

On the boundary value problem for the fourth-order partial differential equation

Yuldasheva A., Sharipova S.

Abstract. This paper investigates a boundary value problem for a fourth-order partial differential equation posed in a rectangular domain. The analysis focuses on determining the conditions on the domain's dimensions that guarantee the existence and uniqueness of the solution. The results provide constraints under which the problem is well-posed in the classical sense.

Keywords: boundary value problem, fourth-order partial differential equation, algebraic number, Fourier series

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1. INTRODUCTION

In recent years, considerable attention has been devoted to the study of ill-posed problems, including ill-posed boundary value problems for partial differential equations. This interest is driven both by practical applications and by purely theoretical considerations.

From the standpoint of boundary value problem formulation, partial differential equations of classical types and their direct generalizations are the most thoroughly studied ([1], [2]).

In the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < \pi, 0 < y < h\}$$

we consider the fourth-order partial differential equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 0, \quad (1.1)$$

with boundary conditions

$$u(x, h) = \psi(x), \quad u_y(x, 0) = u_{yy}(x, 0) = u_{yyy}(x, 0) = 0, \quad (1.2)$$

where $\psi(x)$ is a 2π -periodic and sufficiently smooth function.

Set

$$\alpha = \frac{h}{\pi\sqrt{2}}. \quad (1.3)$$

Boundary value problems with data prescribed on the entire boundary of the domain for partial differential operators are, generally speaking, ill-posed. The question of their solvability is often related to the so-called small denominators problem. A clear example of this is the Dirichlet problem for the wave equation of a vibrating string, which has been extensively studied by both domestic and foreign researchers.

From a mathematical standpoint, the small denominators phenomenon is manifested in the fact that the solutions to the equations of motion, when represented as Fourier series, involve an infinite number of terms with coefficients whose denominators can become arbitrarily small, thereby leading to the divergence of the series.

As first noted by D. Borgin and R. Duffin [3] (see also [1]), the solvability of the Dirichlet problem for the vibrating string equation depends on whether the number (1.3) is rational or not.

It follows from Theorem 1 of [4] that if the number (1.3) is an irrational number, then the L_p -solution of the Dirichlet problem for the vibrating string equation is unique.

In [5] considered the question of the existence of an L_2 -solution of the Dirichlet problem for the wave equation in a rectangle.

In cases where the governing equation involves differences of arbitrary even-order derivatives, related problems have been investigated in [6], [7].

The following statement holds.

Theorem 1.1. *Let the number α , defined by equality (1.3), be an algebraic number of degree $n > 1$. Then, for any function $\psi(x)$ from the Sobolev class $W_2^{n+3}(|x| < \pi)$, the solution of the problem (1.1)-(1.2) exists and unique.*

The proof of Theorem 1.1 is based on finding the solution of the considered problem by the Fourier method.

Let us expand the boundary condition (1.2) into a Fourier series:

$$\psi(x) = \sum_{k=-\infty}^{\infty} \psi_k e^{ikx}.$$

We prove that the function

$$u(x, y) = \sum_{k=-\infty}^{\infty} \frac{\psi_k}{\cosh \frac{kh}{\sqrt{2}} \cos \frac{kh}{\sqrt{2}}} \cosh \frac{ky}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}} e^{ikx} \quad (1.4)$$

is the desired solution. Prior to that, we obtain several estimates.

2. DEPENDENCE OF THE CONVERGENCE OF THE SERIES ON THE DEGREE OF THE ALGEBRAIC NUMBER

Consider the series

$$S = \sum_{k=-\infty}^{\infty} \frac{b_k}{\cos kz}, z \in \mathbb{R}. \quad (2.1)$$

Note that

$$\cos kz = 0 \quad \text{if} \quad kz = \pi m + \frac{\pi}{2},$$

where $m \in \mathbb{Z}$.

Set

$$\gamma_{mk} = \gamma_{mk}(z) = kz - \left(\pi m + \frac{\pi}{2} \right). \quad (2.2)$$

Then

$$kz = \pi \frac{2m+1}{2} + \gamma_{mk}. \quad (2.3)$$

It is admissible to assume that m can always be chosen in such a way that

$$|\gamma_{mk}| \leq \frac{\pi}{2}. \quad (2.4)$$

Set

$$\beta = \frac{z}{\pi}. \quad (2.5)$$

Then from (2.3) we obtain the representation of the variable (2.5)

$$\beta = \frac{2m+1}{2k} + \frac{\gamma_{mk}}{\pi k}. \quad (2.6)$$

Note that if $\beta = z/\pi$ is an algebraic number of degree $n > 1$, then it cannot be rational. Therefore, for any $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$, the quantity

$$\gamma_{mk} \neq 0.$$

In the subsequent analysis, we employ the results of Liouville's theorem. Let us briefly recall the statement of this theorem

Theorem 2.1. (See [8] Chapter 1, §3) *Let β be an algebraic number of degree n . Then, for any fraction p/q , the following estimate*

$$\left| \beta - \frac{p}{q} \right| \geq \frac{c}{q^n}$$

is valid. The constant $c > 0$ does not depend on p and q .

From this theorem, we derive a corollary that is of significant importance for the problem under consideration.

Corollary 2.1. *Let β be an algebraic number of degree n and equality (2.6) holds. Then the following estimate*

$$|\gamma_{mk}| \geq \frac{c}{|k|^{n-1}} \quad (2.7)$$

is valid. The constant $c > 0$ does not depend on m and k .

Proof: Indeed, from (2.6) we get

$$\frac{\gamma_{mk}}{\pi k} = \beta - \frac{2m+1}{2k}.$$

Then, according to Liouville's theorem,

$$|\gamma_{mk}| = \pi k \left| \beta - \frac{2m+1}{2k} \right| \geq \pi k \frac{c}{|2k|^n} = \frac{c_0}{|k|^{n-1}}. \quad (2.8)$$

□

We now return to the estimation of the denominators in series (2.1). Taking into account (2.3), we can write the following

$$\cos kz = \cos \left[\left(\pi m + \frac{\pi}{2} \right) + \gamma_{mk} \right] = (-1)^{m+1} \sin \gamma_{mk}.$$

From estimates (2.4) and (2.8) it follows that, starting from some index k_0 , the inequality holds for all $|k| \geq k_0$

$$|\cos kz| = |\sin \gamma_{mk}| \geq \left| \sin \frac{c_0}{k^{n-1}} \right| \geq \frac{2}{\pi} \frac{c_0}{|k|^{n-1}}. \quad (2.9)$$

Since the condition $0 < |\gamma_{mk}| < \pi/2$ holds, inequality (2.9) is valid for all $k \in \mathbb{Z}$.

From this, we obtain an estimate for the series (2.1)

$$|S| \leq C \sum_{k=-\infty}^{\infty} |b_k| (1 + |k|^2)^{(n-1)/2}.$$

Then

$$|S|^2 \leq C \sum_{k=-\infty}^{\infty} |b_k|^2 (1 + |k|^2)^n \sum_{k=-\infty}^{\infty} \frac{1}{(1 + |k|^2)} \leq C \sum_{k=-\infty}^{\infty} |b_k|^2 (1 + |k|^2)^n. \quad (2.10)$$

Using the obtained estimates, we proceed to the proof of the main result of the paper.

3. ON THE EXISTENCE OF A SOLUTION TO THE PROBLEM

Let us prove the main result of the paper.

Proof: We seek the solution of problem (1.1)-(1.2) in the form

$$u(x, y) = \sum_{k=-\infty}^{\infty} v_k(y) e^{ikx}$$

Then we reduce the following equation

$$v_k^{(4)}(y) + k^4 v_k(y) = 0$$

Set

$$v_k(y) = e^{ay} [A \cos ay + B \sin ay] + e^{-ay} [C \cos ay + D \sin ay]$$

Then

$$v_k'(y) = ae^{ay} [A \cos ay + B \sin ay] + ae^{ay} [-A \sin ay + B \cos ay] -$$

$$\begin{aligned}
& -ae^{-ay}[C \cos ay + D \sin ay] + ae^{-ay}[-C \sin ay + D \cos ay] = \\
& = ae^{ay}[(A + B) \cos ay + (B - A) \sin ay] + ae^{-ay}[(D - C) \cos ay + (-C - D) \sin ay]
\end{aligned}$$

Hence,

$$\begin{aligned}
v_k''(y) &= a^2 e^{ay}[2B \cos ay - 2A \sin ay] + a^2 e^{-ay}[-2D \cos ay + 2C \sin ay] = \\
&= 2a^2 e^{ay}[B \cos ay - A \sin ay] + 2a^2 e^{-ay}[-D \cos ay + C \sin ay]
\end{aligned}$$

and

$$v_k'''(y) = 2a^3 e^{ay}[(B - A) \cos ay - (A + B) \sin ay] + 2a^3 e^{-ay}[(D + C) \cos ay + (D - C) \sin ay]$$

Therefore,

$$v_k^{(4)}(y) = 4a^4 e^{ay}[-A \cos ay - B \sin ay] + 4a^4 e^{-ay}[-C \cos ay - D \sin ay]$$

Consequently,

$$v_k''''(y) = -4a^4 v_k(x)$$

Then

$$u_{xxxx} + u_{yyyy} = k^4 u(x, y) - 4a^4 u(x, y) = 0$$

if

$$a = \frac{k}{\sqrt{2}}$$

According to the conditions (1.2), we get

$$e^{ah}[A \cos(ah) + B \sin(ah)] + e^{-ah}[C \cos(ah) + D \sin(ah)] = \psi_k,$$

$$A + B + D - C = 0,$$

$$B - D = 0,$$

$$A + B + D - C = 0.$$

Hence,

$$A = C = \frac{\psi_k}{2 \cosh(ah) \cos(ah)}, \quad D = B = 0.$$

Consequently, expression (1.4) is a formal solution of problem (1.1)-(1.2).

It remains to prove that the obtained solution and its fourth derivatives with respect to x and y converge uniformly. By direct differentiation of (1.4), we obtain

$$\frac{\partial^4 u(x, y)}{\partial y^4} = \sum_{k=-\infty}^{\infty} \frac{k^4 \psi_k}{\cosh \frac{kh}{\sqrt{2}} \cos \frac{kh}{\sqrt{2}}} \cosh \frac{ky}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}} e^{ikx} \quad (3.1)$$

$$\frac{\partial^4 u(x, y)}{\partial x^4} = \sum_{k=-\infty}^{\infty} \frac{k^4 \psi_k}{\cosh \frac{kh}{\sqrt{2}} \cos \frac{kh}{\sqrt{2}}} \cosh \frac{ky}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}} e^{ikx}.$$

Then the series majorizing series (3.1) has the form

$$\sum_{k=-\infty}^{\infty} \frac{k^4 |\psi_k|}{|\cos \frac{kh}{\sqrt{2}}|}. \quad (3.2)$$

Then, according to Corollary 2.2 starting from some k_0 , the denominator of series (3.2) satisfies estimate (2.9), where $z = \frac{h}{\sqrt{2}}$.

Now, for series (3.2), we can write the following estimate

$$\sum_{k=-\infty}^{\infty} \frac{k^4 |\psi_k|}{|\cos \frac{kh}{\sqrt{2}}|} \leq C \sum_{k=-\infty}^{\infty} |\psi_k| (1 + |k|^2)^{(n+3)/2}$$

The last term converges by virtue of the conditions of Theorem 1.1. Theorem is proved. $\square$

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Yuldasheva A.V.,
 Department of Applied Mathematics and Informatics,
 Moscow State University, Tashkent Branch, Tashkent,
 Uzbekistan
 e-mail: a.v.yuldasheva@mail.ru

Sharipova S.A.,
 National University of Uzbekistan, Uzbekistan
 e-mail: sharipova@mail.ru