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High-accuracy difference schemes for solving non-stationary fourth-order equations and their application to non-classical partial differential equations

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Abstract. In the article, high-accuracy difference schemes for the Cauchy problem for a system of fourth-order equations are obtained. Based on the method of energy inequalities, the stability of the scheme is proved, a priori estimates of the solution to difference schemes are obtained, and their convergence and accuracy are proved. The results obtained for the system are applied to solve the first initial-boundary value problem for the equation of ion-acoustic waves in a "magnetized" plasma for the generalized potential of the electric field. The schemes constructed for this problem have second-order accuracy in spatial variables and fourth-order accuracy in time variables. In energy norms, convergence and accuracy estimates are obtained in classes of smooth solutions.

Keywords: Cauchy problem, fourth-order system of equations, ion-acoustic wave equation, difference schemes, approximation error, stability, convergence, accuracy
MSC (2020): 65M06, 65M12

1. Introduction

In the spatial approximation of partial differential equations using finite difference or finite element methods, we derive systems of ordinary differential equations with large dimensions. Currently, these semi–discrete methods are frequently employed to numerically solve initial–boundary value problems for differential equations, particularly for non–classical Sobolev–type equations. In reference [1], this approach was applied to a system of nonstationary second–order equations, using piecewise cubic interpolation–the finite element method–for approximation. Similar research was conducted in references [2]–[3] for non-stationary first-and second–order equations, resulting in the development of two– or three-parameter vector difference schemes with fourth-order accuracy. References [4]–[5] examined the application of these schemes for numerically solving various high–order Sobolev–type equations. For instance, high–accuracy difference schemes were constructed and analyzed for equations modeling internal waves in a weakly stratified fluid [4] and for equations describing gravitational–gyroscopic waves in a stratified fluid [6]. Additionally, the three–parameter schemes developed in reference [2] were utilized in references [7]–[8] to address various non–classical Sobolev–type equations, where high–accuracy schemes were constructed and investigated within the context of smooth and non–smooth solutions.

The proposed research focuses on the development and examination of two-parameter difference schemes for non-stationary fourth–order equations. These studies were initially conducted in [9]. This work aims to generalize those findings for the numerical solution of the first initial–boundary value problem related to the equation of ion–sound waves in a "magnetized" plasma, incorporating the generalized potential of the electric field. The research presents theorems concerning the convergence and accuracy of the schemes.

2. Statement of the problem

We consider the Cauchy problem for a system of fourth–order operator differential equations:

$$\mathrm{D}\frac{d^4u}{dt^4} + \mathrm{B}\frac{d^2u}{dt^2} + \mathrm{A}u = f, \quad 0 < t \le T, \tag{2.1}$$

$$u(0) = u_{0,0}, \quad \dot{u}(0) = u_{0,1}, \quad \ddot{u}(0) = u_{0,2}, \quad \dddot{u}(0) = u_{0,3}, \tag{2.2}$$

where $\mathrm{D} \ne \mathrm{D}(t)$, $\mathrm{B} \ne \mathrm{B}(t)$, $\mathrm{A} \ne \mathrm{A}(t)$ are operators from $H \to H$, $u = u(t) \in H$, $f = f(t) \in H$ is the Hilbert space with an inner product (u, ϑ) and norm $\|u\| = \sqrt{(u,u)}$. In what follows, we assume that all necessary derivatives of the sought–for solution $u(t)$ exist.

3. Second–order accuracy scheme

Consider problem (2.1), (2.2). On the interval $0 \le t < \infty$, we introduce uniform grid $\overline{\omega}_\tau = \{t_n = n\tau, \quad n = 0, 1, \ldots\}$, $\omega_\tau = \overline{\omega}_\tau \cap \{0\}$ with step τ. We will consider abstract functions $y = y(t_n)$ and $\varphi = \varphi(t_n)$ of discrete argument $t_n \in \omega_\tau$ with values from H is grid space. We denote the space consisting of elements of space H with the inner product $(u, \vartheta)_{\mathrm{A}} = (\mathrm{A}u, \vartheta)$ and energy norm $\|u\|_{\mathrm{A}} = \sqrt{(u,u)}_{\mathrm{A}}$ by H_{A}. Now we approximate problem (2.1), (2.2) with the following difference scheme

$$\mathrm{D}y_{\bar{t}t\bar{t}t} + \mathrm{B}y_{\bar{t}t} + \mathrm{A}y = \varphi, \quad t_n \in \omega_\tau, n = 2, 3, ..., \tag{3.1}$$

$$y^0 = u_{0,0}, \quad y^1 = \overline{u}_{0,1}, \quad y^2 = \overline{u}_{0,2}, \quad y^3 = \overline{u}_{0,3}, \tag{3.2}$$

where $y_{\bar{t}t\bar{t}t} = (y^{n+2} - 4y^{n+1} + 6y^n - 4y^{n-1} + y^{n-2})/\tau^4$, $y_{\bar{t}t} = (y^{n+1} - 2y^n + y^{n-1})/\tau^2$, $y^n = y(t_n)$, $y^{n\pm1} = y(t_n \pm \tau)$, $y^{n\pm2} = y(t_n \pm 2\tau)$,

$$\begin{gathered}
\overline{u}_{0,1} = u_{0,1} + 0.5\tau \left[\mathrm{E} - (\tau^2/12)\mathrm{D}^{-1}\mathrm{B}\right] u_{0,2}, \\
\overline{u}_{0,2} = u_{0,2} + \tau u_{0,3}, \\
\overline{u}_{0,3} = u_{0,3} + (3\tau/2)\mathrm{D}^{-1}\left[f(0) - \mathrm{B}u_{0,2} - \mathrm{A}u_{0,0}\right],
\end{gathered} \tag{3.3}$$

E is the unit operator.

Let us denote the errors by $z = y - u$, where u is the solution to problem (2.1), and y is the solution to scheme (3.1). Then, substituting $y = z + u$ into scheme (3.1), we obtain the problem for the error

$$\mathrm{D}z_{\bar{t}t\bar{t}t} + \mathrm{B}z_{\bar{t}t} + \mathrm{A}z = \psi, \tag{3.4}$$

where $\psi = O(\tau^2)$ is the approximation error of scheme (3.1). The initial conditions (3.2), considering (3.3), also have second-order approximation error, i.e., $O(\tau^2)$.

To study scheme (3.1), we perform the following transformation:

$$\mathrm{D}y^{n+2} - (4\mathrm{D} - \tau^2\mathrm{B})y^{n+1} + (6\mathrm{D} - 2\tau^2\mathrm{B} + \tau^4\mathrm{A})y^n - (4\mathrm{D} - \tau^2\mathrm{B})y^{n-1} + \mathrm{D}y^{n-2} = \tau^4\varphi.$$

Let $y = y^{n+2}$, then from this equality we obtain:

$$B_4 y^{n+4} + B_3 y^{n+3} + B_2 y^{n+2} + B_1 y^{n+1} + B_0 y^n = \tau^4\varphi. \tag{3.5}$$

Here, $B_0 = B_4 = \mathrm{D}$, $B_1 = B_3 = -4\mathrm{D} + \tau^2\mathrm{B}$, $B_2 = 6\mathrm{D} - 2\tau^2\mathrm{B} + \tau^4\mathrm{A}$. Further, following [10], we write scheme (3.5) in the following canonical form:

$$\mathbb{N}y_{\bar{t}} + \tau^2\Re y_{\overset{\circ\circ}{tt}} + \tau^3\Im y_{\bar{t}\overset{\circ}{t}t} + \tau^4\aleph y_{\bar{t}t\bar{t}t} + \mathbb{R}y = \tau^4\varphi, \tag{3.6}$$

where $\mathbb{N} = \tau(2B_4 + B_3 - B_1 - 2B_0)$, $\Re = 2B_0 + 0.5(B_1 + B_3) + 2B_4$, $\Im = 0.5(B_1 - B_3)$,

$$\aleph = -(1/8)(B_1 + B_3), \quad \mathbb{R} = B_4 + B_3 + B_2 + B_1 + B_0. \tag{3.7}$$

Here we use the following notations $y_{\bar{t}} = (y^n - y^{n-1})/\tau$, $y_{\overset{\circ\circ}{tt}} = (y^{n+4} - 2y^{n+2} + y^n)/(4\tau^2)$, $y_{\bar{t}t\overset{\circ}{t}} = (y^{n+4} - 2y^{n+3} + 2y^{n+1} - y^n)/(2\tau^3)$.

Taking into account (3.7), after elementary calculations, from (3.6), we obtain a difference scheme in the canonical form:

$$[\mathrm{D} - (\tau^2/4)\mathrm{B}]y_{\bar{t}t\bar{t}t} + \mathrm{B}y_{\overset{\circ\circ}{tt}} + \mathrm{A}y = \varphi.$$

According to Theorem 2 from [[10], p. 276], an a priori estimate based on the initial data ($\varphi = 0$), holds

$$\|y_{n+1}\|_{\bar{\mathrm{A}}} \le \|y_n\|_{\bar{\mathrm{A}}}, \tag{3.8}$$

if the following conditions are met:

$$\operatorname{Re}\mathbb{N} \ge 0, \tag{3.9}$$

$$\mathbb{R} \ge 0, \tag{3.10}$$

$$\Re - 4\aleph - \mathbb{R} \geq 0, \tag{3.11}$$

$$\mathbb{R} + 16\aleph \geq 0. \tag{3.12}$$

Here

$$\|y_n\|^2_{\bar{\mathrm{A}}} = (1/16)\Big[\|y^n + y^{n+1} + y^{n+2} + y^{n+3}\|^2_{\mathbb{R}} + \|y^{n+3} + y^{n+2} - y^{n+1} - y^n\|^2_{\Re-4\aleph-\mathbb{R}} +$$

$$+ \|y^{n+3} - y^{n+2} - y^{n+1} + y^n\|^2_{\Re-4\aleph-\mathbb{R}} + \|y^{n+3} - y^{n+2} + y^{n+1} - y^n\|^2_{\mathbb{R}+16\aleph}\Big].$$

Let us check the fulfillment of conditions (3.9)-(3.12). Conditions (3.9) and (3.10) are fulfilled, since $\mathbb{N} = 0$ and $\mathbb{R} = \tau^4\mathrm{A}$. $(\mathrm{A} = \mathrm{A}^* > 0)$. Condition (3.11) will be fulfilled if $4\mathrm{D} + \tau^4\mathrm{A} \leq 2\tau^2\mathrm{B}$, and, finally, the last condition (3.12) will be fulfilled if $16\mathrm{D} + \tau^4\mathrm{A} \geq 4\tau^2\mathrm{B}$. These two conditions will be fulfilled if

$$\mathrm{D} \geq (\tau^4/4)\mathrm{A}, \tag{3.13}$$

which is the stability condition of scheme (3.1), (3.2).

Thus, the following theorem holds.

Theorem 3.1. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (3.13) are fulfilled, an apriori estimate for the initial data (3.8) is valid for the solution of scheme (3.1), (3.2).

To prove the stability of the right-hand side of scheme (3.1)-(3.3), we represent it as an equivalent two-layer scheme in space H^4 [10]:

$$\mathbb{C}y_t + \mathbb{Q}y = \phi, \tag{3.14}$$

where $y_t = \Big\{y_{\bar{t}},\ \tau y_{\overset{\circ}{t}\overset{\circ}{t}},\ (\tau^2/2)y_{\bar{t}t\overset{\circ}{t}},\ (\tau/2)y_{\overset{\circ}{t}\overset{\circ}{t}} + (\tau^3/8)y_{\bar{t}t\bar{t}t}\Big\}$, $\quad \phi = \{\varphi, 0, 0, 0\}$,

$$\mathbb{Q} = \begin{pmatrix} \mathbb{R} & 0 & 0 & 0 \\ 0 & \Re - 2\aleph - \mathbb{R} & 0 & 0 \\ 0 & 0 & \Re - 2\aleph - \mathbb{R} & 0 \\ 0 & 0 & 0 & \mathbb{R} + 16\aleph \end{pmatrix},$$

$$\mathbb{C} = \begin{pmatrix} \mathbb{N} + 0.5\tau\mathbb{R} & \tau(\Re - 4\aleph - \mathbb{R}) & 0 & 0.5(\mathbb{R} + 16\aleph) \\ -\tau(\Re - 4\aleph - \mathbb{R}) & 0.5\tau(\Re - 4\aleph - \mathbb{R}) & 0.5\tau(\Re - 4\aleph - \mathbb{R}) & 0 \\ 0 & -0.5\tau(\Re - 4\aleph - \mathbb{R}) & 0.5\tau(\Re - 4\aleph - \mathbb{R}) & 0 \\ -0.5(\mathbb{R} + 16\aleph) & 0 & 0 & 0.5(\mathbb{R} + 16\aleph) \end{pmatrix}.$$

By Theorem 4 from ([10], p. 284), for the solution to difference scheme (3.14), the following a priori estimate holds:

$$\|y_{n+1}\|_{\bar{\mathrm{A}}} \leq \|y_0\|_{\bar{\mathrm{A}}} + \|\varphi_0\|_{\bar{\mathrm{A}}^{-1}} + \|\varphi_n\|_{\bar{\mathrm{A}}^{-1}} + \sum_{k=1}^{n} \tau\|\varphi_{k,\bar{t}}\|_{\bar{\mathrm{A}}^{-1}}, \tag{3.15}$$

if the following conditions are met:

$$\operatorname{Re}\mathbb{N} \geq 0, \tag{3.16}$$

$$\mathbb{R} > 0, \tag{3.17}$$

$$\Re - 4\aleph - \mathbb{R} > 0, \tag{3.18}$$

$$\mathbb{R} + 16\aleph > 0. \tag{3.19}$$

In inequality (3.15), $\|\varphi_n\|_{\bar{\mathrm{A}}^{-1}} = \|\varphi^n\|_{\bar{\mathrm{A}}^{-1}}$, $\|\varphi_{k,\bar{t}}\|_{\bar{\mathrm{A}}^{-1}} = \|\varphi^k_{\bar{t}}\|_{\bar{\mathrm{A}}^{-1}} = (\bar{\mathrm{A}}^{-1}\varphi^k_{\bar{t}}, \varphi^k_{\bar{t}})$. Consequently, the following assertion holds.

Theorem 3.2. Let $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$. Then, if (3.13) is satisfied, then for the solution of scheme (3.1)-(3.3), estimate (3.15) holds.

Let us check the fulfillment of conditions (3.16)-(3.19). Conditions (3.16) and (3.17) are fulfilled, since $\mathbb{N} = 0$ and $\mathrm{A} = \mathrm{A}^* > 0$. Conditions (3.18) and (3.19) will also be fulfilled if condition (3.13) is fulfilled.

To prove the convergence of difference scheme (3.1)-(3.3), we obtain a problem for the error $z = y - u$, i.e., substituting $y = z + u$ into (3.1), we obtain:

$$[\mathrm{D} - (\tau^2/4)\mathrm{B}]z_{\bar{t}t\bar{t}t} + \mathrm{B}z_{\overset{\circ}{t}\overset{\circ}{t}} + \mathrm{A}z = \psi$$

with the corresponding initial conditions. Therefore, based on Theorems 3.1 and 3.2, considering (3.4), we have the following result.

Theorem 3.3. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (3.13) are met, the following accuracy estimate for the solution of scheme (3.1)-(3.3) is valid:

$$\|y(t_n) - u(t_n)\|_{\bar{\mathrm{A}}} \leq O(\tau^2), \quad t_n \in \bar{\omega}_\tau. \tag{3.20}$$

4. Fourth-order accuracy scheme

From (3.4), we obtain $\psi = \varphi - \mathrm{D}u_{\bar{t}t\bar{t}t} - \mathrm{B}u_{\bar{t}t} - \mathrm{A}u$ for the error. Then, using the Taylor expansion formula and equation (2.1), we obtain:

$$\mathrm{D}u_{\bar{t}t\bar{t}t} = \mathrm{D}(t_n) + (1/6)\tau^2\mathrm{D}u^{(6)}(t_n) + O(\tau^4), \quad \mathrm{B}u_{\bar{t}t} = \mathrm{B}\ddot{u}(t_n) + (1/12)\tau^2\mathrm{B}\,\ddot{\ddot{u}}\,(t_n) + O(\tau^4).$$

Consequently, $\psi = \varphi - f^n - (\tau^2/6)\ddot{f}^n + (\tau^2/12)\mathrm{B}\,\ddot{\ddot{u}} + (\tau^2/6)\mathrm{A}\ddot{u} + O(\tau^4)$. Then, if we choose

$$\bar{\mathrm{D}} = \mathrm{D} + (\tau^2/12)\mathrm{B}, \quad \bar{\mathrm{B}} = \mathrm{B} + (\tau^2/6)\mathrm{A}, \quad \bar{\varphi} = \varphi + (\tau^2/6)\ddot{f}, \tag{4.1}$$

then we obtain the following difference scheme:

$$\bar{\mathrm{D}}y_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}}y_{\bar{t}t} + \mathrm{A}y = \bar{\varphi}, \; t_n \in \omega_\tau, \; n = 2, 3, ..., \tag{4.2}$$

which has the order of approximation $\psi = O(\tau^4)$. We choose the initial conditions for (4.2) in the form (3.2):

$$y^0 = u_{0,0}, \; y^1 = \bar{u}_{0,1}, \; y^2 = \bar{u}_{0,2}, \; y^3 = \bar{u}_{0,3}, \tag{4.3}$$

where

$$\begin{gathered} \bar{u}_{0,1} = u_{0,1} + 0.5\tau[\mathrm{E} - (\tau^2/12)\mathrm{D}^{-1}\mathrm{B}]u_{0,2} + (\tau^2/6)u_{0,3} + (\tau^3/24)\mathrm{D}^{-1}[f(0) - \mathrm{A}u_{0,0}], \\ \bar{u}_{0,2} = u_{0,2} + \tau u_{0,3} + (\tau^2/2)\mathrm{D}^{-1}[f(0) - \mathrm{B}u_{0,2} - \mathrm{A}u_{0,0}] + \\ +(\tau^3/4)\mathrm{D}^{-1}[\dot{f}(0) - \mathrm{B}\dot{u}_{0,2} - \mathrm{A}\dot{u}_{0,0}], \\ \bar{u}_{0,3} = u_{0,3} + (3\tau/2)\mathrm{D}^{-1}[f(0) - \mathrm{B}u_{0,2} - \mathrm{A}u_{0,0}] + (5\tau^2/4)\mathrm{D}^{-1}[\dot{f}(0) - \mathrm{B}\dot{u}_{0,2} - \mathrm{A}\dot{u}_{0,0}] + \\ +(3\tau^2/4)\mathrm{D}^{-1}[\ddot{f}(0) - \mathrm{B}\ddot{u}_{0,2} - \mathrm{A}\ddot{u}_{0,0}]. \end{gathered} \tag{4.4}$$

The approximation error of the initial conditions coincides with the approximation error of scheme (4.2), i.e. $\psi = O(\tau^4)$.

To study the stability of the initial data of scheme (4.2), we write it in the canonical form:

$$[\bar{\mathrm{D}} - (\tau^2/4)\bar{\mathrm{B}}]y_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}}y_{\overset{\circ}{t}\overset{\circ}{t}} + \mathrm{A}y = \varphi. \tag{4.5}$$

Then, for scheme (4.5), the a priori estimate (3.8) holds if conditions (3.9)-(3.12) with operators (3.7) are satisfied, where:

$$B_0 = B_4 = \bar{\mathrm{D}}, \quad B_1 = B_3 = -4\bar{\mathrm{D}} + \tau^2\bar{\mathrm{B}}, \quad B_2 = 6\bar{\mathrm{D}} - 2\tau^2\bar{\mathrm{B}} + \tau^4\mathrm{A}. \tag{4.6}$$

Checking the fulfillment of conditions (3.9)-(3.12) and considering (4.6), we arrive at the stability condition of difference scheme (4.2):

$$\bar{\mathrm{D}} \geq (\tau^4/4)\mathrm{A}. \tag{4.7}$$

Consequently, the following theorem holds.

Theorem 4.1. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (4.7) are satisfied, for the solution of scheme (4.2)-(4.4), an a priori estimate based on the initial data (3.8) with operators (4.1), (4.6) holds.

Similarly to the second-order accuracy scheme, the following assertion is proved.

Theorem 4.2. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (4.7) are satisfied, for the solution of scheme (4.2)-(4.4) with operators (4.1), (4.6), estimate (3.15) holds.

To prove the convergence of scheme (4.2)-(4.4), we obtain a problem for the error:

$$\bar{\mathrm{D}} z_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}} z_{\bar{t}t} + \mathrm{A} z = \bar{\psi}, \ t_n \in \bar{\omega}_\tau, \ n = 2, 3, ..., \tag{4.8}$$

with the corresponding initial conditions. Here, $\bar{\psi} = O(\tau^4)$. Therefore, based on Theorems 4.1 and 4.2, taking into account (4.8), we obtain the following result.

Theorem 4.3. Under conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (4.7), the accuracy estimate (3.20) is valid for solving scheme (4.2)-(4.4).

5. Scheme with weights

Based on difference schemes (4.2)-(4.4) with operators (4.6), we consider the following family of difference schemes with weights

$$\bar{\mathrm{D}} y_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}} y_{\bar{t}t}^{(\sigma_1,\sigma_2)} + \mathrm{A} y^{(\sigma_3,\sigma_4)} = \bar{\varphi}, \quad t_n \in \bar{\omega}_\tau. \tag{5.1}$$

Here, $y^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{y} + (1-\sigma_1-\sigma_2)y + \sigma_2 \overset{\vee}{y}$, $y^{(\sigma_3,\sigma_4)} = \sigma_3 \hat{y} + (1-\sigma_3-\sigma_4)y + \sigma_4 \overset{\vee}{y}$, where $\sigma_1, \ \sigma_2, \ \sigma_3, \ \sigma_4$ are some constants, the weights of the scheme, the presence of which allows us to select various explicit and implicit schemes and regulate their accuracy.

Let us study the stability and convergence of scheme (5.1) with initial conditions (4.3), (4.4). To do this, we will reduce (5.1) to canonical form. We will perform the following transformation with scheme (5.1):

$$\begin{gathered}\bar{\mathrm{D}} + \tau^2\sigma_1\bar{\mathrm{B}})y^{n+2} - [4\bar{\mathrm{D}} - \tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} + 2\tau^2\sigma_1\bar{\mathrm{B}} - \tau^4\sigma_3\mathrm{A}]y^{n+1} + \\ +[6\bar{\mathrm{D}} + \tau^2\sigma_2\bar{\mathrm{B}} - 2\tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} + \tau^2\sigma_1\bar{\mathrm{B}} + \tau^4(1-\sigma_3-\sigma_4)\mathrm{A}]y^n - \\ - [4\bar{\mathrm{D}} + 2\tau^2\sigma_2\bar{\mathrm{B}} - \tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} - \tau^4\sigma_4\mathrm{A}]y^{n-1} + (\bar{\mathrm{D}} + \tau^2\sigma_2\bar{\mathrm{B}})y^{n-2} = \tau^4\bar{\varphi}.\end{gathered} \tag{5.2}$$

Let in (5.2) $y = y^{n+2}$. Then (5.2) has the following form:

$$B_4 y^{n+4} + B_3 y^{n+3} + B_2 y^{n+2} + B_1 y^{n+1} + B_0 y^n = \tau^4 \bar{\varphi}^n,$$

where

$$\begin{gathered}B_4 = \bar{\mathrm{D}} + \tau^2\sigma_1\bar{\mathrm{B}}, \quad B_3 = -[4\bar{\mathrm{D}} - \tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} + 2\tau^2\sigma_1\bar{\mathrm{B}} - \tau^4\sigma_3\mathrm{A}], \\ B_2 = 6\bar{\mathrm{D}} + \tau^2\sigma_2\bar{\mathrm{B}} - 2\tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} + \tau^2\sigma_1\bar{\mathrm{B}} + \tau^4(1-\sigma_3-\sigma_4)\mathrm{A}, \\ B_1 = -[4\bar{\mathrm{D}} + 2\tau^2\sigma_2\bar{\mathrm{B}} - \tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} - \tau^4\sigma_4\mathrm{A}], B_0 = \bar{\mathrm{D}} + \tau^2\sigma_2\bar{\mathrm{B}}.\end{gathered} \tag{5.3}$$

Now we write scheme (5.3) in the following canonical form:

$$M y_{\bar{t}} + \tau^2 R y_{\overset{\circ}{t}\overset{\circ}{t}} + \tau^3 P y_{\bar{t}t\overset{\circ}{t}} + \tau^4 Q y_{\bar{t}t\bar{t}t} + N y = \tau^4 \bar{\varphi}, \tag{5.4}$$

where

$$\begin{gathered}M = \tau(2B_4 + B_3 - B_1 - 2B_0), \quad R = 2B_0 + 0.5(B_1 + B_3) + 2B_4, \\ P = 0.5(B_1 - B_3), \quad Q = -(1/8)(B_1 + B_3), \quad N = B_4 + B_3 + B_2 + B_1 + B_0.\end{gathered}$$

From here, taking into account (5.3), we obtain

$$\begin{gathered}M = \tau^5(\sigma_3 - \sigma_4)\mathrm{A}, \quad R = \tau^2\left[1 - 2(\sigma_1 + \sigma_2)\right]\bar{\mathrm{B}} + 0.5\tau^4(\sigma_4 + \sigma_3)\mathrm{A}, \\ P = -(\sigma_2 - \sigma_1)\tau^2\bar{\mathrm{B}} + 0.5\tau^4(\sigma_4 - \sigma_3)\mathrm{A}, \\ Q = \bar{\mathrm{D}} - (\tau^2/4)\left[1 - 2(\sigma_2 + \sigma_1)\right]\bar{\mathrm{B}} - (\tau^4/8)(\sigma_4 + \sigma_3)\mathrm{A}, \quad N = \tau^4\mathrm{A}.\end{gathered}$$

Let $\sigma_1=\sigma_2=\sigma$, $\sigma_3=\sigma_4=\theta$, then $M=P=0$. Consequently, after elementary calculations from (5.4), we obtain the difference scheme in the canonical form:

$$\tilde{Q}y_{\bar{t}t\bar{t}t}+\tilde{R}y_{\overset{\circ}{t}\overset{\circ}{t}}+\mathrm{A}y=\bar{\varphi}, \tag{5.5}$$

where

$$\tilde{Q}\equiv\mathrm{D}-(\tau^2/4)(1-4\sigma)\bar{\mathrm{B}}-(\tau^4/8)\theta\mathrm{A}, \quad \tilde{R}=\tau^2(1-4\sigma)\bar{\mathrm{B}}+\tau^4\theta\mathrm{A}.$$

Then, an a priori estimate based on the initial data (3.8) ($\varphi=0$) holds, if the following conditions are satisfied:

$$\operatorname{Re}M\geq 0, \quad N\geq 0, \quad R-4Q-N\geq 0, \quad N+16Q\geq 0, \tag{5.6}$$

where $\|Y^n\|^2_{\bar{A}}$ is defined in section 3.

Let us check the fulfillment of conditions (5.6). The first condition $\operatorname{Re}M\geq 0$ is fulfilled, since $M=0$. The second condition is $N\geq 0$, since the operator is $\mathrm{A}>0$. Condition $R-4Q-N\geq 0$ will be fulfilled if

$$4\bar{\mathrm{D}}+\tau^4(1-2\sigma)\mathrm{A}\leq 2\tau^2(1-4\sigma)\bar{\mathrm{B}} \tag{5.7}$$

and finally the last condition $N+16Q\geq 0$ will be fulfilled if

$$16\bar{\mathrm{D}}+\tau^4(1-2\theta)\mathrm{A}\geq 4\tau^2(1-4\sigma)\bar{\mathrm{B}}. \tag{5.8}$$

Conditions (5.7) and (5.8) will be fulfilled if

$$\theta\leq 1/2, \quad \sigma\leq 1/4, \quad \bar{\mathrm{D}}\geq(\tau^4/8)\mathrm{A}, \tag{5.9}$$

which are the stability conditions for scheme (5.1), (4.3), (4.4).

Thus, the following theorem is proved.

Theorem 5.1. When conditions $\mathrm{D}^*=\mathrm{D}>0$, $\mathrm{B}^*=\mathrm{B}\geq 0$, $\mathrm{A}^*=\mathrm{A}>0$ and (5.9) are fulfilled, an a priori estimate based on the initial data (3.8) holds for the solution of scheme (5.1), (4.3), (4.4).

To prove the stability of the right-hand side of scheme (5.1), (4.3), (4.4)., we will represent it as an equivalent two-layer scheme in space H^4:

$$\mathfrak{A}y_t+\mathfrak{M}y=\Psi,$$

where $y_t=\left\{y_{\bar{t}},\ \tau y_{\overset{\circ}{t}\overset{\circ}{t}},\ (\tau^2/2)y_{\bar{t}t\overset{\circ}{t}},\ (\tau/2)y_{\overset{\circ}{t}\overset{\circ}{t}}+(\tau^3/8)y_{\bar{t}t\bar{t}t}\right\}$, $\Psi=\{\varphi,\,0,\,0,\,0\}$.
Therefore, the following assertion holds.

Theorem 5.2. Let $\mathrm{D}^*=\mathrm{D}>0$, $\mathrm{B}^*=\mathrm{B}\geq 0$, $\mathrm{A}^*=\mathrm{A}>0$ and the following operator inequalities hold:

$$\operatorname{Re}M\geq 0, \quad N>0, \quad R-4Q-N>0, \quad N+16Q>0. \tag{5.10}$$

Then, for the solution of difference scheme (5.1), (4.3), (4.4), the following a priori estimate is true:

$$\|y_{n+1}\|_{\bar{\mathrm{A}}}\leq\|y_0\|_{\bar{\mathrm{A}}}+\|\bar{\varphi}_0\|_{\bar{\mathrm{A}}^{-1}}+\|\bar{\varphi}_n\|_{\bar{\mathrm{A}}^{-1}}+\sum_{k=1}^{n}\tau\|\bar{\varphi}_{k,\bar{t}}\|_{\bar{\mathrm{A}}^{-1}}.$$

From (5.10), the first two conditions are satisfied, since $M=0$ and $\mathrm{A}^*=\mathrm{A}>0$, and the rest will be satisfied if inequalities (5.9) hold.

To prove the convergence of difference scheme (5.5), (4.3), (4.4), we obtain a problem for the error

$$\tilde{Q}z_{\bar{t}t\bar{t}t}+\tilde{R}z_{\overset{\circ}{t}\overset{\circ}{t}}+\mathrm{A}z=\bar{\psi}, \quad t_n\in\bar{\omega}_\tau, \quad n=2,3,...$$

with the corresponding initial conditions. Here, $\bar{\psi}=O(\tau^4)$. Therefore, based on Theorems 5.1 and 5.2, we obtain the following assertion.

Theorem 5.3. When conditions $\mathrm{D}^*=\mathrm{D}>0$, $\mathrm{B}^*=\mathrm{B}\geq 0$, $\mathrm{A}^*=\mathrm{A}>0$ and (5.9) are satisfied, for the solution of scheme (5.1), (4.3), (4.4), the following accuracy estimate is true:

$$\|y(t_n)-u(t_n)\|_{\bar{\mathrm{A}}}\leq O(\tau^4), \quad t_n\in\bar{\omega}_\tau.$$

6. Difference schemes for partial differential equations

In domain

$$\bar{Q}_T = \left\{(x,t) : x = (x_1, x_2, x_3) \in \bar{\Omega} = [0 \le x_\alpha \le l_\alpha, \ \alpha = 1,2,3], \ t \in [0,T]\right\}$$

consider equation [11]

$$\frac{\partial^4}{\partial t^4}\left(\Delta_3 u - \frac{1}{r_{\mathrm{D}}^2}u\right) + \frac{\partial^2}{\partial t^2}\left[(\omega_{Bi}^2 + \omega_{pi}^2)\Delta_3 u - \frac{\omega_{Bi}^2}{r_{\mathrm{D}}^2}u\right] + \omega_{pi}^2\omega_{Bi}^2\frac{\partial^2 u}{\partial x_3^2} = f(x,t) \tag{6.1}$$

with initial

$$\left.\frac{\partial^k}{\partial t^k}u(x,t)\right|_{t=0} = u_{0,k}, \quad k = \overline{0,3}, \quad x \in \Omega \tag{6.2}$$

and boundary conditions of the first kind

$$u(x,t)|_\Gamma = \mu(t), \quad t \in [0,T], \tag{6.3}$$

where $u = u(x,t)$ is the generalized potential of the electric field, Δ_3 is the three-dimensional Laplace operator, $\omega_{\mathrm{B}i}^2 = e\mathrm{B}_0/(Mc)$ is the ion gyro-frequency, $\omega_{pi}^2 = 4\pi e^2 n_0/M$ is the Langmuir frequency for ions, $r_{\mathrm{D}} = \left[T_e/(4\pi n_0 e^2)\right]^{1/2}$ is the Debye radius, M is the ion mass, T_e is the electron temperature, n_0 is the unperturbed particle density, e is the absolute value of the electron charge.

To discretize problem (6.1)-(6.3) in space, we rewrite it in the following form

$$\begin{gathered} L_0\frac{\partial^4 u}{\partial t^4} + L_1\frac{\partial^2 u}{\partial t^2} + L_2 u = f(x,t), \ \ (x,t) \in Q_T, \\ \frac{\partial^k}{\partial t^k}u(x,0) = u_{0,k}, \ \ k = \overline{0,3}, \ \ x \in \Omega, \\ u(x,t)|_\Gamma = \mu(t), \ \ t \in [0,T], \end{gathered}$$

where

$$L_0 = \Delta_3 - \frac{1}{r_{\mathrm{D}}^2}E, \ \ L_1 = \omega_0^2\Delta_3 - \frac{\omega_{Bi}^2}{r_{\mathrm{D}}^2}E, \ \ L_2 = \omega_1^2\frac{\partial^2}{\partial x_3^2}. \tag{6.4}$$

Here, $\omega_0^2 = \omega_{pi}^2 + \omega_{\mathrm{B}i}^2$, $\omega_1^2 = \omega_{pi}^2\omega_{\mathrm{B}i}^2$, Γ is the boundary of domain $\bar{\Omega}$, E is the unit operator.

Let us construct subspace $H_h \subset H$, that approximates the Hilbert space H with the corresponding scalar product and norm. Let us introduce into $\bar{\Omega}$ a grid uniform in each direction $\bar{\omega}_h = \bar{\omega}_{h_1} \times \bar{\omega}_{h_2} \times \bar{\omega}_{h_3}$, where $\bar{\omega}_{h_\alpha} = \left\{x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = \overline{0, N}_\alpha, \ \ h_\alpha = l_\alpha/N_\alpha\right\}$, $\alpha = 1,2,3$. Here, $\bar{\omega}_h = \omega_h + \gamma_h$, γ_h - are the boundary nodes of the grid. Let us define subspace $H_h = \overset{\circ}{W}{}_2^1(\omega_h)$ with norm

$$\|v\|_1^2 = \sqrt{\sum_{i_1}^{N_1}\sum_{i_2}^{N_2}\sum_{i_3}^{N_3} h_1 h_2 h_3\left[(\vartheta_{\bar{x}_1})^2 + (\vartheta_{\bar{x}_2})^2 + (\vartheta_{\bar{x}_3})^2\right]} \le M.$$

Here M does not depend on $h_1, \ h_2, \ h_3$, $\vartheta = \vartheta(i_1h_1, i_2h_2, i_3h_3)$,

$$\begin{aligned} \vartheta_{\bar{x}_1} &= \left[\vartheta(i_1h_1, i_2h_2, i_3h_3) - \vartheta((i_1-1)h_1, i_2h_2, i_3h_3)\right]/h_1, \\ \vartheta_{\bar{x}_2} &= \left[\vartheta(i_1h_1, i_2h_2, i_3h_3) - \vartheta(i_1h_1, (i_2-1)h_2, i_3h_3)\right]/h_2\,, \\ \vartheta_{\bar{x}_3} &= \left[\vartheta(i_1h_1, i_2h_2, i_3h_3) - \vartheta(i_1h_1, i_2h_2, (i_3-1)h_3)\right]/h_3\,, \end{aligned}$$

where $\overset{\circ}{W}{}_2^1(\omega_h)$ is the Sobolev space [12].

Now, approximating operators L_0, L_1, and L_2 by difference relations, we obtain the following problem:

$$\mathrm{D}\frac{d^4u_h}{dt^4} + \mathrm{B}\frac{d^2u_h}{dt^2} + \mathrm{A}u_h(t) = f_h, \quad \frac{d^ku_h}{dt^k}(0) = u_{0,k,h}, \qquad k = \overline{0,3}, \tag{6.5}$$

where u_h approximates $u(x,t)$, D, B, and A are linear constant operators from $H_h \to H_h$, $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0 \ \forall\ t \geq 0$, $u_h = u_h(t) \in H_h$, $f_h = f_h(t) \in H_h$. Here operators are:

$$\mathrm{D} = \Lambda - \frac{1}{r_{\mathrm{D}}^2} E,\ \ \mathrm{B} = \omega_0^2 \Lambda - \frac{\omega_{\mathrm{B}i}^2}{r_{\mathrm{D}}^2} E,\ \ \mathrm{A} = \omega_1^2 \Lambda_3,$$

where $\Lambda = \sum\limits_{\alpha=1}^{3} \Lambda_\alpha$, $\Lambda_m u_h = u_{h,\bar{x}_m x_m}$, $m = 1,2,3$, u_h are the values of function $u(x,t)$ at the fixed node $x = (i_1 h_1, i_2 h_2, i_3 h_3)$,

$$u_{h,\bar{x}_1 x_1} = (u_h((i_1+1)h_1, i_2 h_2, i_3 h_3) - 2u_h(i_1 h_1, i_2 h_2, i_3 h_3) + u_h((i_1-1)h_1, i_2 h_2, i_3 h_3))/\, h_1^2,$$
$$u_{h,\bar{x}_2 x_2} = (u_h(i_1 h_1, (i_2+1)h_2, i_3 h_3) - 2u_h(i_1 h_1, i_2 h_2, i_3 h_3) + u_h(i_1 h_1, (i_2-1)h_2, i_3 h_3))/\, h_2^2,$$
$$u_{h,\bar{x}_3 x_3} = (u_h(i_1 h_1, i_2 h_2, (i_3+1)h_3) - 2u_h((i_1 h_1, i_2 h_2, i_3 h_3)) + u_h(i_1 h_1, i_2 h_2, (i_3-1)h_3))/\, h_3^2.$$

Operators D, B, and A approximate operators L_0, L_1, and L_2 from (6.4) with the second order, respectively, i.e., $O(|h|^2)$, $|h| = \sqrt{h_1^2 + h_2^2 + h_3^2}$.

For approximation of (6.5), we apply difference scheme (5.1) with parameters $\sigma_1 = \sigma_2 = \sigma$, $\sigma_3 = \sigma_4 = \theta$, i.e., we have a two-parameter difference scheme

$$\bar{\mathrm{D}} y_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}} y_{\bar{t}t}^{(\sigma)} + \mathrm{A} y^{(\theta)} = \bar{\varphi}, \quad t_n \in \bar{\omega}_\tau. \tag{6.6}$$

The initial conditions remain the same, i.e., (4.3). Based on Theorem 5.3 and the results of discretization in space, we obtain the following result.

Theorem 6.1. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and $\theta \leq 1/2$, $\sigma \leq 1/4$, $\bar{\mathrm{D}} \geq (\tau^4/8)\mathrm{A}$ are satisfied, for the solution of scheme (6.6), (4.3), (4.4), the following accuracy estimate holds:

$$\|y(x_i, t_n) - u(x_i, t_n)\|_{\bar{\mathrm{A}}} \leq O(|h|^2 + \tau^4),\ \ y,\, u \in H_h,\ \ x_i \in \bar{\omega}_h,\ \ t_n \in \bar{\omega}_\tau.$$

If we choose $h_\alpha = h$, $\alpha = \overline{1,3}$, then condition $\bar{\mathrm{D}} \geq (\tau^4/8)\mathrm{A}$ will be satisfied for $\tau \leq 2h/\sqrt{\omega_1}$.

7. Numerical implementation

If $\sigma = \theta = 0$, then from (6.6) we can derive explicit scheme (4.2), (4.3), which is realized directly, and for other σ and θ, we obtain implicit schemes, which can be implemented using the sweep method.

8. Conclusions

In this article, we have constructed and analyzed fourth-order accurate schemes for a fourth-order non-stationary equation. We established stability conditions and derived a priori estimates. Based on these estimates, we proved theorems regarding the convergence and accuracy of the solutions for the difference schemes. The difference schemes developed for the abstract Cauchy problem were then applied to solve the initial-boundary value problem for the Sobolev sixth-order partial differential equation. We also proved theorems concerning the convergence and accuracy of the constructed difference schemes. These findings pave the way for the development and analysis of difference schemes for other non-classical high-order equations with various boundary conditions.

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