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On the accuracy of approximate solution scheme based on the Taylor formula for a Cauchy problem for ODEs

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Abstract. In this work, we obtain explicit estimates for the accuracy of an approximation solution scheme for ODEs. The scheme is of the fourth order and based on the Taylor formula. Here we derive estimates with explicit constants, which are expressed by the derivatives of the right-hand side.

Keywords: Dynamical system, Cauchy problem, approximate solution, Taylor formula, Gronwall's lemma, solution accuracy, accuracy level, error estimation
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1. Introduction

In many applications of mathematics, we face the problem of approximately solving the Cauchy problem

$$\dot{x}(t) = f[x(t)], \qquad x(0) = x_0 \tag{1.1}$$

Problem (1.1) is usually given for $x \in D \subset \mathbb{R}^d$, D where is a convex domain, $f : D \to \mathbb{R}^d$ is assumed to be smooth. In this work, we assume it is four times continuously differentiable. The effectiveness of an approximate solution scheme is defined by its accuracy. If $x_*(t)$ is the solution of problem (1.1) and $\tilde{x}(t)$ is the approximate solution in $(0 \le t \le T)$, then the level of accuracy is defined by

$$\sup_{0\le t\le T} |x_*(t) - \tilde{x}(t)| \le Ce^{LT}h^s \tag{1.2}$$

where L is the Lipschitz constant of f, h is the mesh size, s is the accuracy of the approximate solution scheme. There is extensive literature devoted to this problem. See for example [1, 2, 3, 4, 5]. The most common one step approximate solution schemes can be divided into two groups.

1. Approximate solution schemes based on the Taylor formula,
2. Runge-Kutta method schemes.

Generally, the Runge-Kutta method is preferred to the Taylor formula method [2]. This conclusion is explained by the necessity to calculate the derivatives of the function $f(x)$ in the approximate solution method based on the Taylor formula. In this article, we consider that approximate solution schemes based on the Taylor has a valuable property: it is possible to obtain accuracy of the method explicitly in terms of the derivatives of the right hand side of the system. Such schemes also have their positive aspects. Today, due to the development of computer algebra, it has become possible to automatically calculate the derivatives of a broad class of functions $f(x)$ [6, 7, 8].

There is another important reason for a deeper study of the Taylor method. Since the 1960s, numerical methods have been applied not only to approximate solutions of dynamic systems but also to study their properties, specifically in proving theorems that represent fixed properties [9, 10]. This requires a deductive proof of inequalities of the form (1.2), as is accepted in mathematics. From this perspective, approximate solution methods based on the Taylor formula are significantly more convenient than Runge-Kutta methods. [11] established such inequality for a scheme with accuracy level 3 in their article. In this article, for problem (1.1) with $s = 4$, the inequality (1.2) is proved, and a fixed expression is found for the corresponding coefficient C.

The issue of assessing the accuracy of approximate solutions is covered in the literature dedicated to numerical methods with varying degrees of detail. Here, we highlight several prominent monographs. In [12] the error assessment issue is not addressed. In the monographs, [5, 13, 14], and [4] only Runge-Kutta methods are examined. Part III of Bakhvalov's fundamental monograph [1] is devoted to approximate solution methods for ordinary differential equations. Specifically, in the first section of this part, the Taylor method is discussed from a methodological point of view. The monograph [15] mentions the Taylor method in the second section of the second part, but the problem of estimating the error terms is not considered. It is only noted that if the first n terms of the Taylor formula are taken when constructing an approximate solution scheme, the error will be

$$\max_{0\leq n\leq N} |x_n - y(x_n)| = \mathrm{O}(h^n).$$

In this article, as noted above, an exact upper bound for the error is found and a strict proof of inequality (1.2) with explicit constants is provided. Such estimates are of vital importance when using numerical solutions to prove mathematical theorems. Notice that, in the last three decades computational methods being applied to obtain rigorous results in dynamical systems. Some of them have already solved long standing open problems, like the celebrated work of Tucker [16, 17], where numerical analysis combined with analytical to obtain the existence of Lorenz attractors. Many other methods have been devised to obtain various other properties, like the existence of homoclic trajectories; see, for example, [18, 19, 20] and references threin.

Here we set up a 4th order explicit numerical scheme, based on the Taylor formula and obtain explicit bounds for the its accuracy. Notice that our constant in (1.2) can be explicitly computed in terms of the right hand side of the equation.

2. Approximate solution scheme based on the Taylor formula

For a continuous function $f(x)$, a solution to problem (1.1) exists and is unique, but it is impossible to determine its domain of definition [21]. With this in mind, we begin the discussion with the following assumption.

Assumption 1. The solution $x_*(t)$ of the Cauchy problem (1.1) exists on the interval $[0;T]$.

In numerical integration, the values of the solution $x_*(t)$ are usually sought on the grid $\{0 = t_0 < t_1 < t_2 < ... < t_n = T\}$. To focus on the essence of the analysis in this article, only the evenly spaced grid $t_n = nh$ will be considered, where $h = \frac{T}{N}$, $\quad N \in \mathbb{R}^+$.

To construct an approximate solution scheme, we take the Taylor series for the exact solution up to an accuracy of $\mathrm{O}(h^4)$.

$$x_*(t+nh) = x_*(t) + h\dot{x}_*(t) + \frac{h^2}{2}\ddot{x}_*(t) + \frac{h^3}{6}{}_*(t) + \frac{h^4}{24}x^{(IV)}{}_*(t) + R_5(t,h), \tag{2.1}$$

where

$$x_*(t) = x_0, \quad \dot{x}_*(t) = f[x_*(t)], \quad \ddot{x}_*(t) = f'[x_*(t)]f[x_*(t)],$$
$$\dddot{x}_*(t) = f''[x_*(t)]f[x_*(t)]f[x_*(t)] + f'[x_*(t)]f'[x_*(t)]f[x_*(t)].$$

The formulation of the main result and its proof is lengthy and involves extensive calculations. To overcome this inconvenient situation, a series of special notations is used in the article. One of these is the interpretation of the function f and its derivatives as operators: $\dot{x}_*(t) = f[x_*(t)]$, $\ddot{x}_*(t) = (f'f)[x_*(t)]$, $\dddot{x}_*(t) = (f''ff + f'f'f)[x_*(t)]$ and so on.

In order shorten the expressions further we introduce the following notation

$$F^0 = f, \;\; F^1 = f'f, \;\; F^2 = f''ff + f'f'f,$$
$$F^3 = f'''fff + 2f''f'ff + f''ff'f + f'f''ff + f'f'f'f. \tag{2.2}$$

Using the above notation allows us to obtain a compact formula for the derivatives:

$$x^{(k)}(t) = (F^k)[x_*(t)], \text{ for } (k = \overline{0,3}).$$

Therefore, the Taylor expansion of the exact solution have the form

$$x_*[(n+1)h] = x_*(nh) + \sum_{k=1}^{4} \frac{h^k}{k!} F^{k-1}[x_*(nh)] + R_5.$$

If we omit the remainder term in the above formula we obtain the following recurrent formula

$$x_{n+1} = x_n + \sum_{k=1}^{4} \frac{h^k}{k!} F^{k-1}(x_n). \tag{2.3}$$

It is intuitively clear that x_n approximates the value of $x_*(nh)$.

Naturally, in general, x_n does not exactly match $x_*(nh)$, and the issue arises of estimating the difference $|x_*(nh) - x_n|$. For practical applications, an estimate of the form (1.2) is sufficient. However, in theoretical problems, it is also necessary to consider the values of the solution $x_*(t)$ between the node points. Clearly, for $t \in [nh, (n+1)h]$ we can use the inequality

$$|x_*(t) - x_n| \le h \max_{nh \le t \le (n+1)h} |f(x_*(t))|.$$

Since, the order of this approximation is h obtaining higher order estimates for the difference $|x_*(nh) - x_n|$ is meaningless. In order to obtain higher order estimates for $|x_*(t) - \tilde{x}(t)|$ we use the following approximate solution

$$\tilde{x}(t) = x_0 + \int_0^t \sum_{k=0}^{3} \frac{(s-\sigma_s)^k}{k!} F^k[\tilde{x}(\sigma_s)]ds \tag{2.4}$$

where $\sigma_t = h[t/h]$ is the step function for the mesh under consideration.

Lemma 2.1. *For every $n \ge 1$ the following equality holds $\tilde{x}(nh) = x_n$.*

It is easily proved by induction on the intervals $[0,\ nh]$.

Assumption 2. Suppose that there exists $K \subset \mathbb{R}^d$ a compact and convex domain such that in the interval $[0;T]$ the solution $x_*(t)$ exists and $\tilde{x}(t) \in K$ for all $t \in [0;T]$.

Introduce the following constants:

$$\begin{gathered} M_0 = \max_{x \in K} |f(x)|, \ \ M_1 = \max_{x \in K} \|f'(x)\|, \ \ M_2 = \max_{x \in K} \|f''(x)\| \\ M_3 = \max_{x \in K} \|f'''(x)\|, \quad M_4 = \max_{x \in K} \|f^{\mathrm{iv}}(x)\|. \end{gathered} \tag{2.5}$$

Below the arguments of the functions $f^{(k)}$, $k = 0,1,2,3$ will be either $\tilde{x}(t)$, or $\tilde{x}(\sigma_t)$. In the first case we write the argument completely, while in the second case we shorten it to $f_t^{(k)} = f^{(k)}[x(\sigma_t)]$. Likewise, $F^k[x(\sigma_t)] = F_t^k$, $k = 0,1,2,3$. Further, we introduce the following constants:

$$l_0 = M_0, \ \ l_1 = M_1 M_0, \ \ l_2 = M_2 {M_0}^2 + {M_1}^2 M_0, \ \ l_3 = M_3 {M_0}^3 + 4 M_2 M_1 {M_0}^2 + {M_1}^3 M_0$$

and

$$\begin{aligned}
L_3 &= \frac{1}{24} M_1 l_3, \\
L_{2,3} &= \frac{1}{24} M_0 M_2 l_2 + \frac{1}{120} M_0 M_2 l_3 h, \\
L_{1,2,3} &= \frac{1}{8} M_2 {l_1}^2 + \frac{1}{30} M_2 l_1 l_2 h + \frac{1}{144} M_2 l_1 l_3 h^2, \\
{H^1}_{1,2,3} &= \frac{1}{24} {M_0}^2 M_3 l_1 + \frac{1}{120} {M_0}^2 M_3 l_2 h + \frac{1}{720} {M_0}^2 M_3 l_3 h^2, \\
{H^1}_{0,1,2,3} &= \frac{1}{8} {M_2}^2 {M_0}^3 + \frac{1}{20} {M_0}^2 {M_2}^2 l_1 h + \frac{1}{72} {M_0}^2 {M_2}^2 l_2 h^2 + \frac{1}{336} {M_0}^2 {M_2}^2 l_3 h^3, \\
{H^2}_{0,1,2,3} &= \frac{1}{8} M_2 {l_1}^2 + \frac{1}{20} M_1 M_2 {l_1}^2 h + \frac{1}{72} M_1 M_2 l_1 l_2 h^2 + \frac{1}{336} M_1 M_2 l_1 l_3 h^3, \\
{H^3}_{0,1,2,3} &= \frac{1}{12} M_3 {M_0}^2 l_1 + \frac{1}{40} M_0 M_3 {l_1}^2 h + \frac{1}{90} M_0 M_3 l_1 l_2 h^2 + \frac{1}{1008} M_0 M_3 l_1 l_3 h^3, \\
{H^4}_{0,1,2,3} &= \frac{1}{8} M_3 {M_0}^2 l_1 + \frac{1}{30} M_0 M_3 {l_1}^2 h + \frac{1}{144} M_0 M_3 l_1 l_2 h^2 + \frac{1}{840} M_0 M_3 l_1 l_3 h^3, \\
{H^5}_{0,1,2,3} &= \frac{1}{4!} {M_0}^4 M_4 + \frac{1}{5!} {M_0}^3 M_4 l_1 h + \frac{1}{6!} {M_0}^3 M_4 l_2 h^2 + \frac{1}{7!} {M_0}^3 M_4 l_3 h^3.
\end{aligned} \tag{2.6}$$

Observe the following inequality

$$\max_{x\in K}\left|F^k(x)\right| \le L_k, \quad k=0,1,2,3.$$

3. The Main Result

To state the main results we need to compute the derivatives of the approximate solution. By the virtue of (2.4) for every $t \neq nh$ and $(k = 1, 2, ..., N-1)$ we have

$$\dot{\tilde{x}}(t) = \sum_{k=0}^{3} \frac{(t-\sigma_t)^k}{k!} {F_t}^k. \tag{3.1}$$

Taking into account $0 \le t - \sigma_t \le h$ we obtain the estimate

$$\left|\dot{\tilde{x}}(t)\right| \le M_0 + l_1 h + l_2\frac{h^2}{2} + l_3\frac{h^3}{6}.$$

Theorem 3.1. *Let $x_*(\cdot)$ be the solution of Cauchy problem* (1.1) *and $\tilde{x}(\cdot)$ be its approximate solution given by* (2.4) *defined on an interval $[0,T]$. The following inequality holds*

$$|x_*(t) - \tilde{x}(t)| \le Ch^4\frac{e^{M_1T}-1}{M_1},$$

where

$$C = L_3 + L_{2,3} + L_{1,2,3} + {H^1}_{1,2,3} + {H^2}_{0,1,2,3} + {H^3}_{0,1,2,3} + {H^4}_{0,1,2,3} + {H^5}_{0,1,2,3}.$$

Proof: For the exact solution we have $x_*(t) = x_0 + \int\limits_0^t f\left[x_*(s)\right]ds$. Let

$$I(s) = f[\tilde{x}(s)] - \sum_{k=0}^{3} \frac{(s-\sigma_s)^k}{k!} {F_s}^k. \tag{3.2}$$

Using (2.4) and the triangle inequality we obtain

$$|x_*(t) - \tilde{x}(t)| \le \int_0^t |I(s)|ds + \int_0^t |f[x_*(s)] - f[\tilde{x}(s)]|ds. \tag{3.3}$$

We start with estimating the $I(s)$ the aim is to obtain an inequality of the form

$$|I(s)| \le Ch^4.$$

Grouping $f[\tilde{x}(s)]$ with the term $F_S^0 = f_s = f[\tilde{x}(\sigma_s)]$, which corresponds to $k=0$ in the right hand side of (3.3) and applying the fundamental rule of calculus implies

$$I(s) = \int_{\sigma_s}^{s} \frac{df(\tilde{x}(r))}{dr}dr - \sum_{k=1}^{3} \frac{(s-\sigma_s)^k}{k!} {F_s}^k.$$

Notice that s belongs to $[0,t]$. From now on we fix s. Below all the new variables take values from the interval (σ_s, s). In particular, $\sigma_s \le r < s$ in the expression of $I(s)$, whence, $\sigma_s = \sigma_r$. It implies that $f_s = f_r$. This applies to the variable t as well. Hence, $\sigma_t = \sigma_s$, $f_t = f_s$.

Further, notice that equality (3.1) implies that

$$\dot{\tilde{x}}(r) = \sum_{k=0}^{3} \frac{(r-\sigma_r)^k}{k!} {F_r}^k.$$

In the above equality we denote the term corresponding to $k = 3$ can be written in the form $B_3 = \int\limits_{\sigma_s}^{s} f'[\tilde{x}(r)] \frac{(r-\sigma_r)^3}{3!} {F_r}^3 dr$. Taking into account these last to equalities $I(s)$ can be rewritten as

$$I(s) = \int\limits_{\sigma_s}^{s} f'[\tilde{x}(r)] \sum_{k=0}^{2} \frac{(r-\sigma_r)^k}{k!} {F_r}^k dr - \sum_{k=1}^{3} \frac{(s-\sigma_s)^k}{k!} {F_s}^k + B_3 =$$

$$= \int\limits_{\sigma_s}^{s} \sum_{k=0}^{2} \frac{(r-\sigma_r)^k}{k!} \left[f'[\tilde{x}(r)] {F_r}^k - {F_s}^{k+1} \right] dr + B_3. \tag{3.4}$$

It is clear that for B_3 the inequality

$$|B_3| \le \frac{(s-\sigma_s)^4}{24} M_1 l_3 \le \frac{h^4}{24} M_1 l_3 = h^4 L_3 \tag{3.5}$$

holds.

Intuitively it is clear that the terms in (3.4) are estimated exactly by the power of h, which is equal to the number of occurrence of letter f (i.e., f, f',f'', ...). Taking this property into we regroup the term of $I(s)$ as follows:

$$I(s) = I_1(s) + I_2(s) + I_3(s) + B_3,$$

where

$$I_1(s) = \int\limits_{\sigma_s}^{s} \{ f'[\tilde{x}(r)] f_r - (f'f)_s \} dr, \tag{3.6}$$

$$I_2(s) = \int\limits_{\sigma_s}^{s} (r-\sigma_r) \left\{ f'[\tilde{x}(r)](f'f)_r - \left((f''ff)_s + (f'f'f)_s \right) \right\} dr, \tag{3.7}$$

$$I_3(s) = \int\limits_{\sigma_s}^{s} \frac{(r-\sigma_r)^2}{2!} \left\{ f'[\tilde{x}(r)] {F_r}^2 - {F_s}^3 \right\} dr. \tag{3.8}$$

We start with the estimation of $I_1(s)$. Using the definition of $\dot{\tilde{x}}(t)$ in equation (3.1) and the equality $f_r = f_s$ we obtain:

$$I_1(s) = \int\limits_{\sigma_s}^{s} \left(\int\limits_{\sigma_r}^{r} \frac{d(f'[\tilde{x}(u)])}{du} du \right) f_r dr = \int\limits_{\sigma_s}^{s} \int\limits_{\sigma_r}^{r} f''[\tilde{x}(u)] \sum_{k=0}^{3} \frac{(u-\sigma_u)^k}{k!} {F_u}^k f_r du dr.$$

Integrating the term corresponding to $k = 2,3$ twice we obtain terms of the order h^4 and h^5 respectively. Indeed, introducing the following notation:

$$C_{2,3} = \int\limits_{\sigma_s}^{s} \int\limits_{\sigma_r}^{r} f''[\tilde{x}(u)] \sum_{k=2}^{3} \frac{(u-\sigma_u)^k}{k!} {F_u}^k f_r du dr.$$

We have

$$|C_{2,3}| = \left| \int\limits_{\sigma_s}^{s} \int\limits_{\sigma_r}^{r} f''[\tilde{x}(u)] \sum_{k=2}^{3} \frac{(u-\sigma_u)^k}{k!} {F_u}^k f_r du dr \right| \le h^4 L_{2,3}. \tag{3.9}$$

Using the notation for $C_{2,3}$ expression of $I_1(s)$ can be rewritten in the form

$$I_1(s) = \int\limits_{\sigma_s}^{s} \int\limits_{\sigma_r}^{r} f''[\tilde{x}(u)] \sum_{k=0}^{1} \frac{(u-\sigma_u)^k}{k!} {F_u}^k f_r du dr + C_{2,3}. \tag{3.10}$$

As it is easily seen from (3.10) it is sufficient to obtain estimates on the order of h^4 for the fitst summand. But we delay this until the end. Since we need some other terms coming from I_2, I_3 to group with the terms corresponding to $k=0$ and $k=1$ in (3.10).

Now, we start estimating the term $I_2(s)$. As above, substituting $\dot{\tilde{x}}(t)$ with the expression in (3.1) we obtain:

$$I_2(s)=\int\limits_{\sigma_s}^{s}(r-\sigma_r)\left(\int\limits_{\sigma_s}^{r}f''[\tilde{x}(u)]\sum_{k=0}^{3}\frac{(u-\sigma_u)^k}{k!}{F_u}^k du\right)(f'f)_r dr-\int\limits_{\sigma_s}^{s}(r-\sigma_r)(f''ff)_r dr$$

In this case we denote by $D_{1,2,3}$ the sum of the terms corresponding $k=1,2,3$ of the first term of $I_2(s)$ and obtain

$$|D_{1,2,3}|=\left|\int\limits_{\sigma_s}^{s}(r-\sigma_r)\left(\int\limits_{\sigma_s}^{r}f''[\tilde{x}(u)]\sum_{k=1}^{3}\frac{(u-\sigma_u)^k}{k!}{F_u}^k du\right)(f'f)_r dr\right|\le h^4L_{1,2,3}. \tag{3.11}$$

Therefore we write

$$I_2(s)=\int\limits_{\sigma_s}^{s}(r-\sigma_r)\left(\int\limits_{\sigma_s}^{r}f''[\tilde{x}(u)]f_u du\right)(f'f)_r dr-\int\limits_{\sigma_s}^{s}(r-\sigma_r)(f''ff)_r dr+D_{1,2,3}. \tag{3.12}$$

Again we leave estimating the remaining terms to future. Finally we apply the same method to $I_3(s)$. First of all using the definition (2.2) we group the elements $f'[\tilde{x}(r)](f''ff)_r$ and $f'[\tilde{x}(r)](f'f'f)_r$ of $f'[\tilde{x}(r)]{F_r}^2$ with the corresponding elements $(f'f''ff)_s$ and $(f'f'f'f)_s$ of ${F_s}^3$ and denote the difference by:

$${E^1}_{0,1,2,3}=\int\limits_{\sigma_s}^{s}\frac{(r-\sigma_r)^2}{2}\left(\int\limits_{\sigma_s}^{r}f''[\tilde{x}(u)]\sum_{k=0}^{3}\frac{(u-\sigma_u)^k}{k!}{F_u}^k du\right)(f''ff)_s dr,$$

$${E^2}_{0,1,2,3}=\int\limits_{\sigma_s}^{s}\frac{(r-\sigma_r)^2}{2!}\left(\int\limits_{\sigma_s}^{r}f''[\tilde{x}(u)]\sum_{k=0}^{3}\frac{(u-\sigma_u)^k}{k!}{F_u}^k du\right)(f'f'f)_s dr.$$

Observe that for ${E^1}_{0,1,2,3}$ and ${E^2}_{0,1,2,3}$ the following estimates hold:

$$\left|{E^1}_{0,1,2,3}\right|\le h^4{H^1}_{0,1,2,3},\quad \left|{E^2}_{0,1,2,3}\right|\le h^4{H^2}_{0,1,2,3}. \tag{3.13}$$

Therefore, for $I_3(s)$ we have

$$I_3(s)={E^1}_{0,1,2,3}+{E^2}_{0,1,2,3}-\int\limits_{\sigma_r}^{s}\frac{(r-\sigma_r)^2}{2!}\left\{(f''f'ff)_s+2(f''f'ff)_s+(f'''fff)_s\right\}dr. \tag{3.14}$$

Now, we start estimating the remaining terms of $I_1(s)$, $I_2(s)$ and $I_3(s)$. We start with groupning the term of (3.10) corresponding to $k=0$ with the second term of (3.12):

$$\int\limits_{\sigma_s}^{s}\int\limits_{\sigma_r}^{r}f''[\tilde{x}(u)]f_rf_u dudr-\int\limits_{\sigma_s}^{s}(r-\sigma_r)(f''ff)_r dr=\int\limits_{\sigma_s}^{s}\int\limits_{\sigma_r}^{r}\int\limits_{\sigma_r}^{u}f'''[\tilde{x}(v)]\sum_{k=0}^{3}\frac{(v-\sigma_v)^k}{k!}{F_v}^k(ff)_r dvdudr.$$

Now, denoting by $E_{1,2,3}$ the sum of terms corresponding to $k=1,2,3$ under the integral in the above formula we obtain

$$|E_{1,2,3}|=\left|\int\limits_{\sigma_s}^{s}\int\limits_{\sigma_r}^{r}\int\limits_{\sigma_r}^{u}f'''[\tilde{x}(v)]\sum_{k=1}^{3}\frac{(v-\sigma_v)^k}{k!}{F_v}^k(ff)_r dvdudr\right|\le h^4{H^1}_{1,2,3}. \tag{3.15}$$

Substituting the inequalities (3.10), (3.12) and (3.14) into $I(s)$ we obtain

$$I(s)=\int_{\sigma_s}^{s}\int_{\sigma_r}^{r}(u-\sigma_u)f''[\tilde{x}(u)]f_r(f'f)_u dudr-\int_{\sigma_r}^{s}\frac{(r-\sigma_r)^2}{2!}(f''f'ff)_s dr+$$
$$+\int_{\sigma_s}^{s}(r-\sigma_r)\left(\int_{\sigma_s}^{s}f''[\tilde{x}(u)]f_u du\right)(f'f)_r dr-\int_{\sigma_r}^{s}\frac{(r-\sigma_r)^2}{2!}(2f''f'ff)_s dr+$$
$$+\int_{\sigma_s}^{s}\int_{\sigma_r}^{r}\int_{\sigma_r}^{u}f'''[\tilde{x}(v)]f_v(ff)_r dvdudr-\int_{\sigma_r}^{s}\frac{(r-\sigma_r)^2}{2!}(f'''fff)_r dr+ \tag{3.16}$$
$$+B_3+C_{2,3}+D_{1,2,3}+E_{1,2,3}+E^1{}_{0,1,2,3}+E^2{}_{0,1,2,3}$$

In equation (3.16) we couple the first and second terms and apply the fundamental rule of calculus, then use equation (3.1) and obtain the following equation:

$$E^3{}_{0,1,2,3}=\int_{\sigma_s}^{s}\int_{\sigma_r}^{r}(u-\sigma_u)\left(\int_{\sigma_s}^{u}f'''[\tilde{x}(v)]\sum_{k=0}^{3}\frac{(v-\sigma_v)^k}{k!}F_v{}^k dv\right)(f'ff)_s dudr.$$

In the same way, by coupling the first term with the fourth, and the fifth term with the sixes we obtain

$$E^4{}_{0,1,2,3}=\int_{\sigma_s}^{s}(r-\sigma_r)\left(\int_{\sigma_s}^{s}\left(\int_{\sigma_s}^{u}f'''[\tilde{x}(v)]\sum_{k=0}^{3}\frac{(v-\sigma_v)^k}{k!}F_v{}^k dv\right)(f'ff)_r du\right)dr,$$
$$E^5{}_{0,1,2,3}=\int_{\sigma_s}^{s}\int_{\sigma_r}^{r}\int_{\sigma_r}^{u}\left(\int_{\sigma_s}^{v}f^{(IV)}[\tilde{x}(z)]\sum_{k=0}^{3}\frac{(z-\sigma_z)^k}{k!}F_z{}^k dz\right)(fff)_r dvdudr.$$

The following inequalities hold:

$$\left|E^3{}_{0,1,2,3}\right|\le h^4H^3{}_{0,1,2,3},\quad \left|E^4{}_{0,1,2,3}\right|\le h^4H^4{}_{0,1,2,3},\quad \left|E^5{}_{0,1,2,3}\right|\le h^4H^5{}_{0,1,2,3}. \tag{3.17}$$

Using the above notations we rewrite $I(s)$ in the form:

$$I(s)=B_3+C_{2,3}+D_{1,2,3}+E_{1,2,3}+E^1{}_{0,1,2,3}+E^2{}_{0,1,2,3}+E^3{}_{0,1,2,3}+E^4{}_{0,1,2,3}+E^5{}_{0,1,2,3}. \tag{3.18}$$

Taking into account the equations (3.18) (3.5), (3.9), (3.11), (3.13), (3.15) and (3.17) from the above equality we deduce the following inequality:

$$|I(s)|\le Ch^4.$$

Since K is a compact set by assumption, we can use the Mean Value theorem (Hadamard lemma) to estimate the difference $f[x_*(s)]-f[\tilde{x}(s)]$. We have

$$|f[x_*(s)]-f[\tilde{x}(s)]|\le M_1\left|x_*(s)-\tilde{x}(s)\right|.$$

Therefore,

$$|x_*(t)-\tilde{x}(t)|\le\int_0^t|I(s)|ds+\int_0^t|f[x_*(s)]-f[\tilde{x}(s)]|ds\le Ch^4t+\int_0^t M_1\left|x_*(s)-\bar{x}(s)\right|ds. \tag{3.19}$$

We apply the following form of the Grönwall inequality: Suppose that $u(t)\ge 0$ is a continuous function for $t\ge 0$, and C,M are non-negative numbers such that $u(t)\le Ct+M\int_0^t u(s)ds$. Then the following estimate holds:

$$u(t)\le C\frac{e^{Mt}-1}{M}.$$

Using the above inequality in (3.19) implies

$$|x_*(t) - \tilde{x}(t)| \le Ch^4 \frac{e^{M_1 T} - 1}{M_1},$$

which finishes the proof.

□

4. Applications to Van der Pol oscillator

Here we apply the results to the Van der Pol oscillator:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2)y - x \end{cases} \tag{4.1}$$

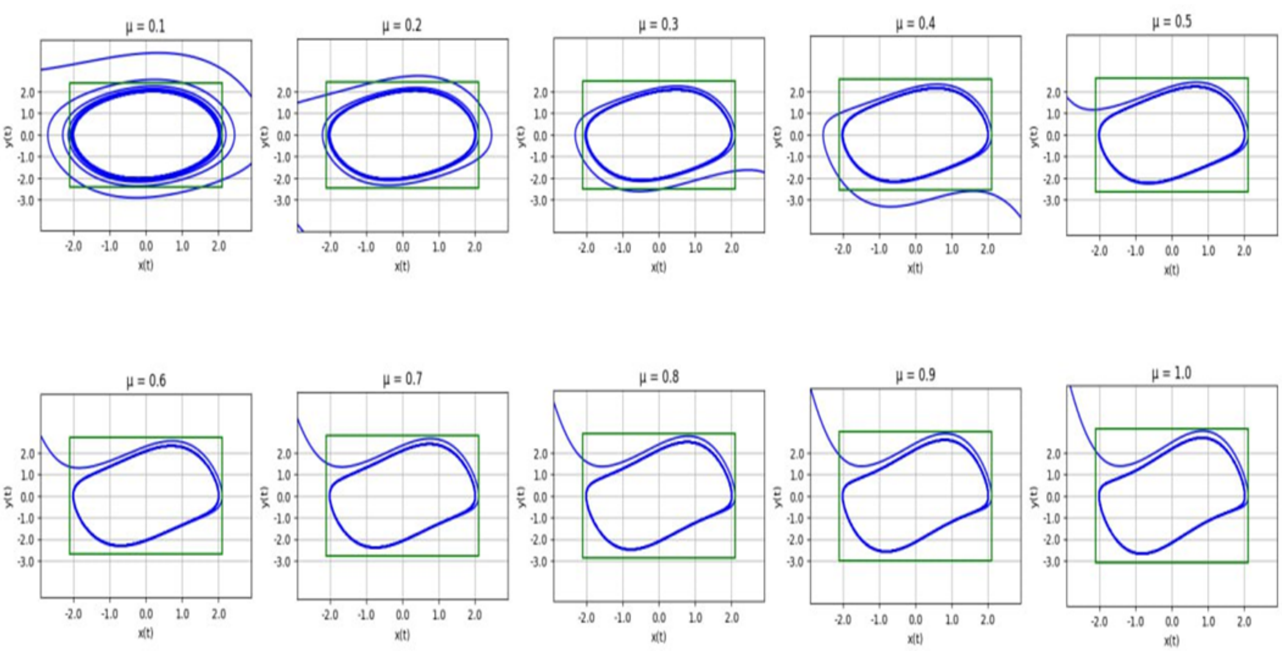
to approximate the period of its periodic trajectories for the parameters $\mu = i/10$, $i = 1, ..., 10$ and compare it with the known formula for the periods:

$$T = \frac{2\pi}{1 - \frac{\mu^2}{16} + \frac{17\mu^4}{3072} + O(\mu^6)}$$

in the rectangle $K = [(x, y) \in R \,||x| \le 2.1,\ \ |y| \le MaxY + 0.001]$

Table 1. Numerical results for different values of μ

μ	Max x	Max y	M_0	M_1	M_2	M_3	M_4	T	Error
0.1	2.0001	2.0116	3.44491	2.12964	0.718523	0.34641	0	6.2871	2.007251 10^{-52}
0.2	2.0004	2.0442	4.05912	2.98222	1.44442	0.69282	0	6.2988	1.188056 10^{-48}
0.3	2.0009	2.0935	4.74375	3.92072	2.1836	1.03923	0	6.3082	6.209675 10^{-45}
0.4	2.0016	2.1554	5.49799	4.93663	2.94036	1.38564	0	6.3457	2.974666 10^{-41}
0.5	2.0025	2.2270	6.32302	6.03082	3.71809	1.73205	0	6.3807	1.327857 10^{-37}
0.6	2.0035	2.3063	7.22075	7.20637	4.5196	2.07846	0	6.4231	5.594606 10^{-34}
0.7	2.0047	2.3919	8.19331	8.46681	5.3474	2.42487	0	6.4728	2.247712 10^{-30}
0.8	2.0059	2.4830	9.24344	9.81633	6.20401	2.77128	0	6.5296	8.680078 10^{-27}
0.9	2.0072	2.5787	10.3729	11.2579	7.09149	3.11769	0	6.5930	3.242253 10^{-23}
1.0	2.0086	2.6784	11.5837	12.7947	8.01186	3.4641	0	6.6627	1.177264 10^{-19}



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