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Lyapunov stability of an upwind difference scheme for a quasilinear hyperbolic system

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Abstract. The paper considers a mixed problem for a quasilinear system of hyperbolic equations in Riemann invariants with dissipative nonlinear boundary conditions. To numerically solve the mixed problem, an initial-boundary difference problem based on an upwind difference scheme is proposed. A discrete Lyapunov function is constructed for the numerical solution of the initial-boundary difference problem for nonlinear problems. A theorem on the exponential stability of the stationary state of a quasilinear system is proven.

Keywords: Exponential stability, hyperbolic system, mixed problem
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1. Introduction

One of the promising directions in the theory of stability of nonlinear difference schemes seems to us to be a generalization of the direct Lyapunov method (see, for example, [1], which allows us to reduce the study of the stability of systems of nonlinear hyperbolic equations to the construction of a positive definite function that monotonically decreases on the solution of this system (function Lyapunov). The method of [2] Lyapunov functions has found its application in the theory of stability of ordinary difference equations [3, 4]. There are generalizations of the direct Lyapunov method to partial differential equations [5, 6, 7, 8, 9] although this method is not so popular here. Note, however, that the method of energy inequalities, widely used in the theory of linear partial differential equations, is, in essence, a special case of the direct [2] Lyapunov method (in this case, the Lyapunov function is constructed in the form of some quadratic form from the solution of the problem). All this allows us to hope to generalize the method of Lyapunov functions to nonlinear difference schemes.

Note that works [10, 11, 12, 13, 14, 15, 16, 17] are devoted to the construction of the Lyapunov function for linear difference schemes.

2. Method

Extensive literature is devoted to the issues of studying the stability of difference schemes. The most complete results were obtained for linear schemes with constant (in time) operators; linear circuits with variable operators, and especially nonlinear circuits, have been studied much less well. Two methods for identifying the stability of linear difference schemes are widely used: the spectral method and the method of energy inequalities, the latter being, in essence, a difference analogue of the Lyapunov method of functions. This paper sets out stability criteria for nonlinear difference schemes. The proposed criterion is a sufficient condition for the stability of the circuit in terms of Lyapunov vector functions.

Well-known methods of Lyapunov stability analysis are based on the qualitative theory of ordinary differential equations. Research on this basis is necessary for the theory and practice of automatic regulation, control and monitoring, and super-operational control. As a rule, sustainability analysis is carried out either a priori, before creating a control system, or a posteriori, based on the results of operation. However, stability monitoring is important for the current state of the system.

The need to study the stability of motion or some state arises at all stages of the design or study of physical systems. For the first time, a rigorous mathematical definition of stability and exact methods for solving the issue of stability for a fairly wide class of systems were given by A.M. Lyapunov in his famous [2].

This work was the logical conclusion of the entire previous stage in the development of the theory of stability. With its advent, stability theory reached the level of an independent discipline, taking its rightful place among other mathematical disciplines. A.M. Lyapunov proposed two methods for analyzing the stability of solutions to ordinary differential equations. The first method consists in constructing the solutions of the differential equations of perturbed motions themselves in the form of certain series. Based on subsequent qualitative research of these solutions, conclusions are drawn about sustainability or instability. The second method is to find some auxiliary function, the properties of which determine the stability or instability of the solution. Currently, these functions are called Lyapunov functions, and the method is called the Lyapunov function method, the second Lyapunov method, or the direct Lyapunov method.

Lyapunov's work was the starting point for research of this kind. His ideas develop and deepen in many directions. New theorems have been established that expand these methods, many questions of the existence of Lyapunov functions and their effective construction have been solved, questions of stability of unsteady and periodic motions, stability of the first approximation, in critical cases, with constantly acting disturbances, and many others have been studied.

A development of the theory of stability in relation to automatic control and regulation systems is the theory of motion stabilization, which explores such system control modes in which some program motion (unperturbed motion) of the system will be stable in one sense or another. In many cases, along with the requirement of stability of undisturbed motion, additional requirements are imposed both on the nature of transient processes and on control actions. Often these requirements can be expressed in the form of a minimum of some integral functional. Stabilization problems with these additional requirements are called optimal stabilization problems, or analytical design of regulators.

3. Results and Discussions

Statement of a quasilinear mixed problem. According to the work [18] in the region $\bar{\omega} \triangleq \{(t,x) : 0 \le t \le T, 0 \le x \le L\}$ we consider a mixed problem for the following quasilinear hyperbolic system

$$\begin{cases} \xi_t + \varphi(\xi,\eta)\,\xi_x = 0, \quad \gamma_t + \varphi(\xi,\eta)\,\gamma_x + \gamma f = 0, \quad \rho_t + \varphi(\xi,\eta)\,\rho_x + \rho f + \gamma f_x = 0, \\ \eta_t - \psi(\xi,\eta)\,\eta_x = 0, \quad \delta_t - \psi(\xi,\eta)\,\delta_x - \delta p = 0, \quad \theta_t - \psi(\xi,\eta)\,\theta_x - \theta p - \delta p_x = 0, \\ 0 < t \le T,\ 0 < x < L \end{cases} \tag{3.1}$$

with boundary conditions

$$\begin{aligned} &\text{at } x = 0: \\ &\begin{cases} \xi(t,0) = a(\eta(t,0)), \quad \varphi(t,0)\,\gamma(t,0) = -a'(\eta(t,0))\,\psi(t,0)\,\delta(t,0), \\ \varphi(t,0)\,\rho(t,0) + \gamma(t,0)\,f(t,0) = -e'(t)\,\delta(t,0) - e(t)\left[\psi(t,0)\,\theta(t,0) + \delta(t,0)\,p(t,0)\right], \end{cases} \\ &\text{at } x = L: \\ &\begin{cases} \eta(t,L) = b(\xi(t,L)), \quad \psi(t,L)\,\delta(t,L) = -b'(\xi(t,L))\,\varphi(t,L)\,\gamma(t,L), \\ \psi(t,L)\,\theta(t,L) + \delta(t,L)\,p(t,L) = -h'(t)\,\gamma(t,L) + h(t)\left[\varphi(t,L)\,\rho(t,L) + \gamma(t,L)\,f(t,L)\right], \end{cases} \end{aligned} \tag{3.2}$$

and with initial data

$$\begin{cases} \xi(0,x) = \xi_0(x), \quad \gamma(0,x) = \xi_0'(x), \quad \rho(0,x) = \xi_0''(x), \\ \eta(0,x) = \eta_0(x), \quad \delta(0,x) = \eta_0'(x), \quad \theta(0,x) = \eta_0''(x), \end{cases} \quad 0 < x < L \tag{3.3}$$

Here $\xi = \xi(t,x)$, $\eta = \eta(t,x)$, $\gamma = \gamma(t,x) = \xi_x$, $\delta = \delta(t,x) = \eta_x$, $\rho = \rho(t,x) = \gamma_x$, $\theta = \theta(t,x) = \delta_x$ are unknown functions to be determined, and $\varphi = \varphi(\xi,\eta)$, $\psi = \psi(\xi,\eta)$ are given functions that have continuous derivatives up to the second order inclusive. We will assume that $a, b \in C^2(\mathrm{R})$.

$$f = \gamma\frac{\partial\varphi}{\partial\xi} + \delta\frac{\partial\varphi}{\partial\eta}, \quad p = \gamma\frac{\partial\psi}{\partial\xi} + \delta\frac{\partial\psi}{\partial\eta}.$$

Here

$$\varphi(t,0) = \varphi(\xi(t,0), \eta(t,0)), \qquad \psi(t,0) = \psi(\xi(t,0), \eta(t,0)),$$
$$\varphi(t,L) = \varphi(\xi(t,L), \eta(t,L)), \qquad \psi(t,L) = \psi(\xi(t,L), \eta(t,L)),$$

$$f(t,0) = f(\xi(t,0), \eta(t,0), \gamma(t,0), \delta(t,0)), \qquad p(t,0) = p(\xi(t,0), \eta(t,0), \gamma(t,0), \delta(t,0)),$$
$$f(t,L) = f(\xi(t,L), \eta(t,L), \gamma(t,L), \delta(t,L)), \qquad p(t,L) = p(\xi(t,L), \eta(t,L), \gamma(t,L), \delta(t,L)).$$

The functions $e(t)$ and $h(t)$ are defined as

$$e(t) := \frac{a'(\eta(t,0))\,\psi(t,0)}{\varphi(t,0)}, \quad h(t) := \frac{b'(\xi(t,L))\,\varphi(t,L)}{\psi(t,L)}.$$

Exponential stability of the numerical solution of a nonlinear initial-boundary difference problem. In this section we establish the exponential stability of the numerical solution of the initial-boundary difference problem for the mixed problem (3.1), (3.2), (3.3).

To obtain the initial-boundary difference problem, we will use an upwind difference scheme for the numerical calculation of system (3.1).

To do this, we cover the spatial region [0, 1] using a uniform grid $\Omega_{\Delta x} = \left\{x_j = j\cdot\Delta x,\ \ j=\overline{0,J}\right\}$, Δx step by x.

To numerically solve the mixed problem (3.1), (3.2), (3.3), we propose the following upwind explicit difference scheme

$$\begin{cases} \xi_j^{k+1} = \left(1-[C_\varphi]_{j-1}^k\right)\xi_j^k + [C_\varphi]_{j-1}^k\,\xi_{j-1}^k,\ k=\overline{0,K-1},\ \ j=\overline{1,J}, \\ \eta_j^{k+1} = \left(1-[C_\psi]_{j+1}^k\right)\eta_j^k + [C_\psi]_{j+1}^k\,\eta_{j+1}^k,\ k=\overline{0,K-1},\ \ j=\overline{0,J-1}, \\ \gamma_j^{k+1} = \left(1-[C_\varphi]_{j-1}^k\right)\gamma_j^k + [C_\varphi]_{j-1}^k\,\gamma_{j-1}^k - \Delta t\cdot\gamma_j^k f_j^k,\ k=\overline{0,K-1},\ \ j=\overline{1,J}, \\ \delta_j^{k+1} = \left(1-[C_\psi]_{j+1}^k\right)\delta_j^k + [C_\psi]_{j+1}^k\,\delta_{j+1}^k + \Delta t\cdot\delta_j^k p_j^k,\ k=\overline{0,K-1},\ \ j=\overline{0,J-1}, \\ \rho_j^{k+1} = \left(1-[C_\varphi]_{j-1}^k\right)\rho_j^k + [C_\varphi]_{j-1}^k\,\rho_{j-1}^k - \Delta t\cdot\left[\rho_j^k f_j^k + \gamma_j^k\left(\frac{\partial f}{\partial x}\right)_j^k\right],\ j=\overline{1,J}, \\ \theta_j^{k+1} = \left(1-[C_\psi]_{j+1}^k\right)\theta_j^k + [C_\psi]_{j+1}^k\,\theta_{j+1}^k + \Delta t\cdot\left[\theta_j^k p_j^k + \delta_j^k\left(\frac{\partial p}{\partial x}\right)_j^k\right],\ j=\overline{0,J-1}, \end{cases} \tag{3.4}$$

with boundary conditions

$$\begin{cases} \xi_0^k = a\left(\eta_0^k\right), \quad \varphi_0^k\gamma_0^k = -a'\left(\eta_0^k\right)\psi_0^k\delta_0^k, \quad \varphi_0^k\rho_0^k + \gamma_0^k f_0^k = -\left(e'(t)\right)^k\delta_0^k - e^k\left[\psi_0^k\theta_0^k + \delta_0^k p_0^k\right], \\ \eta_J^k = b\left(\xi_J^k\right), \quad \psi_J^k\delta_J^k = -b'\left(\xi_J^k\right)\varphi_J^k\gamma_J^k, \quad \psi_J^k\theta_J^k + \delta_J^k p_J^k = -\left(h'(t)\right)^k\gamma_J^k + h^k\left[\varphi_J^k\rho_J^k + \gamma_J^k p_J^k\right], \end{cases} \tag{3.5}$$

and with initial data

$$\begin{aligned} &\xi_j^0 = \xi_0(x_j),\ \ \eta_j^0 = \eta_0(x_j),\ \ \gamma_j^0 = \xi_0'(x_j),\ \ \delta_j^0 = \eta_0'(x_j), \\ &\rho_j^0 = \xi_0''(x_j),\ \ \theta_j^0 = \eta_0''(x_j),\ \ j\in\{0,1,2,\cdots,J\}. \end{aligned} \tag{3.6}$$

Let us introduce the following vectors into consideration

$$\vec{\xi} = (\xi,\gamma,\rho),\ \vec{\eta} = (\eta,\delta,\theta),\ \vec{\xi^*} = (\xi^*,\gamma^*,\rho^*),\ \vec{\eta^*} = (\eta^*,\delta^*,\theta^*),\ \vec{\phi}_1 = (\xi_0,\xi_0',\xi''),\ \vec{\phi}_2 = (\eta_0,\eta_0',\eta'')$$

and the following matrices:

$$\mathsf{U}^k \triangleq diag\left(\vec{\eta}_0^k,\vec{\xi}_1^k,\vec{\eta}_1^k,\cdots\cdots,\vec{\xi}_{J-1}^k,\vec{\eta}_{J-1}^k,\vec{\xi}_J^k\right), \qquad \mathsf{U}^* \triangleq diag\left(\vec{\eta^*},\overbrace{\vec{\xi^*},\vec{\eta^*},\cdots\cdots,\vec{\xi^*},\vec{\eta^*}}^{6J},\vec{\xi^*}\right),$$

$$\mathsf{U}^0 \triangleq diag\left(\vec{\phi}_2(x_0),\vec{\phi}_1(x_1),\vec{\phi}_2(x_1),\cdots\cdots,\vec{\phi}_1(x_{J-1}),\vec{\phi}_2(x_{J-1}),\vec{\phi}_1(x_J)\right).$$

Definition 3.1. The equilibrium state $\mathbf{U}^*$ of the initial-boundary difference problem (3.4), (3.5), (3.6) is stable in the l^2 -norm if there exist positive real constants $n_1,\ n_2$ such that for any initial condition $\mathbf{\Phi}$ the solution to $\mathbf{U}^k$, $k \in \{1, 2, \cdots\}$ the initial-boundary difference problem (3.4), (3.5), (3.6) satisfies the inequality

$$\left\|\mathbf{U}^k - \mathbf{U}^*\right\|_{l^2} \leq n_2 e^{-n_1 t^k} \left\|\mathbf{\Phi} - \mathbf{U}^*\right\|_{l^2}, \quad k \in \{1, 2, \cdots\}, \tag{3.7}$$

where

$$\mathbf{U}^k \triangleq \left(\vec{\eta}_0^k, \vec{\xi}_1^k, \vec{\eta}_1^k, \cdots\cdots, \vec{\xi}_{J-1}^k, \vec{\eta}_{J-1}^k, \vec{\xi}_J^k\right)^T, \qquad \mathbf{U}^* \triangleq \overbrace{\left(\vec{\eta}^*, \vec{\xi}^*, \vec{\eta}^*, \cdots\cdots, \vec{\xi}^*, \vec{\eta}^*, \vec{\xi}^*\right)^T}^{6J},$$

$$\mathbf{\Phi} \triangleq \left(\vec{\phi}_2(x_0), \vec{\phi}_1(x_1), \vec{\phi}_2(x_1), \cdots\cdots, \vec{\phi}_1(x_{J-1}), \vec{\phi}_2(x_{J-1}), \vec{\phi}_1(x_J)\right).$$

and

$$\begin{aligned}
&\left\|\mathbf{U}^k - \mathbf{U}^*\right\|_{l^2}^2 \triangleq \Delta x \left(\left[\vec{\eta}_0^k - \vec{\eta}^*\right]^T, \left[\vec{\eta}_0^k - \vec{\eta}^*\right]^T\right) + h\left(\left[\vec{\xi}_J^k - \vec{\xi}^*\right]^T, \left[\vec{\xi}_J^k - \vec{\xi}^*\right]^T\right) + \\
&\quad + \Delta x \sum_{j=1}^{J-1} \left\{\left(\left[\vec{\xi}_j^k - \vec{\xi}^*\right]^T, \left[\vec{\xi}_j^k - \vec{\xi}^*\right]^T\right) + \left(\left[\vec{\eta}_j^k - \vec{\eta}^*\right]^T, \left[\vec{\eta}_j^k - \vec{\eta}^*\right]^T\right)\right\}, \\
&\left\|\mathbf{\Phi} - \mathbf{U}^*\right\|_{l^2} \triangleq \Delta x \left(\left[\vec{\phi}_2(x_0) - \vec{\eta}^*\right]^T, \left[\vec{\phi}_2(x_0) - \vec{\eta}^*\right]^T\right) + \Delta x \left(\left[\vec{\phi}_1(x_J) - \vec{\xi}^*\right]^T, \left[\vec{\phi}_1(x_J) - \vec{\xi}^*\right]^T\right) + \\
&\quad + \Delta x \sum_{j=1}^{J-1} \left\{\left(\left[\vec{\phi}_1(x_j) - \vec{\xi}^*\right]^T, \left[\vec{\phi}_1(x_j) - \vec{\xi}^*\right]^T\right) + \left(\left[\vec{\phi}_2(x_j) - \vec{\eta}^*\right]^T, \left[\vec{\phi}_2(x_j) - \vec{\eta}^*\right]^T\right)\right\}, \\
&\qquad k \in \{0, 1, \cdots\}.
\end{aligned}$$

Definition 3.2. (Discrete Lyapunov function). It is said that the function $\mathbf{L}^k : \mathrm{R}^{n\times J} \to \mathrm{R}_0^+$ is a discrete Lyapunov function for the initial-boundary difference problem (3.4), (3.5), (3.6), if

(1) there are positive constants h_1 and h_2 that for all $k \in \{0, 1, \cdots\}$:

$$h_1 \left\|\mathbf{U}^k - \mathbf{U}^*\right\|_{l^2}^2 \leq \mathbf{L}^k\left(\mathbf{U}^k\right) \leq h_2 \left\|\mathbf{U}^k - \mathbf{U}^*\right\|_{l^2}^2, \tag{3.8}$$

(2) there is a positive constant n such that for all $k \in \{0, 1, \cdots\}$:

$$\frac{\mathbf{L}^k\left(\mathbf{U}^{k+1}\right) - \mathbf{L}^k\left(\mathbf{U}^k\right)}{\Delta t} \leq -n\mathbf{L}^k\left(\mathbf{U}^k\right). \tag{3.9}$$

To simplify the notation, in what follows we define a sequence of discrete values $\mathcal{L}^k$ as

$$\mathcal{L}^k = \mathbf{L}^k\left(\mathbf{U}^k\right), \quad k \in \{0, 1, \cdots\}$$

where $\mathbf{U}^k$ a given solution of the initial-boundary difference problem (3.4), (3.5), (3.6).

It should be noted that the presence of a discrete Lyapunov function ensures the stability of the equilibrium state of $\mathbf{U}^*$ the initial-boundary difference problem (3.4), (3.5), (3.6) in the l^2 -norm.

First, let us separately consider the difference equations for ξ_j^k and η_j^k system (3.4):

$$\left\{\begin{array}{ll}
\xi_j^{k+1} = \left(1 - \left[C_\varphi\right]_{j-1}^k\right)\xi_j^k + \left[C_\varphi\right]_{j-1}^k \xi_{j-1}^k, & k = \overline{0, K-1}, \quad j = \overline{1, J}, \\
\eta_j^{k+1} = \left(1 - \left[C_\psi\right]_{j+1}^k\right)\eta_j^k + \left[C_\psi\right]_{j+1}^k \eta_{j+1}^k, & k = \overline{0, K-1}, \quad j = \overline{0, J-1},
\end{array}\right. \tag{3.10}$$

where

$$\left[C_\varphi\right]_j^k = \varphi_j^k \frac{\Delta t}{\Delta x}, \quad \left[C_\psi\right]_j^k = \psi_j^k \frac{\Delta t}{\Delta x}, \quad C_j^k = \max\left(\left[C_\varphi\right]_j^k, \left[C_\psi\right]_j^k\right).$$

$$\bar{\varphi} = \varphi(0,0) > 0, \quad \bar{\psi} = \psi(0,0) > 0$$

As a discrete Lyapunov function for (3.10), we consider discrete function

$$\mathcal{L}^k = \mathcal{L}_1^k + \mathcal{L}_2^k, \qquad \mathcal{L}_1^k = \frac{\mathrm{A}}{\bar{\varphi}}\Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 \exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right), \quad \mathcal{L}_2^k = \frac{\mathrm{B}}{\bar{\psi}}\Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \exp\left(\frac{m}{\bar{\psi}}x_{j+1}\right).$$

The difference time derivative of the discrete Lyapunov function on solutions of difference equations (3.4) is

$$\frac{\mathcal{L}^{k+1}-\mathcal{L}^{k}}{\Delta t}=\frac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}+\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^{k}}{\Delta t} \tag{3.11}$$

Let us calculate the difference ratios the right side of (3.11) separately.

Lemma 3.3. *For grid functions* ξ_j^k *satisfying difference equations* (3.10), *the following inequality holds:*

$$\begin{gathered}\tfrac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}\leq-\tfrac{\mathrm{A}}{\bar\varphi}\left[\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)\right]\Big|_0^J-m\tfrac{\mathrm{A}}{\bar\varphi^2}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)+\\ +\tfrac{\mathrm{A}}{\bar\varphi}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2f_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right).\end{gathered} \tag{3.12}$$

with

$$f_j^k=\frac{\varphi_j^k-\varphi_{j-1}^k}{\Delta x}=\frac{\partial\varphi\left(\xi_j^k,\eta_j^k\right)}{\partial\xi}\gamma_j^k+\frac{\partial\varphi\left(\xi_{j-1}^k,\eta_j^k\right)}{\partial\eta}\delta_j^k$$

Proof. Substituting the value of the expression for ξ_j^{k+1} from (3.10) into expression $\frac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}$ we get

$$\begin{gathered}\tfrac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}=\tfrac{\mathrm{A}}{\bar\varphi}\Delta x\sum_{j=1}^{J}\left[\tfrac{\left(\xi_j^{k+1}\right)^2-\left(\xi_j^k\right)^2}{\Delta t}\right]\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)=\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left(\xi_j^{k+1}\right)^2-\left(\xi_j^k\right)^2\right]\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)=\\ =\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left\{\left(1-[C_\varphi]_{j-1}^k\right)\xi_j^k+[C_\varphi]_{j-1}^k\,\xi_{j-1}^k\right\}^2-\left(\xi_j^k\right)^2\right]\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right).\end{gathered} \tag{3.13}$$

Using Jensen's inequality to evaluate the expression $\left\{\left(1-[C_\varphi]_{j-1}^k\right)\xi_j^k+[C_\varphi]_{j-1}^k\,\xi_{j-1}^k\right\}^2$ from above we have

$$\frac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}=\frac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left(\xi_{j-1}^k\right)^2-\left(\xi_j^k\right)^2\right][C_\varphi]_{j-1}^k\exp\left(-\frac{m}{\bar\varphi}x_{j-1}\right). \tag{3.14}$$

Using the difference formula

$$\begin{gathered}\left(u_j-u_{j-1}\right)v_{j-1}w_{j-1}=\left(u_jv_jw_j-u_{j-1}v_{j-1}w_{j-1}\right)-u_j\left(v_jw_j-v_{j-1}w_{j-1}\right)=\\ =\left(u_jv_jw_j-u_{j-1}v_{j-1}w_{j-1}\right)-u_j\left[\left(v_j-v_{j-1}\right)w_j+v_{j-1}\left(w_j-w_{j-1}\right)\right]\end{gathered} \tag{3.15}$$

and assuming $u_j=\left(\xi_j^k\right)^2,\ v_j=[C_\varphi]_j^k,\ w_j=\exp\left(-\frac{m}{\bar\varphi}x_j\right)$ from (3.14) we obtain the inequality

$$\begin{gathered}\tfrac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}\leq\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left(\xi_{j-1}^k\right)^2[C_\varphi]_{j-1}^k\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)-\left(\xi_j^k\right)^2[C_\varphi]_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)\right]+\\ +\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left\{\left[[C_\varphi]_j^k-[C_\varphi]_{j-1}^k\right]\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)+[C_\varphi]_{j-1}^k\left[\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)-\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)\right]\right\}.\end{gathered}$$

Therefore, taking into account equality

$$\begin{gathered}\left[\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)-\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)\right]=\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)\left(\exp\left(-\tfrac{m}{\bar\varphi}\Delta x\right)-1\right)=\\ =\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)\left[-\tfrac{m}{\bar\varphi}\Delta x+O\left((\Delta x)^2\right)\right].\end{gathered}$$

with accuracy $O\left(\Delta x\right)$ from inequality (3.14) we obtain

$$\begin{gathered}\tfrac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}\leq-\tfrac{\mathrm{A}}{C_\varphi}\left[\left(\xi_j^k\right)^2[C_\varphi]_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)\right]\Big|_0^J-m\tfrac{\mathrm{A}}{\bar\varphi C_\varphi}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2[C_\varphi]_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)+\\ +\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left[[C_\varphi]_j^k-[C_\varphi]_{j-1}^k\right]\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)=-\tfrac{\mathrm{A}}{\bar\varphi}\left[\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)\right]\Big|_0^J-\\ -m\tfrac{\mathrm{A}}{\bar\varphi^2}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_{j-1}\right)+\tfrac{\mathrm{A}}{\bar\varphi}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2f_j^k\exp\left(-\tfrac{m}{\bar\varphi}x_j\right)\end{gathered}$$

Lemma 3.3 is proven. □

Lemma 3.4. *For grid functions η_j^k satisfying difference equations* (3.10)*, the following inequality holds:*

$$\begin{aligned}\tfrac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} \le \tfrac{\mathrm{B}}{\psi}\left[\left(\eta_j^k\right)^2\psi_j^k\exp\left(\tfrac{m}{\psi}x_j\right)\right]\Big|_0^J - m\tfrac{\mathrm{B}}{\psi^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\psi_{j+1}^k\exp\left(\tfrac{m}{\psi}x_{j+1}\right)-\\ -\tfrac{\mathrm{B}}{\psi}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2 p_j^k\exp\left(\tfrac{m}{\psi}x_j\right)\end{aligned} \tag{3.16}$$

with

$$p_j^k = \frac{\psi_{j+1}^k-\psi_j^k}{\Delta x} = \frac{\partial\psi\left(\xi_j^k,\eta_{j+1}^k\right)}{\partial\xi}\gamma_j^k + \frac{\partial\psi\left(\xi_j^k,\eta_j^k\right)}{\partial\eta}\delta_j^k.$$

Proof. Substituting the value of the expression for η_j^{k+1} from (3.10) into expression $\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t}$ we get

$$\begin{aligned}\tfrac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} = \tfrac{\mathrm{B}}{\psi}\Delta x\sum_{j=0}^{J-1}\left[\tfrac{\left(\eta_j^{k+1}\right)^2-\left(\eta_j^k\right)^2}{\Delta t}\right]\exp\left(\tfrac{m}{\psi}x_{j+1}\right) = \tfrac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left(\eta_j^{k+1}\right)^2-\left(\eta_j^k\right)^2\right]\exp\left(\tfrac{m}{\psi}x_{j+1}\right) =\\ = \tfrac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left\{\left(1-\left[C_\psi\right]_{j+1}^k\right)\eta_j^k+\left[C_\psi\right]_{j+1}^k\eta_{j+1}^k\right\}^2-\left(\eta_j^k\right)^2\right]\exp\left(\tfrac{m}{\psi}x_{j+1}\right).\end{aligned} \tag{3.17}$$

Using Jensen's inequality (for convex mappings $y\to y^2$ the inequality $\left[q_1y_1+q_2y_2\right]^2\le q_1\left(y_1\right)^2+q_2\left(y_2\right)^2$ holds, $q_1,\ q_2>0$ and $q_1+q_2=1$), to evaluate the expression $\left\{\left(1-\left[C_\psi\right]_{j+1}^k\right)\eta_j^k+\left[C_\psi\right]_{j+1}^k\eta_{j+1}^k\right\}^2$ from above we have

$$\begin{aligned}\tfrac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} \le \tfrac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left\{\left(1-\left[C_\psi\right]_{j+1}^k\right)\left(\eta_j^k\right)^2+\left[C_\psi\right]_{j+1}^k\left(\eta_{j+1}^k\right)^2\right\}-\left(\eta_j^k\right)^2\right]\exp\left(\tfrac{m}{\psi}x_{j+1}\right) =\\ = \tfrac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left(\eta_{j+1}^k\right)^2-\left(\eta_j^k\right)^2\right]\left[C_\psi\right]_{j+1}^k\exp\left(\tfrac{m}{\psi}x_{j+1}\right)\end{aligned} \tag{3.18}$$

Using the difference formula

$$\begin{aligned}\left(u_{j+1}-u_j\right)v_{j+1}w_{j+1} = \left(u_{j+1}v_{j+1}w_{j+1}-u_jv_jw_j\right)-u_j\left(v_{j+1}w_{j+1}-v_jw_j\right) =\\ = \left(u_{j+1}v_{j+1}w_{j+1}-u_jv_jw_j\right)-u_j\left[\left(v_{j+1}-v_j\right)w_j+v_{j+1}\left(w_{j+1}-w_j\right)\right]\end{aligned} \tag{3.19}$$

and taking into $u_j=\left(\eta_j^k\right)^2,\ \ v_j=\left[C_\psi\right]_j^k,\ \ w_j=\exp\left(\frac{m}{\psi}x_{j+1}\right)$ account equality (3.19) from inequality (3.18) we obtain

$$\begin{aligned}\tfrac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} \le \tfrac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left(\eta_{j+1}^k\right)^2\left[C_\psi\right]_{j+1}^k\exp\left(\tfrac{m}{\psi}x_{j+1}\right)-\left(\eta_j^k\right)^2\left[C_\psi\right]_j^k\exp\left(\tfrac{m}{\psi}x_j\right)\right]-\\ -\tfrac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\left[C_\psi\right]_{j+1}^k-\left[C_\psi\right]_j^k\right]\exp\left(\tfrac{m}{\psi}x_j\right)-\tfrac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[C_\psi\right]_{j+1}^k\left[\exp\left(\tfrac{m}{\psi}x_{j+1}\right)-\exp\left(\tfrac{m}{\psi}x_j\right)\right].\end{aligned}$$

we obtain with accuracy $O\left(\Delta x\right)$

$$\begin{aligned}\tfrac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} \le \tfrac{\mathrm{B}}{C_\psi}\left[\left(\eta_j^k\right)^2\left[C_\psi\right]_j^k\exp\left(\tfrac{m}{\psi}x_j\right)\right]\Big|_0^J - \tfrac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\left[C_\psi\right]_{j+1}^k-\left[C_\psi\right]_j^k\right]\exp\left(\tfrac{m}{\psi}x_j\right)-\\ -\tfrac{\mathrm{B}}{C_\psi}\tfrac{m}{\psi}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[C_\psi\right]_{j+1}^k\exp\left(\tfrac{m}{\psi}x_{j+1}\right) = \tfrac{\mathrm{B}}{\psi}\left[\left(\eta_j^k\right)^2\psi_j^k\exp\left(\tfrac{m}{\psi}x_j\right)\right]\Big|_0^J-\\ -m\tfrac{\mathrm{B}}{\psi^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\psi_{j+1}^k\exp\left(\tfrac{m}{\psi}x_{j+1}\right) - \tfrac{\mathrm{B}}{\psi}\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\psi_{j+1}^k-\psi_j^k\right]\exp\left(\tfrac{m}{\psi}x_j\right) =\\ = \tfrac{\mathrm{B}}{\psi}\left[\left(\eta_j^k\right)^2\psi_j^k\exp\left(\tfrac{m}{\psi}x_j\right)\right]\Big|_0^J - m\tfrac{\mathrm{B}}{\psi^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\psi_{j+1}^k\exp\left(\tfrac{m}{\psi}x_{j+1}\right)-\\ -\tfrac{\mathrm{B}}{\psi}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\tfrac{\partial\psi\left(\xi_j^k,\eta_{j+1}^k\right)}{\partial\xi}\gamma_{j+1}^k-\tfrac{\partial\psi\left(\xi_j^k,\eta_j^k\right)}{\partial b}\delta_j^k\right]\exp\left(\tfrac{m}{\psi}x_j\right).\end{aligned}$$

Lemma 3.4 is proven. ☐

From Lemmas 3.3-3.4 taking into account equality (3.11), we obtain the following inequality

$$\frac{\mathcal{L}^{k+1}-\mathcal{L}^{k}}{\Delta t} \leq \Upsilon_1^k+\Upsilon_2^k+\Upsilon_3^k. \tag{3.20}$$

Here

$$\begin{aligned}
\Upsilon_1^k &= -\tfrac{\mathrm{A}}{\bar{\varphi}}\left[\left(\xi_j^k\right)^2\varphi_j^k \exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)\right]\Big|_0^J+\tfrac{\mathrm{B}}{\bar{\psi}}\left[\left(\eta_j^k\right)^2\psi_j^k \exp\left(\tfrac{m}{\bar{\psi}}x_j\right)\right]\Big|_0^J,\\
\Upsilon_2^k &= -m\left[\tfrac{\mathrm{A}}{\bar{\varphi}^2}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)+\tfrac{\mathrm{B}}{\bar{\psi}^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\psi_{j+1}^k\exp\left(\tfrac{m}{\bar{\psi}}x_{j+1}\right)\right],\\
\Upsilon_3 &= \tfrac{\mathrm{A}}{\bar{\varphi}}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2 f_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)-\tfrac{\mathrm{B}}{\bar{\psi}}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2 p_j^k\exp\left(\tfrac{m}{\bar{\psi}}x_j\right).
\end{aligned}$$

Let us assume that the functions of boundary conditions (3.5) $a, b: \ \mathrm{R}\to\mathrm{R}$, satisfy the inequalities:

$$\begin{aligned}
\max_{0\le k\le K}\left|a\left(\eta_0^k\right)\right|<+\infty,\quad \max_{0\le k\le K}\left|a'\left(\eta_0^k\right)\right|<+\infty,\\
\max_{0\le k\le K}\left|b\left(\xi_J^k\right)\right|<+\infty,\quad \max_{0\le k\le K}\left|b'\left(\xi_J^k\right)\right|<+\infty
\end{aligned} \tag{3.21}$$

and denote

$$\kappa_0=a'(0), \qquad \kappa_L=b'(0).$$

Then we have the following lemma

Lemma 3.5. *If* $|\kappa_0\kappa_L|<1$, *if coefficients* m, A, B *satisfy inequalities* $\mathrm{A}\kappa_0^2-\mathrm{B}<0$ *and* $\mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\kappa_L^2-\mathrm{A}\exp\left(-\frac{m}{\bar{\varphi}}L\right)<0$, *then there are positive real constants* $\mathrm{K}_1,\ d_1,\ l_1$ *such that if* $\left|\xi_j^k\right|+\left|\eta_j^k\right|\le d_1 \quad \forall j\in\{0,1,\cdots,J\}$, *then along the solution of system* (3.4) *with boundary conditions* (3.5) *the following inequality holds*

$$\frac{\mathcal{L}^{k+1}-\mathcal{L}^{k}}{\Delta t}\le -l_1\mathcal{L}^k+\mathrm{K}_1\left\{\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left[\left|\gamma_j^k\right|+\left|\delta_j^k\right|\right]+\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\left|\gamma_j^k\right|+\left|\delta_j^k\right|\right]\right\}.$$

Proof. For expression Υ_1^k the following equalities are true

$$\begin{aligned}
\Upsilon_1^k &= \mathrm{A}\frac{\left(\xi_0^k\right)^2\varphi_0^k}{\bar{\varphi}}-\mathrm{A}\frac{\left(\xi_J^k\right)^2\varphi_J^k}{\bar{\varphi}}\exp\left(-\tfrac{m}{\bar{\varphi}}L\right)+\mathrm{B}\frac{\left(\eta_J^k\right)^2\psi_J^k}{\bar{\psi}}\exp\left(\tfrac{m}{\bar{\psi}}L\right)-\mathrm{B}\frac{\left(\eta_0^k\right)^2\psi_0^k}{\bar{\psi}}=\\
&= A\frac{a^2\left(\eta_0^k\right)\varphi_0^k}{\bar{\varphi}}-\mathrm{A}\frac{\left(\xi_J^k\right)^2\varphi_J^k}{\bar{\varphi}}\exp\left(-\tfrac{m}{\bar{\varphi}}L\right)+\mathrm{B}\frac{b^2\left(\xi_J^k\right)\psi_J^k}{\bar{d}}\exp\left(\tfrac{m}{\bar{\psi}}L\right)-\mathrm{B}\frac{\left(\eta_0^k\right)^2\psi_0^k}{\bar{\psi}}.
\end{aligned}$$

This expression can be represented as

$$\Upsilon_1^k=\Upsilon_{01}^k+\Delta\Upsilon_1^k$$

where, through Υ_{01}^k for small ξ_J^k and η_0^k denoted the terms of the second order of smallness with

$$\varphi_j^k\simeq\bar{\varphi},\ \ \psi_j^k\simeq\bar{\psi},\ \ a\left(\eta_0^k\right)\simeq\kappa_0\eta_0^k,\ \ b\left(\xi_J^k\right)\simeq\kappa_L\xi_J^k.$$

Then it is obvious that for Υ_{01}^k we have

$$\Upsilon_{01}^k=\left[\mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\kappa_L^2-\mathrm{A}\exp\left(-\frac{m}{\bar{\varphi}}L\right)\right]\left(\xi_J^k\right)^2+\left[\mathrm{A}\kappa_0^2-\mathrm{B}\right]\left(\eta_0^k\right)^2$$

which, as was shown earlier in [6], is a non-negative expression $\Upsilon_{01}^k\le 0$. Moreover, the remainder $\Delta\Upsilon_{01}^k$ is denoted by terms of third order of smallness with respect to ξ_J^k and η_0^k (i.e, $\Delta\Upsilon_1^k\approx O\left(\left(\xi_J^k\right)^3,\ \left(\eta_0^k\right)^3\right)$ for small $\left|\xi_J^k\right|$ and $\left|\eta_0^k\right|$. Now let's look at the expression for Υ_2^k. Let us introduce the following notation:

$$\varphi\left(\xi_j^k,\eta_j^k\right)=\bar{\varphi}+\tilde{\varphi}\left(\xi_j^k,\eta_j^k\right),\ \ \psi\left(\xi_j^k,\eta_j^k\right)=\bar{\psi}+\tilde{\psi}\left(\xi_j^k,\eta_j^k\right).$$

Then for the expression Υ_2^k we have

$$\Upsilon_2^k = -mL^k + \Delta\Upsilon_2^k$$

with

$$\Delta\Upsilon_2^k = -m\left[\frac{\mathrm{A}}{\bar{\varphi}^2}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\tilde{\varphi}_j^k\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)+\frac{\mathrm{B}}{\bar{\psi}^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\tilde{\psi}_{j+1}^k\exp\left(\frac{m}{\bar{\psi}}x_{j+1}\right)\right].$$

According to the definitions of $\tilde{\varphi}_j^k$ and $\tilde{\psi}_{j+1}^k$, it is easy to verify that the expression for $\Delta\Upsilon_2$ will take place of the members third order of smallness relative to ξ_j^k and η_j^k:

$$\Delta\Upsilon_2^k \approx O\left(\Delta x\sum_{j=1}^{J}\left|\left(\xi_j^k\right)^3\right|,\ \Delta x\sum_{j=0}^{J-1}\left|\left(\eta_j^k\right)^3\right|\right) \text{ for small } \Delta x\sum_{j=1}^{J}\left|\left(\xi_j^k\right)^3\right| \text{ and } \Delta x\sum_{j=0}^{J-1}\left|\left(\eta_j^k\right)^3\right|.$$

For ξ_j^k and η_j^k satisfying inequalities $\left|\xi_j^k\right|+\left|\eta_j^k\right|\le d_1 \quad \forall j\in\{0,1,\cdots,J\}$

Select a parameter l_1 so $l_1>m$ and $\Upsilon_1^k+\Upsilon_2^k\le -l_1\mathcal{L}^k$ for all k.

One can choose a sufficiently large positive real number K_1 such that

$$\Upsilon_3^k \le \mathrm{K}_1\left\{\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left[\left|\gamma_j^k\right|+\left|\delta_j^k\right|\right]+\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\left|\gamma_j^k\right|+\left|\delta_j^k\right|\right]\right\}.$$

This completes the proof of Lemma 3.5. □

Now consider the difference equations for γ_j^k and δ_j^k from the system (3.7)

$$\left\{\begin{aligned}&\gamma_j^{k+1}=\left(1-\left[C_\varphi\right]_{j-1}^k\right)\gamma_j^k+\left[C_\varphi\right]_{j-1}^k\gamma_{j-1}^k-\Delta t\cdot\gamma_j^kf_j^k, \qquad k=\overline{0,K-1},\ \ j=\overline{1,J},\\&\delta_j^{k+1}=\left(1-\left[C_\psi\right]_{j+1}^k\right)\delta_j^k+\left[C_\psi\right]_{j+1}^k\delta_{j+1}^k+\Delta t\cdot\delta_j^kp_j^k,\ \ k=\overline{0,K-1},\ \ j=\overline{0,J-1},\end{aligned}\right. \tag{3.22}$$

with appropriate boundary conditions

$$\varphi_0^k\gamma_0^k=-a'\left(\eta_0^k\right)\psi_0^k\delta_0^k,\ \ \psi_J^k\delta_J^k=-b'\left(\xi_J^k\right)\varphi_J^k\gamma_J^k. \tag{3.23}$$

Here

$$\varphi_0^k=\varphi\left(\xi_0^k,\eta_0^k\right),\quad \psi_0^k=\psi\left(\xi_0^k,\eta_0^k\right),\ \ \varphi_J^k=\varphi\left(\xi_J^k,\eta_J^k\right),\quad \psi_J^k=\psi\left(\xi_J^k,\eta_J^k\right).$$

Next, consider the system of difference equations for ρ_j^k,θ_j^k

$$\left\{\begin{aligned}&\rho_j^{k+1}=\left(1-\left[C_\varphi\right]_{j-1}^k\right)\rho_j^k+\left[C_\varphi\right]_{j-1}^k\rho_{j-1}^k-\Delta t\cdot\left[\rho_j^kf_j^k+\gamma_j^k\left(\tfrac{\partial f}{\partial x}\right)_j^k\right],\quad j=\overline{1,J},\\&\theta_j^{k+1}=\left(1-\left[C_\psi\right]_{j+1}^k\right)\theta_j^k+\left[C_\psi\right]_{j+1}^k\theta_{j+1}^k+\Delta t\cdot\left[\theta_j^kp_j^k+\delta_j^k\left(\tfrac{\partial p}{\partial x}\right)_j^k\right],\ \ j=\overline{0,J-1},\end{aligned}\right.\quad k=\overline{0,K-1}, \tag{3.24}$$

with appropriate boundary conditions

$$\begin{aligned}&\varphi_0^k\rho_0^k+\gamma_0^kf_0^k=-\left(e'\left(t\right)\right)^k\delta_0^k-e^k\left[\psi_0^k\theta_0^k+\delta_0^kp_0^k\right],\\&\psi_J^k\theta_J^k+\delta_J^kp_J^k=-\left(h'\left(t\right)\right)^k\gamma_J^k+h^k\left[\varphi_J^k\rho_J^k+\gamma_J^kp_J^k\right].\end{aligned} \tag{3.25}$$

Here

$$f_0^k=f\left(\xi_0^k,\ \eta_0^k,\ \gamma_0^k,\ \delta_0^k\right),\ \ p_0^k=p\left(\xi_0^k,\ \eta_0^k,\ \gamma_0^k,\ \delta_0^k\right),$$

$$f_J^k=f\left(\xi_J^k,\ \eta_J^k,\ \gamma_J^k,\ \delta_J^k\right),\ \ p_J^k=p\left(\xi_J^k,\ \eta_J^k,\ \gamma_J^k,\ \delta_J^k\right).$$

The functions e^k and h^k are defined as $e^k:=a'\left(\eta_0^k\right)\psi_0^k/\varphi_0^k,\quad h^k:=b'\left(\xi_J^k\right)\varphi_J^k/\psi_J^k$.

Linearization of systems of difference equations (3.24) and (3.25) (around the origin) have the following form:

$$\gamma_j^{k+1} = (1 - C_\varphi)\,\gamma_j^k + C_\varphi \gamma_{j-1}^k,\ k = \overline{0, K-1},\ \ j = \overline{1, J},$$

$$\delta_j^{k+1} = (1 - C_\psi)\,\delta_j^k + C_\psi \delta_{j+1}^k,\ k = \overline{0, K-1},\ \ j = \overline{0, J-1}$$

respectively

$$\rho_j^{k+1} = (1 - C_\varphi)\,\rho_j^k + C_\varphi \rho_{j-1}^k, \qquad k = \overline{0, K-1},\ \ j = \overline{1, J},$$

$$\theta_j^{k+1} = (1 - C_\psi)\,\theta_j^k + C_\psi \theta_{j+1}^k, \qquad k = \overline{0, K-1},\ \ j = \overline{0, J-1}.$$

It is easy to see that the resulting systems coincide with the linear system with other notations for the dependent variables. That is why this circumstance suggests that we should take as the Lyapunov function

$$\mathbf{L}^k = \mathcal{L}^k + \mathsf{L}^k + \mathfrak{L}^k,$$

where L^k and $\mathfrak{L}^k$ have the format $\mathcal{L}^k$:

$$\mathsf{L}^k = \mathsf{L}_1^k + \mathsf{L}_2^k, \quad \mathsf{L}_1^k = \bar\varphi \mathrm{A} \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \exp\left(-\frac{m}{\bar\varphi} x_{j-1}\right),\ \mathsf{L}_2^k = \bar\psi\,\mathrm{B}\,\Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \exp\left(\frac{m}{\bar\psi} x_{j+1}\right)$$

and

$$\mathfrak{L}^k = \mathfrak{L}_1^k + \mathfrak{L}_2^k,\ \ \mathfrak{L}_1^k = \bar\varphi^3 \mathrm{A} \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 \exp\left(-\frac{m}{\bar\varphi} x_{j-1}\right),\ \ \mathfrak{L}_2^k = \bar\psi^3\,\mathrm{B}\,\Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \exp\left(\frac{m}{\bar\psi} x_{j+1}\right).$$

Now let's study the difference derivatives with respect to time of functions L^k and $\mathfrak{L}^k$ along solutions of the closed-loop system (3.22)-(3.23)-(3.24)-(3.25).

Lemma 3.6. *If* $|\kappa_0 \kappa_L| < 1$*, if positive real constants* $m, \mathrm{A}, \mathrm{B}$ *satisfy inequalities* $\mathrm{A}\kappa_0^2 - \mathrm{B} < 0$ *and* $\mathrm{B}\exp\left(\frac{m}{\bar\psi}L\right)\kappa_L^2 - \mathrm{A}\exp\left(-\frac{m}{\bar\varphi}L\right) < 0$*, then there exist positive real constants* K_2, d_2, l_2 *such that if* $\left|\xi_j^k\right| + \left|\eta_j^k\right| \le d_2 \quad \forall j \in \{0, 1, \cdots, J\}$ *then*

$$\frac{\mathsf{L}^{k+1} - \mathsf{L}^k}{\Delta t} \le -l_2 \mathsf{L}^k + \mathrm{K}_2 \left\{\ \Delta x \sum_{j=1}^{J}\ \left(\gamma_j^k\right)^2 \left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] \right\}$$

along solutions of systems (3.22),(3.24) *with boundary conditions* (3.23),(3.25).

Proof. The proof of Lemma 3.6 is similar to the proof of Lemma 3.5. Therefore, we omit it. □

Lemma 3.7. *If* $|\kappa_0 \kappa_L| < 1$*, if positive real constants* $m, \mathrm{A}, \mathrm{B}$ *satisfy inequalities* $\mathrm{A}\kappa_0^2 - \mathrm{B} < 0$ *and* $\mathrm{B}\exp\left(\frac{m}{\bar\psi}L\right)\kappa_L^2 - \mathrm{A}\exp\left(-\frac{m}{\bar\varphi}L\right) < 0$*, then there are positive real constants* K_3, d_3, l_3 *such that if* $\left|\xi_j^k\right| + \left|\eta_j^k\right| \le d_3 \quad \forall j \in \{0, 1, \cdots, J\}$ *then*

$$\frac{\mathfrak{L}^{k+1} - \mathfrak{L}^k}{\Delta t} \le -l_3 \mathfrak{L}^k + \mathrm{K}_3 \left\{\ \Delta x \sum_{j=1}^{J}\ \left(\rho_j^k\right)^2 \left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] + +\Delta x \sum_{j=0}^{J-1}\ \left(\theta_j^k\right)^2\ \left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] \right\} +$$
$$+\mathrm{K}_3 \left\{\ \Delta x \sum_{j=1}^{J}\ \left(\gamma_j^k\right)^2 \left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] \right\}.$$

along solutions of systems (3.22), (3.24) *with boundary conditions* (3.23), (3.25).

Proof. The proof of Lemma 3.7 is similar to the proof of Lemma 3.5. Therefore, we omit it. □

Lemma 3.8. *If* $|\kappa_0\kappa_L| < 1$, *if positive real constants* $m, \mathrm{A}, \mathrm{B}$ *satisfy inequalities* $\mathrm{A}\kappa_0^2 - \mathrm{B} < 0$ *and* $\mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\kappa_L^2 - \mathrm{A}\exp\left(-\frac{m}{\bar{\varphi}}L\right) < 0$, *then there are positive real constants* l_0 *such* d_0 *that if* $\mathbf{L}^k < d_0$, *then*

$$\left(\mathbf{L}^{k+1} - \mathbf{L}^k\right)\Big/\Delta t \leq -l_0\mathbf{L}^k$$

along closed loop solutions system (3.22), (3.23), (3.24),(3.25).

Proof. In the process of proof, we use discrete versions of some inequalities (a continuous analogue of which can be found in [9]), valid for l^2-functions $\sigma, \varsigma :\ [0, L] \to \mathrm{R}$ and some positive real constant Ξ:

$$\Delta x \sum_{j=0}^{J} \sigma_j^2 |\varsigma_j| \leq \max_{0\leq j\leq J} |\varsigma_j|\, \Delta x \sum_{j=0}^{J} \sigma_j^2, \tag{3.26}$$

$$\Delta x \sum_{j=0}^{J} \sigma_j^2 |\varsigma_j| \leq \max_{0\leq j\leq J} |\sigma_j|^2\, \Delta x \sum_{j=0}^{J} |\varsigma_j|\,. \tag{3.27}$$

According to ([3], p.109) we have

$$\|\sigma\|_C^2 \leq 2\left(L\|\,\sigma_{\bar{x}}]|^{\,2} + \sigma_0^2\right), \quad \|\sigma\|_C^2 \leq 2\left(L\|\,\sigma_{\bar{x}}]|^{\,2} + \sigma_J^2\right), \quad \sigma_{\bar{x}} = \frac{\sigma_j - \sigma_{j-1}}{\Delta x},$$

$$\|\sigma\|_C = \max_{0\leq j\leq J} |\sigma_j|\,, \ \|\,\sigma_{\bar{x}}]| = \left(\sigma_{\bar{x}}, \sigma_{\bar{x}}\right]^{1/2} = \left(\Delta x \sum_{j=1}^{J} \left(\sigma_{\bar{x}}\right)^2\right)^{1/2}$$

hence

$$\max_{0\leq j\leq J} |\sigma_j| \leq C\left[\left(\Delta x \sum_{j=0}^{J} \sigma_j^2\right)^{1/2} + \left(\Delta x \sum_{j=0}^{J} \left\{\frac{d\sigma\left(x_j\right)}{dx}\right\}^2\right)^{1/2}\right]. \tag{3.28}$$

Discrete Cauchy-Schwarz inequality:

$$\Delta x \sum_{j=0}^{J} |\sigma_j| \leq \sqrt{L}\left(\Delta x \sum_{j=0}^{J} \sigma_j^2\right)^{1/2}. \tag{3.29}$$

From Lemmas 3.5, 3.6 and 3.7 we conclude that $\mathbf{L}^k = \mathcal{L}^k + \mathsf{L}^k + \mathfrak{L}^k$, it satisfies the following inequality:

$$\begin{aligned}
&\tfrac{\mathbf{L}^{k+1}-\mathbf{L}^k}{\Delta t} \leq -l_4\mathbf{L}^k + \mathrm{K}_1\left\{\ \Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2\left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2\left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right]\right\} + \\
&+\mathrm{K}_2\left\{\ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2\left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2\left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right]\right\} + \\
&+\mathrm{K}_3\left\{\ \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2\left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2\left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right]\right\} + \\
&+\mathrm{K}_3\left\{\ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2\left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2\left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right]\right\}.
\end{aligned}$$

with $l_4 = \min\{l_1, l_2, l_3\}$. Next we use the following inequalities. Regardless, $\xi, \eta, \gamma, \delta, \rho, \theta$ there are real positive constants Ξ_1, Ξ_2 such that

$$\begin{aligned}
&\left\{ \Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} \leq \\
&\leq \left(\|\gamma^k\|_C + \|\delta^k\|_C\right) \left\{ \Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 + h \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \right\} \leq \\
&\leq \Xi_1 \left\{ \begin{array}{l} \left[\Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2\right]^{\frac{1}{2}} + \\ + \left[\Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2\right]^{\frac{1}{2}} \end{array} \right\} \left\{ \Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \right\} \leq \\
&\leq \ \Xi_2 \left[\left(\mathsf{L}_1^k\right)^{1/2} + \left(\mathfrak{L}_1^k\right)^{1/2} + \left(\mathsf{L}_2^k\right)^{1/2} + \left(\mathfrak{L}_2^k\right)^{1/2}\right] \left[\mathcal{L}_1^k + \mathcal{L}_2^k\right].
\end{aligned} \tag{3.30}$$

We obtain this chain of inequalities as a result of applying inequalities (3.26) and (3.28). Likewise, regardless of $\xi, \eta, \gamma, \delta, \rho, \theta$ $\exists$ $\Xi_1' > 0$ and $\exists$ $\Xi_2' > 0$ such that

$$\begin{aligned}
&\left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} \leq \\
&\leq \left(\|\gamma^k\|_C + \|\delta^k\|_C\right) \left[\Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \right] \leq \\
&\leq \Xi'_1 \left\{ \begin{array}{l} \left[\Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2\right]^{\frac{1}{2}} + \\ + \left[\Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2\right]^{\frac{1}{2}} \end{array} \right\} \left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \right\} \leq \\
&\leq \Xi'_2 \left[\left(\mathsf{L}_1^k\right)^{1/2} + \left(\mathfrak{L}_1^k\right)^{1/2} + \left(\mathsf{L}_2^k\right)^{1/2} + \left(\mathfrak{L}_2^k\right)^{1/2}\right] \left[\mathsf{L}_1^k + \mathsf{L}_2^k\right].
\end{aligned} \tag{3.31}$$

Inequalities (3.26) and (3.28) were also applied here. Likewise, regardless of $\xi, \eta, \gamma, \delta, \rho, \theta$ $\exists$ $\Xi''_1 > 0$ and $\exists$ $\Xi_2'' > 0$ such that

$$\begin{aligned}
&\left\{ \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} \ \leq \\
&\leq \ \left(\|\gamma^k\|_C + \|\delta^k\|_C\right) \left[\Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \right] \leq \\
&\leq \ \Xi''_1 \left\{ \begin{array}{l} \left[\Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2\right]^{\frac{1}{2}} + \\ + \left[\Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2\right]^{\frac{1}{2}} \end{array} \right\} \left\{ \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \right\} \leq \\
&\leq \Xi''_2 \left[\left(\mathsf{L}_1^k\right)^{1/2} + \left(\mathfrak{L}_1^k\right)^{1/2} + \left(\mathsf{L}_2^k\right)^{1/2} + \left(\mathfrak{L}_2^k\right)^{1/2}\right] \left[\mathfrak{L}_1^k + \mathfrak{L}_2^k\right].
\end{aligned} \tag{3.32}$$

To obtain this inequality, we applied inequalities (3.26) and (3.28). Regardless $\xi, \eta, \gamma, \delta, \rho, \theta$ $\exists$ $\Xi_1''' > 0$ and $\exists$ $\Xi_2''' > 0$ such that

$$\begin{gathered}
\left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] \right\} \leq \\
\leq \left(\left\| \left(\gamma^k\right)^2 \right\|_C + \left\| \left(\delta^k\right)^2 \right\|_C \right) \left\{ \Delta x \sum_{j=1}^{J} \left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] \right\} \leq \\
\leq \Xi'''_1 \left\{ \begin{array}{l} \Delta x \sum\limits_{j=1}^{J} \left(\gamma_j^k\right)^2 + \Delta x \sum\limits_{j=1}^{J} \left(\rho_j^k\right)^2 + \\ + \Delta x \sum\limits_{j=0}^{J-1} \left(\delta_j^k\right)^2 + \Delta x \sum\limits_{j=0}^{J-1} \left(\theta_j^k\right)^2 \end{array} \right\} \left\{ \left[\Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 \right]^{\frac{1}{2}} + \left[\Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \right]^{\frac{1}{2}} \right\} \leq \qquad (3.33) \\
\leq \Xi'''_2 \left[\left(\mathsf{L}_1^k\right)^{1/2} + \left(\mathfrak{L}_1^k\right)^{1/2} + \left(\mathsf{L}_2^k\right)^{1/2} + \left(\mathfrak{L}_2^k\right)^{1/2} \right] \left[\mathfrak{L}_1^k + \mathfrak{L}_2^k \right].
\end{gathered}$$

Here we also applied inequalities (3.26), (3.27) and (3.28). So from inequality (3.30)-(3.31)-(3.32)-(3.33), we obtain the inequality

$$\frac{\mathbf{L}^{k+1} - \mathbf{L}^k}{\Delta t} \leq -l_4 \mathbf{L}^k + \mathrm{K}_4 \left(\mathbf{L}^k\right)^{3/2}.$$

$\forall l_0, \quad 0 < l_0 < l_4 \quad \exists d_0$ that

$$K_4 \left(\mathbf{L}^k\right)^{3/2} < \left(l_4 - l_0\right) \mathbf{L}^k \qquad \forall \mathbf{L}^k < d_0.$$

Taking into account this inequality we have

$$\frac{\mathbf{L}^{k+1} - \mathbf{L}^k}{\Delta t} \leq -l_0 \mathbf{L}^k \qquad \forall \mathbf{L}^k < d_0. \qquad (3.34)$$

Note that if d_0 we take small enough, then from (3.34) we conclude that
$\left|\xi_j^k\right| + \left|\eta_j^k\right| < \min\left(d_1, d_2, d_3\right) \quad \forall j \in \{0, 1, \cdots, J\}$. Finally, this fact gives us the right to use Lemmas 3.1, 3.2, 3.3 in the process of proving Lemma 3.4. □

Theorem 3.9. *(Discrete stability for the case $\mathsf{U}^* \geq 0$). Let us assume that the Courant Friedrichs Lewy (CFL) type condition*

$$C = \max\left(C_\varphi, C_\psi\right) < 1, \quad \text{where} \quad C_\varphi = \bar{\varphi} \frac{\Delta t}{\Delta x}, \quad C_\psi = \bar{\psi} \frac{\Delta t}{\Delta x},$$

is satisfied for (3.4)*. For each $\mathbf{U}^*$ satisfying the matrix inequality $\mathsf{U}^* \geq 0$, each κ_0, κ_L satisfying the inequality $0 < |\kappa_0 \kappa_L| < 1$, each $\mathcal{U} > 0$ and for any initial vector function $\mathbf{\Phi}$ satisfying the matrix inequality $\mathsf{U}^{\mathbf{0}} \geq \mathbf{0}$, and*

$$\left\|\mathbf{\Phi} - \mathbf{U}^*\right\|_{l^2} < \mathcal{U} \qquad (3.35)$$

the solution $\mathbf{U}^k$ of the initial-boundary value problem (3.4), (3.5), (3.6) *satisfies the matrix inequalities $\mathsf{U}^k \geq 0, \quad k \in \{0, 1, \cdots\}$, and the stationary state $\mathbf{U}^*$of the initial-boundary difference problem* (3.4), (3.5), (3.6) *is stable in the l^2 -norm.*

Let's go to $\mathsf{U}^* = 0$. Then inequality (3.35) in Theorem 3.9 is now expressed as

$$\|\mathbf{\Phi}\|_{l^2} < \mathcal{U}. \qquad (3.36)$$

Note that inequality (3.7) can be rewritten as

$$\left\|\mathbf{U}^k\right\|_{l^2} \leq \nu_2 e^{-\nu_1 t^k} \|\mathbf{\Phi}\|_{l^2}, \quad k \in \{1, 2, \cdots\},$$

Proof. Further, in the process of proving Theorem 3.9, we consider only the case of the matrix inequality

$$\mathsf{U}^* > 0. \qquad (3.37)$$

Since the initial data are $\mathsf{U}^0 \geq 0$, then according to the discrete system (3.4), (3.5), (3.6) and the CFL condition in equation (3.4), (3.5), (3.6) , we have $\mathsf{U}^k \geq 0,\ \ k \in \{0, 1, \cdots\}$.

Consider the following candidate for the discrete Lyapunov function for any $\mathbf{U}^k \in \mathrm{R}^{6\times J}$

$$\mathbf{L}\left(\mathbf{U}^k\right) = \Delta x \sum_{j=1}^{J} \left[\tfrac{\mathrm{A}}{\bar{\varphi}}\left(\xi_j^k\right)^2 + \bar{\varphi}\mathrm{A}\left(\gamma_j^k\right)^2 + \bar{\varphi}^3\mathrm{A}\left(\rho_j^k\right)^2\right] \exp\left(-\tfrac{m}{\bar{\varphi}} x_{j-1}\right) +$$

$$+ \Delta x \sum_{j=0}^{J-1} \left[\tfrac{\mathrm{B}}{\bar{\psi}}\left(\eta_j^k\right)^2 + \bar{\psi}\mathrm{B}\left(\delta_j^k\right)^2 + \bar{\psi}^3\mathrm{B}\left(\theta_j^k\right)^2\right] \exp\left(\tfrac{m}{\bar{\psi}} x_{j+1}\right).$$

Discrete weight norm $\mathbf{L}\left(\mathbf{U}^k\right)$ is equivalent to the discrete W_2^2-norm , for all $k \geq 0$.

$$\min\left\{\exp\left(-\tfrac{m}{\bar{\varphi}}L\right)\min\left(\tfrac{\mathrm{A}}{\bar{\varphi}}, \bar{\varphi}\mathrm{A}, \bar{\varphi}^3\mathrm{A}\right) + \min\left(\tfrac{\mathrm{B}}{\bar{\psi}}, \bar{\psi}\mathrm{B}, \bar{\psi}^3\mathrm{B}\right)\right\} \left\|\mathbf{U}^k\right\|_{w_2^2}^2 \leq \mathbf{L}\left(\mathbf{U}^k\right) \leq$$

$$\max\left\{\exp\left(\tfrac{m}{\bar{\psi}}L\right)\max\left(\tfrac{\mathrm{A}}{\bar{\varphi}}, \bar{\varphi}\mathrm{A}, \bar{\varphi}^3\mathrm{A}\right) + \max\left(\tfrac{\mathrm{B}}{\bar{\psi}}, \bar{\psi}\mathrm{B}, \bar{\psi}^3\mathrm{B}\right)\right\} \left\|\mathbf{U}^k\right\|_{w_2^2}^2$$

where

$$\left\|\mathbf{U}^k\right\|_{w_2^2}^2 = \Delta x \sum_{j=1}^{J}\left[\left(\xi_j^k\right)^2 + \left(\gamma_j^k\right)^2 + \left(\rho_j^k\right)^2\right] + \Delta x \sum_{j=0}^{J-1}\left[\left(\eta_j^k\right)^2 + \left(\delta_j^k\right)^2 + \left(\theta_j^k\right)^2\right].$$

As a second step, we evaluate a finite difference approximation of the time derivative of $\mathbf{L}\left(\mathbf{U}^k\right)$ time. For this purpose, we use inequality (3.34)

$$\frac{\mathbf{L}\left(\mathbf{U}^{k+1}\right) - \mathbf{L}\left(\mathbf{U}^k\right)}{\Delta t} \leq -e\,\mathbf{L}\left(\mathbf{U}^k\right),$$

Inequality (3.34) means the existence of a discrete Lyapunov function $\mathbf{L}\left(\mathbf{U}^k\right)$, which provides an exponential decrease $\mathbf{L}\left(\mathbf{U}^k\right)$.
This completes the proof of Theorem 3.9. □

4. Conclusion

So, in this work we studied the problem of exponential stability of the numerical solution of an upwind difference scheme for a quasilinear hyperbolic system with dissipative boundary conditions. An upwind difference scheme is constructed for the numerical solution of the initial boundary value problem. The definition of exponential stability of a numerical solution with respect to the equilibrium state of an initial-boundary difference problem is given. For the first time, a discrete Lyapunov function for a numerical solution was constructed and a theorem on the exponential stability of the equilibrium state of an initial-boundary difference problem for a quasilinear hyperbolic system was proved.

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