + \lambda_k^b u_k(t) \right] v_k = f.$$

Using condition (4.7), we arrive at the following Cauchy problem:

$$\begin{cases} D_t^\rho u_k(t) + \frac{\lambda_k^b}{1 + \lambda_k^a} u_k(t) = \frac{1}{1 + \lambda_k^a} f_k, \\ u_k(0+) = 0. \end{cases}$$

The solution of this problem has the following form (see [16]):

$$u_k(t) = \frac{f_k}{1 + \lambda_k^a} t^\rho E_{\rho, \rho+1}(-\mu_k t^\rho).$$

Using the additional condition (4.8), we obtain

$$\int_0^T u(\tau) \omega(\tau) d\tau = \frac{f_k}{1 + \lambda_k^a} \int_0^T \tau^\rho E_{\rho, \rho+1}(-\mu_k \tau^\rho) \omega(\tau) d\tau = 0.$$

Since

$$\tau^\rho E_{\rho, \rho+1}(-\mu_k \tau^\rho) \omega(\tau) \neq 0 \quad \text{for } \tau > 0, \quad 0 < \rho < 1,$$

it follows that $f_k = 0$ for all $k \geq 1$. Since $\{v_k\}$ forms a complete orthonormal system in H , we obtain $f = 0$. From $f_k = 0$ it follows that $u_k(t) = 0$ for all $k \geq 1$. Again, using the completeness of $\{v_k\}$ in H , we conclude that $u(t) \equiv 0$.

Theorem (4.1) is completely proved. $\square$

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