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Pauli Gaussian Leonardo quaternions

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Abstract. In this paper, a new family of Pauli quaternions whose components are the Gaussian Leonardo numbers is defined. These new Pauli quaternions are called Pauli Gaussian Leonardo quaternions. Furthermore, several properties of Pauli Gaussian Leonardo quaternions, including relations with the Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions are investigated. In addition, Binet-like formula, (ordinary) generating function, exponential generating function, and some summation formulas for these Pauli quaternions are given. Moreover, some results are illustrated with examples.

Keywords: Leonardo number, Gaussian Leonardo number, Pauli matrix, Pauli quaternion, Pauli Gaussian Leonardo quaternion
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1. Introduction

The sequences of Fibonacci and Lucas numbers [1] are defined by the relations

$$F_0 = 0, F_1 = 1; F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{1.1}$$

and

$$L_0 = 2, L_1 = 1; L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \tag{1.2}$$

respectively. The Binet formulas for Fibonacci and Lucas numbers are given as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n,$$

respectively, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Here, α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$ of (1.1) and (1.2). For more information, we refer to [1].

Over the past few years, the sequence of Leonardo numbers, which is listed as $A001595$ in the OEIS [2], has garnered enormous interest. In [3], Catarino and Borges studied some properties of the Leonardo number sequence.

The Leonardo sequence is defined by the relations

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2$$

or

$$Le_n = 2Le_{n-1} - Le_{n-3}, \quad n \geq 3$$

with initial conditions $Le_0 = Le_1 = 1$ and $Le_2 = 3$. Here, Le_n represents the n-th Leonardo number [3].

The Leonardo numbers have a strong relationship with the well-known Fibonacci numbers. Let Le_n be the n-th Leonardo number and F_{n+1} be the $(n+1)$-th Fibonacci number. Then, the Leonardo and Fibonacci numbers are related in the way that follows [3]:

$$Le_n = 2F_{n+1} - 1.$$

Furthermore, the n-th Leonardo number is given as

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (see [3]).

A Gaussian number is a complex number with integer coefficients that was investigated by Gauss [4]. The concept of complex Fibonacci numbers was first introduced and studied by Horadam [5]. Then, the Gaussian Fibonacci and Gaussian Lucas numbers were studied by Jordan [6].

The n-th Gaussian Fibonacci number is defined recursively by the relation

$$GF_n = GF_{n-1} + GF_{n-2}, \quad n \geq 2$$

with $GF_0 = i$ and $GF_1 = 1$. Similarly, the n-th Gaussian Lucas number is defined recursively by

$$GL_n = GL_{n-1} + GL_{n-2}, \quad n \geq 2$$

with $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$ (see [6]).

Exactly like the Fibonacci numbers, the complex Leonardo numbers [7] and the Gaussian Leonardo numbers [8, 9, 10] are introduced and studied. Then, in [11], some new results involving the Gaussian Leonardo numbers are given.

The n-th Gaussian Leonardo number is defined by

$$GLe_n = GLe_{n-1} + GLe_{n-2} + (1+i), \quad n \geq 2 \tag{1.3}$$

or

$$GLe_n = 2GLe_{n-1} - GLe_{n-3}, \quad n \geq 3 \tag{1.4}$$

with $GLe_0 = 1 - i$, $GLe_1 = 1 + i$ and $GLe_2 = 3 + i$. Moreover, for non-negative integer n, the n-th Gaussian Leonardo number is given as

$$GLe_n = \frac{2(\alpha^{n+1} - \beta^{n+1}) + 2i(\alpha^n - \beta^n)}{\alpha - \beta} - (1+i), \tag{1.5}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (see [8, 9, 10]).

Besides, for Gaussian Leonardo numbers, the followings hold [8, 9, 10, 11]:

$$GLe_n = Le_n + Le_{n-1}i, \tag{1.6}$$

$$GLe_n = 2GF_{n+1} - (1+i), \tag{1.7}$$

$$GLe_{n-1} + GLe_{n+1} = 2GL_{n+1} - 2(1+i), \tag{1.8}$$

$$GLe_{n+1} - GLe_n = 2GF_n, \tag{1.9}$$

$$GLe_{n+2} - GLe_{n-2} = 2GL_{n+1}, \tag{1.10}$$

$$GLe_n + GF_n + GL_n = 2GLe_n + (1+i), \tag{1.11}$$

$$\sum_{k=0}^{n} GLe_k = GLe_{n+2} - (n+2)(1+i), \tag{1.12}$$

$$\sum_{k=0}^{n} GLe_{2k} = GLe_{2n+1} - n - (n+2)i, \tag{1.13}$$

$$\sum_{k=0}^{n} GLe_{2k+1} = GLe_{2n+2} - (n+2) - ni, \tag{1.14}$$

where Le_n is the n-th Leonardo number, GLe_n is the n-th Gaussian Leonardo number, GF_n is the n-th Gaussian Fibonacci number, and GL_n is the n-th Gaussian Lucas number.

A (real) quaternion, introduced by W. R. Hamilton in 1843, is a hyper-complex number. A quaternion q is represented as follows:

$$q = q_0 + q_1 i + q_2 j + q_3 k,$$

where q_0, q_1, q_2, and q_3 are real numbers, and i, j, and k are quaternionic units such that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

For further information, we refer to [12, 13].

The Pauli quaternions are quaternions formed by using the Pauli matrices. The Pauli matrices comprise a collection of 2×2 complex matrices as follows [14, 15]:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{\mathbf{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\mathbf{2}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{\mathbf{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with multiplication rules given by

$$\sigma_{\mathbf{1}}{}^2 = \sigma_{\mathbf{2}}{}^2 = \sigma_{\mathbf{3}}{}^2 = \mathbf{1},$$

$$\sigma_{\mathbf{1}}\sigma_{\mathbf{2}} = -\sigma_{\mathbf{2}}\sigma_{\mathbf{1}} = i\sigma_{\mathbf{3}}, \quad \sigma_{\mathbf{2}}\sigma_{\mathbf{3}} = -\sigma_{\mathbf{3}}\sigma_{\mathbf{2}} = i\sigma_{\mathbf{1}}, \quad \sigma_{\mathbf{3}}\sigma_{\mathbf{1}} = -\sigma_{\mathbf{1}}\sigma_{\mathbf{3}} = i\sigma_{\mathbf{2}}. \tag{1.15}$$

Moreover, the Pauli matrices, named after the German physicist Wolfgang E. Pauli, are Hermitian and unitary (see [14, 15]). The Pauli matrices have found wide applications in various areas, including mathematics, physics, and mathematical physics (see, e.g., [16, 17, 18, 14, 15, 19, 20, 21]).

The set of $\{\mathbf{1}, i\sigma_{\mathbf{1}}, i\sigma_{\mathbf{2}}, i\sigma_{\mathbf{3}}\}$ is the basis of Pauli quaternions. This set is isomorphic to the set of (real) quaternions [15].

A Pauli quaternion is defined by Kim [15] as

$$p = p_0\mathbf{1} + p_1\sigma_{\mathbf{1}} + p_2\sigma_{\mathbf{2}} + p_3\sigma_{\mathbf{3}},$$

where $\sigma_{\mathbf{1}}$, $\sigma_{\mathbf{2}}$, and $\sigma_{\mathbf{3}}$ satisfy the rules (1.15).

The conjugate of a Pauli quaternion p, denoted by $\overline{p}$, is

$$\overline{p} = p_0\mathbf{1} - p_1\sigma_{\mathbf{1}} - p_2\sigma_{\mathbf{2}} - p_3\sigma_{\mathbf{3}}.$$

Furthermore, in [15], Kim investigated algebraic and analytic properties of Pauli quaternions.

In [22], Aydın introduced the Pauli Fibonacci quaternions and obtained some properties involving these Pauli quaternions. Then, in [23], İşbilir et al. defined and studied the Pauli Leonardo quaternions.

The n-th Pauli Leonardo quaternion is defined as

$$Q_P Le_n = Le_n\mathbf{1} + Le_{n+1}\sigma_{\mathbf{1}} + Le_{n+2}\sigma_{\mathbf{2}} + Le_{n+3}\sigma_{\mathbf{3}}, \tag{1.16}$$

where Le_n is the n-th Leonardo number (see [23]).

More recently, in [24], Azak defined the Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions as follow:

The n-th Pauli Gaussian Fibonacci quaternion is

$$Q_P GF_n = GF_n\mathbf{1} + GF_{n+1}\sigma_{\mathbf{1}} + GF_{n+2}\sigma_{\mathbf{2}} + GF_{n+3}\sigma_{\mathbf{3}}, \tag{1.17}$$

where GF_n is the n-th Gaussian Fibonacci number.

The n-th Pauli Gaussian Lucas quaternion is

$$Q_P GL_n = GL_n \mathbf{1} + GL_{n+1}\sigma_{\mathbf{1}} + GL_{n+2}\sigma_{\mathbf{2}} + GL_{n+3}\sigma_{\mathbf{3}}, \tag{1.18}$$

where GL_n is the n-th Gaussian Lucas number.

Furthermore, the author obtained some identities and formulas involving the Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions (see [24]).

Quaternions are an extension of complex numbers. Both complex numbers and quaternions have a rich representation capacity in many scientific disciplines, especially in applied sciences. Also, Pauli matrices are very useful in quantum mechanics as well as in classical mechanics. On the other hand, due to the extraordinary patterns of Fibonacci numbers in nature, the Fibonacci sequence and other sequences that are closely related to this sequence are the focus of many researchers' studies. The Leonardo sequence, a non-homogeneous extension of the Fibonacci sequence, has recently attracted considerable attention by researchers.

Inspired and motivated by some of the above mentioned papers, by bringing together complex (Gaussian) numbers, quaternions, Pauli matrices, and Leonardo numbers, we aim to introduce a new family of quaternions. These numbers will be referred to as Pauli Gaussian Leonardo quaternions. The Pauli Gaussian Leonardo quaternions are an extended description of the Pauli Leonardo quaternions in [23] to the complex case. For our purpose, we first define Pauli quaternions with Gaussian Leonardo number coefficients. Then, we derivate some identities and formulas involving these Pauli quaternions.

2. Main Results

Definition 2.1. For $n \geq 0$, the n-th Pauli Gaussian Leonardo quaternion, denoted by $Q_P GLe_n$, is defined by

$$Q_P GLe_n = GLe_n \mathbf{1} + GLe_{n+1}\sigma_{\mathbf{1}} + GLe_{n+2}\sigma_{\mathbf{2}} + GLe_{n+3}\sigma_{\mathbf{3}}, \tag{2.1}$$

where GLe_n is the n-th Gaussian Leonardo number, and $\sigma_{\mathbf{1}}$, $\sigma_{\mathbf{2}}$, and $\sigma_{\mathbf{3}}$ satisfy the rules (1.15).

Note that, by virtue of (1.6) and (1.16), it is easy to see that

$$Q_P GLe_n = Q_P Le_n + Q_P Le_{n-1} i.$$

By considering the definition of the conjugate of a Pauli quaternion, the conjugate of $Q_P GLe_n$ is defined as

$$\overline{Q_P GLe_n} = GLe_n \mathbf{1} - GLe_{n+1}\sigma_{\mathbf{1}} - GLe_{n+2}\sigma_{\mathbf{2}} - GLe_{n+3}\sigma_{\mathbf{3}}. \tag{2.2}$$

Throughout the paper, let $P = (1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}})$. From the definition of the Pauli Gaussian Leonardo quaternions, we can get the recurrence relations as

$$Q_P GLe_n = Q_P GLe_{n-1} + Q_P GLe_{n-2} + P, \quad n \geq 2 \tag{2.3}$$

or

$$Q_P GLe_n = 2Q_P GLe_{n-1} - Q_P GLe_{n-3}, \quad n \geq 3 \tag{2.4}$$

with

$$\begin{aligned}
Q_P GLe_0 &= (1-i)\mathbf{1} + (1+i)\sigma_{\mathbf{1}} + (3+i)\sigma_{\mathbf{2}} + (5+3i)\sigma_{\mathbf{3}}, \\
Q_P GLe_1 &= (1+i)\mathbf{1} + (3+i)\sigma_{\mathbf{1}} + (5+3i)\sigma_{\mathbf{2}} + (9+5i)\sigma_{\mathbf{3}}, \\
Q_P GLe_2 &= (3+i)\mathbf{1} + (5+3i)\sigma_{\mathbf{1}} + (9+5i)\sigma_{\mathbf{2}} + (15+9i)\sigma_{\mathbf{3}}.
\end{aligned}$$

Theorem 2.2. *Let Q_PGLe_n be the n-th Pauli Gaussian Leonardo quaternion. Then, for $n \geq 0$, we have*

$$Q_PGLe_n + \overline{Q_PGLe_n} = 2GLe_n\mathbf{1}, \tag{2.5}$$

$$(Q_PGLe_n)^2 = Q_PGLe_n\left(2GLe_n\mathbf{1} - \overline{Q_PGLe_n}\right), \tag{2.6}$$

$$Q_PGLe_n\mathbf{1} - Q_PGLe_{n+1}\sigma_{\mathbf{1}} - Q_PGLe_{n+2}\sigma_{\mathbf{2}} - Q_PGLe_{n+3}\sigma_{\mathbf{3}} = -\left(GLe_{n+1} + 2GL_{n+6} - (1+i)\right)\mathbf{1}. \tag{2.7}$$

Proof. (2.5): From (2.1) and (2.2), it is straightforward.
(2.6): By considering (2.5), we get

$$(Q_PGLe_n)^2 = Q_PGLe_n.Q_PGLe_n = Q_PGLe_n\left(2GLe_n\mathbf{1} - \overline{Q_PGLe_n}\right).$$

(2.7): By virtue of the multiplication rules (1.15) and Equation (2.1), we obtain

$$\begin{aligned}
&Q_PGLe_n\mathbf{1} - Q_PGLe_{n+1}\sigma_{\mathbf{1}} - Q_PGLe_{n+2}\sigma_{\mathbf{2}} - Q_PGLe_{n+3}\sigma_{\mathbf{3}} \\
&\quad = \left(GLe_n\mathbf{1} + GLe_{n+1}\sigma_{\mathbf{1}} + GLe_{n+2}\sigma_{\mathbf{2}} + GLe_{n+3}\sigma_{\mathbf{3}}\right)\mathbf{1} \\
&\qquad - \left(GLe_{n+1}\mathbf{1} + GLe_{n+2}\sigma_{\mathbf{1}} + GLe_{n+3}\sigma_{\mathbf{2}} + GLe_{n+4}\sigma_{\mathbf{3}}\right)\sigma_{\mathbf{1}} \\
&\qquad - \left(GLe_{n+2}\mathbf{1} + GLe_{n+3}\sigma_{\mathbf{1}} + GLe_{n+4}\sigma_{\mathbf{2}} + GLe_{n+5}\sigma_{\mathbf{3}}\right)\sigma_{\mathbf{2}} \\
&\qquad - \left(GLe_{n+3}\mathbf{1} + GLe_{n+4}\sigma_{\mathbf{1}} + GLe_{n+5}\sigma_{\mathbf{2}} + GLe_{n+6}\sigma_{\mathbf{3}}\right)\sigma_{\mathbf{3}} \\
&\quad = \left((GLe_n - GLe_{n+2}) - (GLe_{n+4} + GLe_{n+6})\right)\mathbf{1}.
\end{aligned}$$

From (1.3) and (1.8), we have

$$Q_PGLe_n\mathbf{1} - Q_PGLe_{n+1}\sigma_{\mathbf{1}} - Q_PGLe_{n+2}\sigma_{\mathbf{2}} - Q_PGLe_{n+3}\sigma_{\mathbf{3}} = -\left(GLe_{n+1} + 2GL_{n+6} - (1+i)\right)\mathbf{1}.$$

This completes the proof. □

We now give some identities involving the Pauli Gaussian Leonardo quaternion Q_PGLe_n, including relations with the Pauli Gaussian Fibonacci quaternion Q_PGF_n and Pauli Gaussian Lucas quaternion Q_PGL_n.

Theorem 2.3. *For $n \geq 0$, let Q_PGLe_n be the n-th Pauli Gaussian Leonardo quaternion. Then, the followings hold true:*

$$Q_PGLe_{n-1} + Q_PGLe_{n+1} = 2Q_PGL_{n+1} - 2P, \tag{2.8}$$

$$Q_PGLe_{n+1} - Q_PGLe_n = 2Q_PGF_n, \tag{2.9}$$

$$Q_PGLe_{n+2} - Q_PGLe_{n-2} = 2Q_PGL_{n+1}, \tag{2.10}$$

$$Q_PGLe_n + Q_PGF_n + Q_PGL_n = 2Q_PGLe_n + P. \tag{2.11}$$

Proof. (2.8): By virtue of (1.8), (1.18), and (2.1), we have

$$\begin{aligned}
Q_PGLe_{n-1} + Q_PGLe_{n+1} &= GLe_{n-1}\mathbf{1} + GLe_n\sigma_{\mathbf{1}} + GLe_{n+1}\sigma_{\mathbf{2}} + GLe_{n+2}\sigma_{\mathbf{3}} \\
&\quad + GLe_{n+1}\mathbf{1} + GLe_{n+2}\sigma_{\mathbf{1}} + GLe_{n+3}\sigma_{\mathbf{2}} + GLe_{n+4}\sigma_{\mathbf{3}} \\
&= (GLe_{n-1} + GLe_{n+1})\mathbf{1} + (GLe_n + GLe_{n+2})\sigma_{\mathbf{1}} \\
&\quad + (GLe_{n+1} + GLe_{n+3})\sigma_{\mathbf{2}} + (GLe_{n+2} + GLe_{n+4})\sigma_{\mathbf{3}} \\
&= 2\left(GL_{n+1}\mathbf{1} + GL_{n+2}\sigma_{\mathbf{1}} + GL_{n+3}\sigma_{\mathbf{2}} + GL_{n+4}\sigma_{\mathbf{3}}\right) \\
&\quad - 2(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) \\
&= 2Q_PGL_{n+1} - 2P.
\end{aligned}$$

(2.9): By virtue of (1.9), (1.17), and (2.1), the desired result can be obtained in a similar manner as Equation (2.8).

(2.10): By virtue of (1.10), (1.18), and (2.1), the proof is similar as Equation (2.8).
(2.11): By virtue of (1.11), (1.17), (1.18), and (2.1), we have

$$\begin{aligned} Q_PGLe_n + Q_PGF_n + Q_PGL_n &= (GLe_n + GF_n + GL_n)\mathbf{1} + (GLe_{n+1} + GF_{n+1} + GL_{n+1})\sigma_\mathbf{1} \\ &\quad + (GLe_{n+2} + GF_{n+2} + GL_{n+2})\sigma_\mathbf{2} + (GLe_{n+3} + GF_{n+3} + GL_{n+3})\sigma_\mathbf{3} \end{aligned}$$

$$\begin{aligned} &= (2GLe_n + (1+i))\mathbf{1} + (2GLe_{n+1} + (1+i))\sigma_\mathbf{1} + (2GLe_{n+2} + (1+i))\sigma_\mathbf{2} + (2GLe_{n+3} + (1+i))\sigma_\mathbf{3} \\ &= 2\left(GLe_n\mathbf{1} + GLe_{n+1}\sigma_\mathbf{1} + GLe_{n+2}\sigma_\mathbf{2} + GLe_{n+3}\sigma_\mathbf{3}\right) + (1+i)(\mathbf{1} + \sigma_\mathbf{1} + \sigma_\mathbf{2} + \sigma_\mathbf{3}) = 2Q_PGLe_n + P \end{aligned}$$

which completes the proof. □

Example 2.4. If we take $n = 1$ in Equation (2.8), $n = 0$ in Equations (2.9) and (2.11), and $n = 2$ in Equation (2.10) in Theorem 2.3 then, we get

$$\begin{aligned} Q_PGLe_0 + Q_PGLe_2 &= (1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3} \\ &\quad + (3+i)\mathbf{1} + (5+3i)\sigma_\mathbf{1} + (9+5i)\sigma_\mathbf{2} + (15+9i)\sigma_\mathbf{3} \\ &= 4.\mathbf{1} + (6+4i)\sigma_\mathbf{1} + (12+6i)\sigma_\mathbf{2} + (20+12i)\sigma_\mathbf{3} \end{aligned}$$

$$\begin{aligned} 2Q_PGL_2 - 2P &= 2\left((3+i)\mathbf{1} + (4+3i)\sigma_\mathbf{1} + (7+4i)\sigma_\mathbf{2} + (11+7i)\sigma_\mathbf{3}\right) - 2(1+i)(\mathbf{1} + \sigma_\mathbf{1} + \sigma_\mathbf{2} + \sigma_\mathbf{3}) \\ &= 4.\mathbf{1} + (6+4i)\sigma_\mathbf{1} + (12+6i)\sigma_\mathbf{2} + (20+12i)\sigma_\mathbf{3}, \end{aligned}$$

$$\begin{aligned} Q_PGLe_1 - Q_PGLe_0 &= (1+i)\mathbf{1} + (3+i)\sigma_\mathbf{1} + (5+3i)\sigma_\mathbf{2} + (9+5i)\sigma_\mathbf{3} \\ &\quad - ((1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3}) \\ &= 2i\mathbf{1} + 2\sigma_\mathbf{1} + (2+2i)\sigma_\mathbf{2} + (4+2i)\sigma_\mathbf{3} \\ 2Q_PGF_0 &= 2\left(i\mathbf{1} + 1\sigma_\mathbf{1} + (1+i)\sigma_\mathbf{2} + (2+i)\sigma_\mathbf{3}\right) \\ &= 2i\mathbf{1} + 2\sigma_\mathbf{1} + (2+2i)\sigma_\mathbf{2} + (4+2i)\sigma_\mathbf{3}, \end{aligned}$$

$$\begin{aligned} Q_PGLe_4 - Q_PGLe_0 &= (9+5i)\mathbf{1} + (15+9i)\sigma_\mathbf{1} + (25+15i)\sigma_\mathbf{2} + (41+25i)\sigma_\mathbf{3} \\ &\quad - ((1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3}) \\ &= (8+6i)\mathbf{1} + (14+8i)\sigma_\mathbf{1} + (22+14i)\sigma_\mathbf{2} + (36+22i)\sigma_\mathbf{3} \\ 2Q_PGL_3 &= 2\left((4+3i)\mathbf{1} + (7+4i)\sigma_\mathbf{1} + (11+7i)\sigma_\mathbf{2} + (18+11i)\sigma_\mathbf{3}\right) \\ &= (8+6i)\mathbf{1} + (14+8i)\sigma_\mathbf{1} + (22+14i)\sigma_\mathbf{2} + (36+22i)\sigma_\mathbf{3} \end{aligned}$$

and

$$\begin{aligned} Q_PGLe_0 + Q_PGF_0 + Q_PGL_0 &= (1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3} \\ &\quad + i\mathbf{1} + 1\sigma_\mathbf{1} + (1+i)\sigma_\mathbf{2} + (2+i)\sigma_\mathbf{3} \\ &\quad + (2-i)\mathbf{1} + (1+2i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (4+3i)\sigma_\mathbf{3} \\ &= (3-i)\mathbf{1} + (3+3i)\sigma_\mathbf{1} + (7+3i)\sigma_\mathbf{2} + (11+7i)\sigma_\mathbf{3} \end{aligned}$$

$$\begin{aligned} 2Q_PGLe_0 + P &= 2\left((1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3}\right) + (1+i)(\mathbf{1} + \sigma_\mathbf{1} + \sigma_\mathbf{2} + \sigma_\mathbf{3}) \\ &= (3-i)\mathbf{1} + (3+3i)\sigma_\mathbf{1} + (7+3i)\sigma_\mathbf{2} + (11+7i)\sigma_\mathbf{3}, \end{aligned}$$

respectively.

Theorem 2.5. *For $n \geq 0$, the Binet-like formula for the Pauli Gaussian Leonardo quaternions is given by*

$$Q_PGLe_n = \frac{2(\alpha+i)\alpha^*\alpha^n - 2(\beta+i)\beta^*\beta^n}{\alpha-\beta} - P, \tag{2.12}$$

where $\alpha^* = \mathbf{1} + \alpha\sigma_\mathbf{1} + \alpha^2\sigma_\mathbf{2} + \alpha^3\sigma_\mathbf{3}$ and $\beta^* = \mathbf{1} + \beta\sigma_\mathbf{1} + \beta^2\sigma_\mathbf{2} + \beta^3\sigma_\mathbf{3}$.

Proof. From (1.5) and (2.1), we have

$$\begin{aligned}
Q_PGLe_n &= GLe_n\mathbf{1} + GLe_{n+1}\sigma_{\mathbf{1}} + GLe_{n+2}\sigma_{\mathbf{2}} + GLe_{n+3}\sigma_{\mathbf{3}} \\
&= \left(\frac{2(\alpha^{n+1}-\beta^{n+1})+2i(\alpha^n-\beta^n)}{\alpha-\beta} - (1+i)\right)\mathbf{1} \\
&\quad + \left(\frac{2(\alpha^{n+2}-\beta^{n+2})+2i(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta} - (1+i)\right)\sigma_{\mathbf{1}} \\
&\quad + \left(\frac{2(\alpha^{n+3}-\beta^{n+3})+2i(\alpha^{n+2}-\beta^{n+2})}{\alpha-\beta} - (1+i)\right)\sigma_{\mathbf{2}} \\
&\quad + \left(\frac{2(\alpha^{n+4}-\beta^{n+4})+2i(\alpha^{n+3}-\beta^{n+3})}{\alpha-\beta} - (1+i)\right)\sigma_{\mathbf{3}} \\
&= \frac{2\alpha^{n+1}(\mathbf{1}+\alpha\sigma_{\mathbf{1}}+\alpha^2\sigma_{\mathbf{2}}+\alpha^3\sigma_{\mathbf{3}}) - 2\beta^{n+1}(\mathbf{1}+\beta\sigma_{\mathbf{1}}+\beta^2\sigma_{\mathbf{2}}+\beta^3\sigma_{\mathbf{3}})}{\alpha-\beta} \\
&\quad + \frac{2i\alpha^{n}(\mathbf{1}+\alpha\sigma_{\mathbf{1}}+\alpha^2\sigma_{\mathbf{2}}+\alpha^3\sigma_{\mathbf{3}}) - 2i\beta^{n}(\mathbf{1}+\beta\sigma_{\mathbf{1}}+\beta^2\sigma_{\mathbf{2}}+\beta^3\sigma_{\mathbf{3}})}{\alpha-\beta} \\
&\quad - (1+i)(\mathbf{1}+\sigma_{\mathbf{1}}+\sigma_{\mathbf{2}}+\sigma_{\mathbf{3}}) \\
&= \frac{2(\alpha+i)\alpha^*\alpha^n - 2(\beta+i)\beta^*\beta^n}{\alpha-\beta} - P.
\end{aligned}$$

Thus, the proof is completed. □

Example 2.6. If we take $n = 2$ in Theorem 2.5 , then Q_PGLe_2 can be obtained as

$$\begin{aligned}
Q_PGLe_2 &= \frac{2(\alpha+i)\alpha^*\alpha^2 - 2(\beta+i)\beta^*\beta^2}{\alpha-\beta} - P \\
&= \frac{2\left(\frac{1+\sqrt{5}}{2}+i\right)\left(\frac{3+\sqrt{5}}{2}\right)\left(\mathbf{1}+\left(\frac{1+\sqrt{5}}{2}\right)\sigma_{\mathbf{1}}+\left(\frac{3+\sqrt{5}}{2}\right)\sigma_{\mathbf{2}}+\left(2+\sqrt{5}\right)\sigma_{\mathbf{3}}\right)}{\sqrt{5}} \\
&\quad - \frac{2\left(\frac{1-\sqrt{5}}{2}+i\right)\left(\frac{3-\sqrt{5}}{2}\right)\left(\mathbf{1}+\left(\frac{1-\sqrt{5}}{2}\right)\sigma_{\mathbf{1}}+\left(\frac{3-\sqrt{5}}{2}\right)\sigma_{\mathbf{2}}+\left(2-\sqrt{5}\right)\sigma_{\mathbf{3}}\right)}{\sqrt{5}} - (1+i)(\mathbf{1}+\sigma_{\mathbf{1}}+\sigma_{\mathbf{2}}+\sigma_{\mathbf{3}}) \\
&= \frac{\left(4\sqrt{5}.\mathbf{1}+6\sqrt{5}\sigma_{\mathbf{1}}+10\sqrt{5}\sigma_{\mathbf{2}}+16\sqrt{5}\sigma_{\mathbf{3}}\right)+i\left(2\sqrt{5}.\mathbf{1}+4\sqrt{5}\sigma_{\mathbf{1}}+6\sqrt{5}\sigma_{\mathbf{2}}+10\sqrt{5}\sigma_{\mathbf{3}}\right)}{\sqrt{5}} - (1+i)(\mathbf{1}+\sigma_{\mathbf{1}}+\sigma_{\mathbf{2}}+\sigma_{\mathbf{3}}) \\
&= (4+2i)\mathbf{1} + (6+4i)\sigma_{\mathbf{1}} + (10+6i)\sigma_{\mathbf{2}} + (16+10i)\sigma_{\mathbf{3}} - (1+i)(\mathbf{1}+\sigma_{\mathbf{1}}+\sigma_{\mathbf{2}}+\sigma_{\mathbf{3}}) \\
&= (3+i)\mathbf{1} + (5+3i)\sigma_{\mathbf{1}} + (9+5i)\sigma_{\mathbf{2}} + (15+9i)\sigma_{\mathbf{3}}.
\end{aligned}$$

Theorem 2.7. *The (ordinary) generating function of the Pauli Gaussian Leonardo quaternions is*

$$g(x) = \frac{Q_PGLe_0 + (Q_PGLe_1 - 2Q_PGLe_0)x + (Q_PGLe_2 - 2Q_PGLe_1)x^2}{1-2x+x^3}.$$

Proof. Let $g(x)$ be the generating function of the Pauli Gaussian Leonardo quaternions. From the definition of the generating function of a sequence, we can write

$$g(x) = \sum_{n=0}^{\infty} Q_PGLe_n x^n. \tag{2.13}$$

By virtue of (2.4) and (2.13), we have

$$\begin{aligned}
g(x) &= Q_PGLe_0 + Q_PGLe_1x + Q_PGLe_2x^2 + \sum_{n=3}^{\infty} Q_PGLe_n x^n \\
&= Q_PGLe_0 + Q_PGLe_1x + Q_PGLe_2x^2 + \sum_{n=3}^{\infty} (2Q_PGLe_{n-1} - Q_PGLe_{n-3})x^n \\
&= Q_PGLe_0 + Q_PGLe_1x + Q_PGLe_2x^2 + 2x\sum_{n=3}^{\infty} Q_PGLe_{n-1}x^{n-1} - x^3\sum_{n=3}^{\infty} Q_PGLe_{n-3}x^{n-3} \\
&= Q_PGLe_0 + (Q_PGLe_1 - 2Q_PGLe_0)x + (Q_PGLe_2 - 2Q_PGLe_1)x^2 \\
&\quad + 2x\sum_{n=0}^{\infty} Q_PGLe_n x^n - x^3\sum_{n=0}^{\infty} Q_PGLe_n x^n.
\end{aligned}$$

Then, it follows that

$$g(x)(1-2x+x^3) = Q_PGLe_0 + (Q_PGLe_1 - 2Q_PGLe_0)x + (Q_PGLe_2 - 2Q_PGLe_1)x^2$$

which completes the proof. □

Theorem 2.8. *The exponential generating function of the Pauli Gaussian Leonardo quaternions is*

$$g(t) = \frac{2(\alpha+i)\alpha^* e^{\alpha t} - 2(\beta+i)\beta^* e^{\beta t}}{\alpha-\beta} - Pe^t.$$

Proof. Let $g(t) = \sum_{n=0}^{\infty} Q_PGLe_n \dfrac{t^n}{n!}$ be the exponential generating function for the Pauli Gaussian Leonardo quaternions. Then, from (2.12), we have

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty}\left(\frac{2(\alpha+i)\alpha^*\alpha^n - 2(\beta+i)\beta^*\beta^n}{\alpha-\beta} - P\right)\frac{t^n}{n!} \\ &= \frac{2(\alpha+i)\alpha^*}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!} - \frac{2(\beta+i)\beta^*}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!} - P\sum_{n=0}^{\infty}\frac{t^n}{n!} \\ &= \frac{2(\alpha+i)\alpha^*}{\alpha-\beta}e^{\alpha t} - \frac{2(\beta+i)\beta^*}{\alpha-\beta}e^{\beta t} - Pe^t. \end{aligned}$$

Hence, the proof is completed. □

Theorem 2.9. *Let Q_PGLe_n be the n-th Pauli Gaussian Leonardo quaternion. Then, we have*

$$\sum_{k=0}^{n} Q_PGLe_k = Q_PGLe_{n+2} - (n+2)P - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}), \tag{2.14}$$

$$\sum_{k=0}^{n} Q_PGLe_{2k} = Q_PGLe_{2n+1} - (n+2)P + 2(\mathbf{1} + i\sigma_{\mathbf{1}} - \sigma_{\mathbf{3}}), \tag{2.15}$$

$$\sum_{k=0}^{n} Q_PGLe_{2k+1} = Q_PGLe_{2n+2} - (n+2)P + 2(i\mathbf{1} - \sigma_{\mathbf{2}} - (2+i)\sigma_{\mathbf{3}}). \tag{2.16}$$

Proof. (2.14): By virtue of (1.12), and (2.1), we get

$$\begin{aligned} \sum_{k=0}^{n} Q_PGLe_k &= \sum_{k=0}^{n}\left(GLe_k\mathbf{1} + GLe_{k+1}\sigma_{\mathbf{1}} + GLe_{k+2}\sigma_{\mathbf{2}} + GLe_{k+3}\sigma_{\mathbf{3}}\right) \\ &= \left(\sum_{k=0}^{n} GLe_k\right)\mathbf{1} + \left(\sum_{k=0}^{n} GLe_{k+1}\right)\sigma_{\mathbf{1}} + \left(\sum_{k=0}^{n} GLe_{k+2}\right)\sigma_{\mathbf{2}} + \left(\sum_{k=0}^{n} GLe_{k+3}\right)\sigma_{\mathbf{3}} \\ &= \left(GLe_{n+2} - (n+2)(1+i)\right)\mathbf{1} + \left(GLe_{n+3} - (n+2)(1+i) - 2\right)\sigma_{\mathbf{1}} \\ &\quad + \left(GLe_{n+4} - (n+2)(1+i) - (4+2i)\right)\sigma_{\mathbf{2}} + \left(GLe_{n+5} - (n+2)(1+i) - (8+4i)\right)\sigma_{\mathbf{3}} \\ &= GLe_{n+2}\mathbf{1} + GLe_{n+3}\sigma_{\mathbf{1}} + GLe_{n+4}\sigma_{\mathbf{2}} + GLe_{n+5}\sigma_{\mathbf{3}} \\ &\quad - (n+2)(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}) \\ &= Q_PGLe_{n+2} - (n+2)P - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}). \end{aligned}$$

By considering (1.13) and (1.14), Equations (2.15) and (2.16) can be obtained similarly. □

Example 2.10. For $n=3$ in Theorem 2.9, we obtain

$$\begin{aligned} \sum_{k=0}^{3} Q_PGLe_k &= Q_PGLe_0 + Q_PGLe_1 + Q_PGLe_2 + Q_PGLe_3 \\ &= (1-i)\mathbf{1} + (1+i)\sigma_{\mathbf{1}} + (3+i)\sigma_{\mathbf{2}} + (5+3i)\sigma_{\mathbf{3}} \\ &\quad + (1+i)\mathbf{1} + (3+i)\sigma_{\mathbf{1}} + (5+3i)\sigma_{\mathbf{2}} + (9+5i)\sigma_{\mathbf{3}} \\ &\quad + (3+i)\mathbf{1} + (5+3i)\sigma_{\mathbf{1}} + (9+5i)\sigma_{\mathbf{2}} + (15+9i)\sigma_{\mathbf{3}} \\ &\quad + (5+3i)\mathbf{1} + (9+5i)\sigma_{\mathbf{1}} + (15+9i)\sigma_{\mathbf{2}} + (25+15i)\sigma_{\mathbf{3}} \\ &= (10+4i)\mathbf{1} + (18+10i)\sigma_{\mathbf{1}} + (32+18i)\sigma_{\mathbf{2}} + (54+32i)\sigma_{\mathbf{3}} \end{aligned}$$

$$
\begin{aligned}
&Q_PGLe_5 - 5P - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}) \\
&\quad = (15+9i)\mathbf{1} + (25+15i)\sigma_{\mathbf{1}} + (41+25i)\sigma_{\mathbf{2}} + (67+41i)\sigma_{\mathbf{3}} \\
&\qquad - 5(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}) \\
&\quad = (10+4i)\mathbf{1} + (18+10i)\sigma_{\mathbf{1}} + (32+18i)\sigma_{\mathbf{2}} + (54+32i)\sigma_{\mathbf{3}},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{3} Q_PGLe_{2k} &= Q_PGLe_0 + Q_PGLe_2 + Q_PGLe_4 + Q_PGLe_6 \\
&= (1-i)\mathbf{1} + (1+i)\sigma_{\mathbf{1}} + (3+i)\sigma_{\mathbf{2}} + (5+3i)\sigma_{\mathbf{3}} \\
&\quad + (3+i)\mathbf{1} + (5+3i)\sigma_{\mathbf{1}} + (9+5i)\sigma_{\mathbf{2}} + (15+9i)\sigma_{\mathbf{3}} \\
&\quad + (9+5i)\mathbf{1} + (15+9i)\sigma_{\mathbf{1}} + (25+15i)\sigma_{\mathbf{2}} + (41+25i)\sigma_{\mathbf{3}} \\
&\quad + (25+15i)\mathbf{1} + (41+25i)\sigma_{\mathbf{1}} + (67+41i)\sigma_{\mathbf{2}} + (109+67i)\sigma_{\mathbf{3}} \\
&= (38+20i)\mathbf{1} + (62+38i)\sigma_{\mathbf{1}} + (104+62i)\sigma_{\mathbf{2}} + (170+104i)\sigma_{\mathbf{3}}
\end{aligned}
$$

$$
\begin{aligned}
&Q_PGLe_7 - 5P + 2(\mathbf{1} + i\sigma_{\mathbf{1}} - \sigma_{\mathbf{3}}) = (41+25i)\mathbf{1} + (67+41i)\sigma_{\mathbf{1}} + (109+67i)\sigma_{\mathbf{2}} + (177+109i)\sigma_{\mathbf{3}} \\
&\qquad - 5(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) + 2(\mathbf{1} + i\sigma_{\mathbf{1}} - \sigma_{\mathbf{3}}) \\
&\quad = (38+20i)\mathbf{1} + (62+38i)\sigma_{\mathbf{1}} + (104+62i)\sigma_{\mathbf{2}} + (170+104i)\sigma_{\mathbf{3}}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{3} Q_PGLe_{2k+1} &= Q_PGLe_1 + Q_PGLe_3 + Q_PGLe_5 + Q_PGLe_7 \\
&= (1+i)\mathbf{1} + (3+i)\sigma_{\mathbf{1}} + (5+3i)\sigma_{\mathbf{2}} + (9+5i)\sigma_{\mathbf{3}} \\
&\quad + (5+3i)\mathbf{1} + (9+5i)\sigma_{\mathbf{1}} + (15+9i)\sigma_{\mathbf{2}} + (25+15i)\sigma_{\mathbf{3}} \\
&\quad + (15+9i)\mathbf{1} + (25+15i)\sigma_{\mathbf{1}} + (41+25i)\sigma_{\mathbf{2}} + (67+41i)\sigma_{\mathbf{3}} \\
&\quad + (41+25i)\mathbf{1} + (67+41i)\sigma_{\mathbf{1}} + (109+67i)\sigma_{\mathbf{2}} + (177+109i)\sigma_{\mathbf{3}} \\
&= (62+38i)\mathbf{1} + (104+62i)\sigma_{\mathbf{1}} + (170+104i)\sigma_{\mathbf{2}} + (278+170i)\sigma_{\mathbf{3}}
\end{aligned}
$$

$$
\begin{aligned}
&Q_PGLe_8 - 5P + 2(i\mathbf{1} - \sigma_{\mathbf{2}} - (2+i)\sigma_{\mathbf{3}}) = (67+41i)\mathbf{1} + (109+67i)\sigma_{\mathbf{1}} + (177+109i)\sigma_{\mathbf{2}} \\
&\qquad + (287+177i)\sigma_{\mathbf{3}} - 5(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) + 2(i\mathbf{1} - \sigma_{\mathbf{2}} - (2+i)\sigma_{\mathbf{3}}) \\
&\quad = (62+38i)\mathbf{1} + (104+62i)\sigma_{\mathbf{1}} + (170+104i)\sigma_{\mathbf{2}} + (278+170i)\sigma_{\mathbf{3}},
\end{aligned}
$$

respectively.

3. Conclusions

In the current study, complex (Gaussian) numbers, quaternions, Pauli matrices, and Leonardo numbers are combined to develop a new class of quaternions. These newly defined quaternions are referred to as Pauli Gaussian Leonardo quaternions. Moreover, many formulas for these quaternions, such as the recurrence relation, Binet-like formula, ordinary generating function, exponential generating function, and some summation formulas, are presented. Also, some relationships between the Pauli Gaussian Leonardo quaternions, Pauli Gaussian Fibonacci quaternions, and Pauli Gaussian Lucas quaternions are established. Additionally, examples of some results obtained in this paper are provided.

Pauli matrices and quaternions are extensively used in mathematics and physics, especially in quantum mechanics. We believe that the new number family we propose may offer a new perspective to researchers working in the related fields.

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