Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 34–48
DOI: 10.29229/uzmj.2026-1-4

On a coefficient inverse problem with nonlocal boundary conditions of periodic type for the three-dimensional Tricomi equation in a parallelepiped

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Abstract. This article addresses the well-posedness of a coefficient inverse problem for the three-dimensional Tricomi equation in a parallelepiped. For this problem with nonlocal boundary conditions in anisotropic Sobolev spaces, we prove existence and uniqueness theorems for a regular solution using Fourier methods, $\varepsilon-$ regularization, a priori estimates, successive approximations, and contraction mappings

Keywords: three-dimensional Tricomi equation, coefficient inverse problem with nonlocal boundary conditions of periodic type, well-posedness, Fourier methods, $\varepsilon-$ regularization, a priori estimates, successive approximations
MSC (2020): 35M10

1. Introduction

In the study of nonlocal problems, a close relationship has been observed between problems with nonlocal boundary conditions and inverse problems [1]. By now, inverse problems for classical equations of mathematical physics have been studied extensively [2, 3, 4, 5]. Linear inverse problems (involving the determination of solutions and right-hand sides of equations) for equations of mixed type, both of the first and the second kind, have been considered in [6, 7, 8, 9, 10].

In contrast, coefficient inverse problems (involving the determination of solutions, coefficients, and right-hand sides of equations) for equations of mixed type have not been studied in detail. In this work, we aim to partially fill this gap.

In particular, we investigate the unique solvability of a coefficient inverse problem for the three-dimensional Tricomi equation in a parallelepiped. Our approach is based on reducing the coefficient inverse problem to a family of direct problems for loaded Tricomi differential equations with nonlocal boundary conditions of periodic type in a bounded rectangular domain [5, 11].

Recall that a loaded equation is a partial differential equation whose coefficients or right-hand side involve certain functionals of the solution itself [8, 9].

2. Coefficient inverse problem

In the domain

$$G = Q \times (0, \ell) = \{(x, t, y)| \ -1 < x < 1, \ 0 < t < T, \ 0 < y < \ell\}$$

we consider the three-dimensional Tricomi equation:

$$Lu = xu_{tt} - au_{xx} - u_{yy} + \alpha(x, t)u_t = c(x, t)u + \psi(x, t, y), \tag{2.1}$$

here $\psi(x, t, y) = g(x, t, y) + h(x, t) \cdot f(x, t, y)$, where $g(x, t, y)$ and $f(x, t, y)$ are given functions, while $h(x, t)$ and $c(x, t)$ are unknown. For simplicity, we omit function arguments from now on.

Problem statement. Find functions $\{u(x, t, y), \ h(x, t), \ c(x, t)\}$, satisfying equation (2.1) almost everywhere in domain G, such that $u(x, t, y)$ satisfies the following boundary conditions:

1) non-local boundary conditions of periodic type:

$$\gamma D_t^p u|_{t=0} = D_t^p u|_{t=T}, \tag{2.2}$$

$$D_x^p u|_{x=-1} = D_x^p u|_{x=1}, \quad p = 0, 1 \tag{2.3}$$

$$u|_{y=0} = u|_{y=l} = 0, \tag{2.4}$$

where γ is some constant number, such that $|\gamma| > 1$;

2) additional conditions

$$u(x,t,\ell_1) = \varphi_1(x,t), \tag{2.5}$$

$$u(x,t,\ell_2) = \varphi_2(x,t), \tag{2.6}$$

$$0 < \ell_1 < \ell_2 < \ell < +\infty$$

and together with the functions $h(x,t)$, $c(x,t)$ belong to the class

$$U = \left\{(u,\ h,\ c)|\ u \in W_2^{2,3}(G),\ h \in W_2^2(Q),\ c \in W_2^2(Q)\right\}.$$

Here $W_2^{2,3}(G)$ is an anisotropic Sobolev space with norm

$$[u]_{l,3}^2 \equiv |\|u\||^2_{W_2^{l,3}(G)} = \sum_{k=1}^{\infty} \left(1+\lambda_k^2\right)^3 \|u_k\|^2_{W_2^l(Q)}, \quad l = 0,1,2,...$$

where $u_k(x,t)$ denote the coefficients of the Fourier expansion of the functions $u(x,t,y)$ in the system $\{Y_k(y)\} = \left\{\sqrt{\frac{2}{\ell}}\sin\lambda_k y\right\}$, $\lambda_k = \frac{\pi k}{\ell}$, $k = 1,2,3,...$; $W_2^l(Q)$ is the Sobolev space with the norm

$$\|u\|^2_{W_2^l(Q)} = \|u\|_l^2 = \sum_{|\alpha|=0}^{l} \int_Q (D^\alpha u)^2 dxdt,$$

where $l = 0,1,2,...$, $D^\alpha-$ generalized derivative with respect to the variables x and t. Further through $C^l(Q)$ let us denote the spaces of continuously differentiable functions up to order l inclusive with norm

$$\|u\|_{C^l(Q)} = \sum_{|\alpha|=0}^{l} \max_{(x,t)\in Q} |D^\alpha u|.$$

For further study of the inverse problem, we need the following auxiliary theorems and notations in a simplified form.

Theorem A.1. (S.L. Sobolev). There is a continuous embedding $W_2^2(Q) \subset C(Q)$, i.e.

$$\|u\|^2_{C(Q)} \le c_2 \|u\|^2_{W_2^2(Q)},$$

where c_2 is a positive constant [12, 13].

Theorem A.2. For any function $u(x,t) \in W_2^1(Q)$ holds the following inequality holds

$$\|u\|_{L_4(Q)} \le c_3 \|u\|_{W_2^1(Q)},$$

where c_3 is a positive constant [12, 13].

Let us introduce the following notations:

$$g_j(x,t) = g(x,t,\ell_j), \quad f_j(x,t) = f(x,t,\ell_j), \quad \forall j = 1,2; \quad H = \det\begin{pmatrix}\varphi_1 & f_1\\ \varphi_2 & f_2\end{pmatrix},$$

$$\gamma_0 = m_1\,[g]^2_{1,3}, \quad \gamma_1 = m_2\,[f]^2_{2,3}, \quad \gamma_2 = m_2;$$

$$m_1 = 60\delta_0^{-1}\delta^{-1}e^{\mu T}\left(4\left(1+\mu^2\right) + \delta^{-1}e^{\mu T}\left(\|\alpha_x\|_C^2 + \|\alpha_t\|_C^2\right)\right),$$

$$m_2 = 192\delta_0^{-1}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)$$

where $\mu = \frac{2}{T}\ln|\gamma| > 0$, $|\gamma| > 1$, $\delta = min\left\{\frac{\delta_0}{2}, \delta_1, a\mu, \mu\left(\frac{\pi}{\ell}\right)^2\right\}$, δ_1 will be defined in the theorem,

$$\mathfrak{F} = \max\left(\|f_1\|^2_{C^{0,1}_{x,t}(\bar{Q})}, \|f_2\|^2_{C^{0,1}_{x,t}(\bar{Q})}, \|\varphi_1\|^2_{C^{0,1}_{x,t}(\bar{Q})}, \|\varphi_2\|^2_{C^{0,1}_{x,t}(\bar{Q})}\right),$$

$$c_1 = \sum_{m=1}^{\infty} \frac{\lambda_m^4}{\left(1+\lambda_m^2\right)^3}$$

and c_2, c_3 are as defined above.

Definition 2.1. A generalized solution of problem (2.1)-(2.6) will be a functions $\{u, h, c\}$ from class U satisfying equation (2.1) almost everywhere in domain G, with conditions (2.2)-(2.6).

Assume all coefficients of equation (2.1) are sufficiently smooth in $\overline{Q}$, and the following conditions are satisfied with respect to the coefficients, right-hand side, and given functions $\varphi_i(x,t)$, $i = 1, 2$:
Conditions 1:
nonlocal conditions: $\alpha(x,0) = \alpha(x,T), \quad \alpha(-1,t) = \alpha(1,t);$

$$\gamma g(x,0,y) = g(x,T,y), \quad \gamma f(x,0,y) = f(x,T,y);$$

smoothness: $f_i \in C^{0,1}_{x,t}(\bar{Q}),\ g_i \in C^{0,1}_{x,t}(\bar{Q}),\ f \in W^{2,3}_2(G),\ g \in W^{2,3}_2(G).$

coefficient conditions: $2\alpha(x,t) + \mu x > \delta_0 > 1.$

Conditions 2:
Functions $\varphi_i \in W^3_2(Q)$, $i = 1, 2$ is the solution to the following problem:

$$L_1\varphi_i = g_i,$$

$$\gamma D^p_t \varphi_i|_{t=0} = D^p_t \varphi_i|_{t=T},$$

$$D^p_x \varphi_i|_{x=-1} = D^p_x \varphi_i|_{x=1}, p = 0, 1$$

where $L_1\varphi_i = x\varphi_{itt} - a\varphi_{ixx} + \alpha\varphi_{it}$. Besides this let

$$|H| = |\varphi_1 f_2 - \varphi_2 f_1| \geq \eta > 0.$$

Without loss of generality, we may take $\eta = 1$.

To prove the solvability of problem (2.1)−(2.6), we first use the Fourier method. Namely, we search the solution to problem (2.1)-(2.6) in the form

$$u(x,t,y) = \sum_{k=1}^{\infty} u_k(x,t) Y_k(y), \tag{A}$$

where functions $\{Y_k(y)\} = \left\{\sqrt{\frac{2}{\ell}} \sin \lambda_k y\right\}$, $\lambda_k = \frac{\pi k}{\ell}$, $k = 1, 2, 3, ...$ are solutions of the Sturm-Liouville spectral problem with Dirichlet conditions, $u_k(x,t)$ are Fourier coefficients of the function $u(x,t,y)$. It is known that the system of eigenfunctions $\{Y_k(y)\}$ is complete in the space $L_2(0,\ell)$ and forms an orthonormal basis [2, 9, 10, 14, 15]. In order to determine unknown functions $u_k(x,t)$, it is necessary to perform some construction formalities. Let us consider the traces of equation (2.1) at $y = \ell_i$, $i = 1, 2$,

$$Lu\big|_{y=l_i} = x\varphi_{itt} - a\varphi_{ixx} - u_{yy}(x,t,\ell_i) + \alpha\varphi_{it} = c\varphi_i + hf_i + g_i,\ i = 0, 1. \tag{2.7}$$

Now, taking into account conditions (2.5), (2.6) and condition $H \neq 0$, we define formally unknown functions $h(x,t)$ and $c(x,t)$ from the system of equations (2.7):

$$\begin{cases} c\varphi_1 + hf_1 = \Phi, \\ c\varphi_2 + hf_2 = \Psi, \end{cases}$$

where $\Phi = \sum\limits_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1$, $\Psi = \sum\limits_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_2$, in the form

$$h(x,t) = \frac{1}{H}\begin{vmatrix} \varphi_1 & \Phi \\ \varphi_2 & \Psi \end{vmatrix} = \frac{1}{H}\left[\varphi_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_2 - \varphi_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1\right]$$

and

$$c(x,t) = \frac{1}{H}\begin{vmatrix} \Phi & f_1 \\ \Psi & f_2 \end{vmatrix} = \frac{1}{H}\left[f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1 - f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_2\right],$$

and to determine the functions $u_k(x,t)$ in the domain Q we obtain infinite systems of loaded nonlinear differential Tricomi equations:

$$\begin{gathered} L\,u_k = xu_{k\,tt} - au_{k\,xx} + \alpha u_{k\,t} + \lambda_k^2 u_k \\ = \frac{u_k}{H}\left[f_2\sum_{m=1}^{\infty}\lambda_m^2 u_m(x,t)\sin\lambda_m\ell_1 - f_1\sum_{m=1}^{\infty}\lambda_m^2 u_m(x,t)\sin\lambda_m\ell_2\right] \\ +\frac{f_k}{H}\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2 u_m(x,t)\sin\lambda_m\ell_2 - \varphi_2\sum_{m=1}^{\infty}\lambda_m^2 u_m(x,t)\sin\lambda_m\ell_1\right] + g_k \equiv F(u_k) \end{gathered} \tag{2.8}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^p u_k\big|_{t=0} = D_t^p u_k\big|_{t=T}, \tag{2.9}$$

$$D_x^p u_k\big|_{x=-1} = D_x^p u_k\big|_{x=1}, \quad p=0,1, \tag{2.10}$$

where f_k and g_k are the coefficients of the Fourier expansion of the functions f and g, $k=1,2,....$

3. Main result

Let us introduce the following notation

$$q \equiv 4m_1m_2\left(2m_1\,[g]_{1,3}^2 + [f]_{2,3}^2\right).$$

Theorem 3.1. *Let Conditions 1 and 2 above be satisfied for the coefficients of equation* (2.1)*. Suppose there exists a*δ_1 *such that*

$$a - \frac{5\delta_0}{48} > \delta_1 > 1,$$

and assume

$$q < \frac{1}{2}.$$

Then problem (2.1)–(2.6) *admits a unique solution in the class* U.

Remark 3.2. The inequality $q < \frac{1}{2}$ can be achieved by choosing the domain sufficiently small and by requiring that g_i and their derivatives are sufficiently small.

Proof. Let us prove the theorem step by step. Let there exist a solution to problem (2.1) – (2.4) from the class U. First, let us show that the function $u(x,t,y)$ satisfies the boundary conditions (2.5) – (2.6) for any $i=1,2$, i.e.

$$u(x,t,\ell_i) = \varphi_i(x,t), \quad i=1,2.$$

We prove that these conditions are satisfied using the converse assumptions. Let there exist functions $\vartheta_i(x,t)$ satisfying conditions (2.5), (2.6):

$$u(x,t,\ell_i) = \sum_{k=1}^{\infty} u_k(x,t)\sin\lambda_k\ell_i = \vartheta_i(x,t) \neq \varphi_i(x,t).$$

To make it easier to understand, let's consider each case separately. First, let's consider the case when $i=1$. Then for the function $z_1(x,t) = \vartheta_1(x,t) - \varphi_1(x,t)$ in the domain Q, taking into account conditions (2.9), (2.10), multiplying the systems of equations (2.8) by $\sin\mu_k\ell_1$ and summing over k from 1 to ∞, we obtain the following loaded equations

$$\begin{gathered} x\vartheta_{1\ tt} - a\vartheta_{1\ xx} + \alpha\vartheta_{1\ t} + \sum_{k=1}^{\infty}\lambda_k^2 u_k\sin\lambda_k\ell_1 \\ = \frac{1}{H}\vartheta_1\left[f_2\sum_{m=1}^{\infty}\lambda_m^2 u_m(x,t)\sin\lambda_m\ell_1 - f_1\sum_{m=1}^{\infty}\lambda_m^2 u_m(x,t)\sin\lambda_m\ell_2\right] \\ +\frac{1}{H}f_1\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2 u_m(x,t)\sin\lambda_m\ell_2 - \varphi_2\sum_{m=1}^{\infty}\lambda_m^2 u_m(x,t)\sin\lambda_m\ell_1\right] + g_1. \end{gathered} \tag{3.1}$$

Let us substitute expression $z_1+\varphi_1=\vartheta_1=\sum\limits_{m=1}^{\infty}u_m(x,t)\sin\lambda_m\ell_1$ instead of ϑ_1. Then from (3.1) it follows

$$x(z_1+\varphi_1)_{tt}-a(z_1+\varphi_1)_{xx}+\alpha(z_1+\varphi_1)_t+\sum_{m=1}^{\infty}\lambda_m^2u_m(x,t)\sin\lambda_m\ell_1 \\ =g_1+cz_1+\frac{1}{H}\sum_{k=1}^{\infty}\lambda_k^2u_k\sin\lambda_k\ell_1[\varphi_1f_2-f_1\varphi_2]. \tag{3.2}$$

Based on formulas (2.8)−(3.1) for the function $z_1(x,t)=\vartheta_1(x,t)-\varphi_1(x,t)$ in the domain Q we obtain the following problem

$$L_0z_1\equiv xz_{1\ tt}-az_{1\ xx}+\alpha z_{1\ t}=cz_1, \tag{3.3}$$

$$\gamma D_t^pz_1|_{t=0}=D_t^pz_1|_{t=T},\quad p=0,1, \tag{3.4}$$

$$z_{1x}|_{x=-1}=z_{1x}|_{x=1}=0. \tag{3.5}$$

Now we prove the uniqueness of the solution to problem (3.3)−(3.5) by the energy integral method [10]. To do this, consider the identity

$$2(L_0z_1,e^{-\mu t}(z_{1\,t}+z_1))=2(cz_1,e^{-\mu t}(z_{1\,t}+z_1)); \tag{3.6}$$

integrating by parts identity (3.6) taking into account the conditions of Theorem 3.1 and boundary conditions (3.4), (3.5) for $|\gamma|>1$, and applying Sobolev embedding theorems, we obtain the following inequality

$$\delta e^{\mu T}\|z_1\|_1^2\le 2\|c\|_{C(Q)}^2\|z_1\|_1^2,$$

$$\left(\delta e^{\mu T}-2\|c\|_{C(Q)}^2\right)\|z_1\|_1^2\le 0. \tag{3.7}$$

At the end we will show the correctness of the inequality $\|c\|_{C(Q)}^2<r$, where $r=\frac{1}{2}\delta e^{\mu T}$. For now we will just use it and obtain $\|z_1\|_1^2\le 0$, i.e. $z_1=0$, from which it follows $\vartheta_1=\varphi_1$. Thus, for $y=\ell_1$ the solution of equation (2.1) satisfies the condition $u(x,t,\ell_1)=\varphi_1(x,t)$. Similarly, $u(x,t,\ell_2)=\varphi_2(x,t)$ is proved in the case when $i=2$.

Now we will prove the solvability of problem (2.8)−(2.10) by the methods of "$\varepsilon-$ regularization", a priori estimates and successive approximations [10, 16], namely, in domain Q we will consider the following family of nonlinear equations of the third order with a small parameter:

$$L_\varepsilon u_{k,\varepsilon}=-\varepsilon u_{k,\varepsilon\,ttt}+L_0u_{k,\varepsilon}+\lambda_k^2u_{k,\varepsilon} \\ =\frac{u_{k,\varepsilon}}{H}\left[f_1\sum_{m=1}^{\infty}\lambda_m^2u_m(x,t)\sin\lambda_m\ell_1-f_1\sum_{m=1}^{\infty}\lambda_m^2u_m(x,t)\sin\lambda_m\ell_2\right] \\ +\frac{f_k}{H}\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2u_m(x,t)\sin\lambda_m\ell_2-\varphi_2\sum_{m=1}^{\infty}\lambda_m^2u_m(x,t)\sin\lambda_m\ell_1\right]+g_k\equiv Fu_{k,\varepsilon}, \tag{3.8}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^pu_{k,\varepsilon}|_{t=0}=D_t^pu_{k,\varepsilon}|_{t=T},\quad p=0,1,2, \tag{3.9}$$

$$D_x^qu_{k,\varepsilon}|_{x=-1}=D_x^qu_{k,\varepsilon}|_{x=1},\quad q=0,1, \tag{3.10}$$

where ε is a small positive number.

To prove the unique solvability of problem (3.8)−(3.10), we need the following notations and lemmas.

Let us define the spaces of vector functions

$$W_s(Q)=\{v=(v_1,v_2,...,v_k,...)|\ v_k\in W_2^s(Q),\ k=1,2,3,...\},\quad s=0,1,2$$

with the norm

$$\langle \upsilon\rangle_s^2 \equiv \sum_{k=1}^{\infty} \left(1+\lambda_k^2\right)^3 \|\upsilon_k\|^2_{W_2^s(Q)}, \tag{B}$$

where $W_2^s(Q)-$ Sobolev spaces. From the definition of the spaces $W_s(Q)$, $s=0,1,2$ with a certain norm (B) it follows that $W_2(Q)\subset W_1(Q)\subset W_0(Q)$ and they are Banach spaces.

Now by $W(Q)=\{\{\upsilon_k\}_{k=1}^{\infty}|\quad \{\upsilon_{k\,ttt}\}_{k=1}^{\infty}\in L_2(Q),\ \{\upsilon_k\}_{k=1}^{\infty}\in W_2(Q)\}$ we denote the class of vector functions satisfying the corresponding boundary conditions (3.9)−(3.10).

Definition 3.3. A generalized solution of problem (3.8)−(3.10) will be a vector of functions $\{\upsilon_k^{(\theta)}\}_{k=\overline{1,\infty}}^{\theta=\overline{0,\infty}}\subset W(Q)$ satisfying equation (3.8) almost everywhere in Q.

We prove the solvability of problem (3.8)−(3.10) by the method of successive approximations, namely, we consider the following system of nonlinear loaded equations of the third order with a small parameter

$$\begin{gathered} L_\varepsilon u_{k,\varepsilon}^{(\theta)} = -\varepsilon u_{k,\varepsilon\ ttt}^{(\theta)} + L_0 u_{k,\varepsilon}^{(\theta)} + k^2 u_{k,\varepsilon}^{(\theta)} \\ = \frac{u_{k,\varepsilon}^{(\theta-1)}}{H}\left[f_2\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1 - f_1\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin m\ell_2\right] \\ +\frac{f_k}{H}\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_2 - \varphi_2\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1\right] + g_k \equiv F(u_{k,\varepsilon}^{(\theta-1)}) \end{gathered} \tag{3.11}$$

with semi-nonlocal boundary conditions

$$\gamma D_t^p u_{k,\varepsilon}^{(\theta)}\Big|_{t=0} = D_t^p u_{k,\varepsilon}^{(\theta)}\Big|_{t=T} \quad p=0,1,2, \tag{3.12}$$

$$u_{k,\varepsilon x}^{(\theta)}\Big|_{x=-1} = u_{k,\varepsilon x}^{(\theta)}\Big|_{x=1} = 0, \tag{3.13}$$

where $\varepsilon>0$, $\theta=0,1,2,...,\quad k=1,2,3...,$ $u_{k,\varepsilon}^{(-1)}\equiv 0$.

Lemma 3.4. *Let all the conditions of Theorem 3.1 be satisfied, then the following estimates are valid for the solutions of the problem* (3.11) − (3.13)*:*

I) $\frac{\varepsilon}{\delta}\left\langle u_{\varepsilon\ tt}^{(\theta)}\right\rangle_0^2 + \left\langle u_\varepsilon^{(\theta)}\right\rangle_1^2 < 2\gamma_0$,

II) $\frac{\varepsilon}{\delta}\left\langle u_{\varepsilon\ ttt}^{(\theta)}\right\rangle_0^2 + \left\langle u_\varepsilon^{(\theta)}\right\rangle_2^2 < 2\gamma_0$.

Proof. Consider the identity.

$$2\left(L_\varepsilon u_{k,\varepsilon}^{(\theta)}, e^{-\mu t}u_{k,\varepsilon\ t}^{(\theta)}\right)_0 = 2\left(F_\varepsilon(u_{k,\varepsilon}^{(\theta-1)}), e^{-\mu t}u_{k,\varepsilon\ t}^{(\theta)}\right)_0. \tag{3.14}$$

Let us estimate it from below. Integrating by parts we get:

$$\begin{gathered} \varepsilon e^{-\mu T}\left\|u_{k,\varepsilon\ tt}^{(\theta)}\right\|_0^2 + \int_Q (2\alpha(x,t)+\mu x)u_{k,\varepsilon\ t}^{2(\theta)}e^{-\mu t}dQ + \int_Q a\mu u_{k,\varepsilon\ x}^{2(\theta)}e^{-\mu t}dQ + \mu\left(\frac{\pi}{\ell}\right)^2\int_Q u_{k,\varepsilon}^{2(\theta)}e^{-\mu t}dQ \\ -\int_{\partial Q}(2\varepsilon u_{k,\varepsilon\ t}^{(\theta)}u_{k,\varepsilon\ tt}^{(\theta)} + x u_{k,\varepsilon\ t}^{2(\theta)} - \mu u_{k,\varepsilon\ t}^{2(\theta)} + \lambda_k^2 u_{k,\varepsilon\ t}^{2(\theta)} + u_{k,\varepsilon\ x}^{2(\theta)})e^{-\mu t}e_t dS \\ -\int_{\partial Q} 2u_{k,\varepsilon\ t}^{(\theta)}u_{k,\varepsilon\ x}^{(\theta)}e_x e^{-\mu t}dS \le 2\left(L_\varepsilon u_{k,\varepsilon}^{(\theta)}, e^{-\mu t}u_{k,\varepsilon\ t}^{(\theta)}\right)_0, \end{gathered} \tag{3.15}$$

where $\vec{e}=((e_x,e_t)|\ e_x=(\vec{e},x),\ e_t=(\vec{e},t))$ is the unit internal normal vector to the boundary ∂Q. Let us estimate it from above. Using Cauchy's inequalities with σ to (3.15) we have

$$\begin{gathered} 2\left(F_\varepsilon(u_{k,\varepsilon}^{(\theta-1)}), e^{-\mu t}u_{k,\varepsilon\ t}^{(\theta)}\right)_0 \le \sigma\|g_k\|_0^2 + \sigma c_1c_3^2\left(\|f_0\|_C^2+\|f_1\|_C^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right) \\ +\sigma c_1c_3^2\left(\|\varphi_0\|_C^2+\|\varphi_1\|_C^2\right)\|f_k\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right) + 5\sigma^{-1}\int_Q u_{k,\varepsilon\ t}^{2(\theta)}e^{-\mu t}dQ, \end{gathered} \tag{3.16}$$

where c_3 is the coefficient from the Sobolev embedding theorem, i.e. $\|u\|^2_{L_4(Q)} \leq c_3 \|u\|^2_{W_2^1(Q)}$ [12, 17]. Taking into account the boundary conditions (3.12), (3.13) and $\gamma^2 = e^{\mu T}$, we obtain that the boundary integrals will vanish. After performing some simple operations we get

$$\begin{gathered}\varepsilon e^{-\mu T}\left\|u^{(\theta)}_{k,\varepsilon\ tt}\right\|^2_0 + \delta e^{-\mu T}\left\|u^{(\theta)}_{k,\varepsilon}\right\|^2_1 \leq \sigma\|g_k\|^2_0 + 2\sigma c_1 c_3^2 \mathfrak{F}\|f_k\|^2_1\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u^{(\theta-1)}_{m,\varepsilon}\right\|^2_1\right)\\ +2\sigma c_1 c_3^2\mathfrak{F}\left\|u^{(\theta-1)}_{k,\varepsilon}\right\|^2_1\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u^{(\theta-1)}_{m,\varepsilon}\right\|^2_1\right),\end{gathered} \tag{3.17}$$

where $\delta = \min\left(\delta_0/2, \mu a, \mu(\frac{\pi}{\ell})^2\right)$. Here we took σ equal to $\sigma = \frac{10}{\delta_0}$, which follows $\delta - 5\sigma^{-1} = \frac{\delta_0}{2}$. Now multiplying (3.17) by $(1+\lambda_k^2)^3$ and summing the equations by k from 1 up to ∞ we obtain the first recurrent formula

$$\frac{\varepsilon}{\delta}\left\langle u^{(\theta)}_{\varepsilon\ tt}\right\rangle^2_0 + \left\langle u^{(\theta)}_{\varepsilon}\right\rangle^2_1 < \gamma_{00} + \gamma_{11}\left\langle u^{(\theta-1)}_{\varepsilon}\right\rangle^2_1 + \gamma_{22}\left\langle u^{(\theta-1)}_{\varepsilon}\right\rangle^4_1, \tag{3.18}$$

here

$$\gamma_{00} = 30{\delta_0}^{-1}\delta^{-1}e^{\mu T}[g]^2_{0,3},\ \gamma_{11} = 80{\delta_0}^{-1}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\,[f]^2_{1,3}\,,\ \gamma_{22} = 80{\delta_0}^{-1}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}.$$

For convenience, in inequality (3.18) we introduce the notation through $J_1^{(\theta)} = \langle u^{(\theta)}_\varepsilon\rangle^2_1$ and $I_2^{(\theta)} = \left\langle u^{(\theta)}_{\varepsilon\ tt}\right\rangle^2_0$. Then from (3.18) we obtain the following recurrent formula

$$\frac{\varepsilon}{\delta}I_2^{(\theta)} + J_1^{(\theta)} \leq \gamma_{00} + \gamma_{11}J_1^{(\theta-1)} + \gamma_{22}J_1^{2(\theta-1)}. \tag{3.19}$$

Remark 3.5. Note that for linear inverse problems in the recurrent formula (3.19) the coefficient γ_2 will always be zero. However, for nonlinear inverse problems γ_{22} is not zero. This complicates the limited nature of the iterative process.

Now we will prove the boundedness of the iterative process. Let us introduce notations For the initial approximation we take $u^{(-1)}_{k,\varepsilon} = 0$. Then from problem (3.11)–(3.13) for the zero approximation we have

$$\frac{\varepsilon}{\delta}I_2^{(0)} + J_1^{(0)} \leq \gamma_{00} < 2\gamma_{00}.$$

From here, from the recurrent formula (3.19), taking into account the condition of Theorem 3.1, we obtain the following estimate

$$\frac{\varepsilon}{\delta}I_2^{(1)} + J_1^{(1)} < \gamma_{00} + 2\gamma_{00}\gamma_{11} + 4\gamma_0^2\gamma_{22} < 2\gamma_{00}.$$

Continuing this process, by induction from (3.19), taking into account the inequality $\gamma_{11} + 2\gamma_{00}\gamma_{22} < q < \frac{1}{2}$ following from the condition of Theorem 3.1, we obtain the first a priori estimate for $\forall\,\theta$, $\theta = 2, 3, ...,$

$$\frac{\varepsilon}{\delta}I_2^{(\theta)} + J_1^{(\theta)} < \gamma_{00} + \gamma_{11}J_1^{(\theta-1)} + \gamma_{22}J_1^{2(\theta-1)} < \gamma_{00} + 2\gamma_{11}\gamma_{00} + 4\gamma_{22}\gamma_{00}^2 < 2\gamma_{00}. \tag{3.20}$$

Thus, the validity of the first a priori estimate of Lemma 3.4 is proven.
Now we will prove the second a priori estimate.
Let us consider the identities

$$\left|-2\left(L_\varepsilon u^{(\theta)}_{k,\varepsilon}, Pu^{(\theta)}_{k,\varepsilon}e^{-\mu t}\right)_0\right| = \left|-2\left(F_\varepsilon(u^{(\theta-1)}_{k,\varepsilon}), Pu^{(\theta)}_{k,\varepsilon}e^{-\mu t}\right)_0\right|, \tag{3.21}$$

where $Pu^{(\theta)}_{k,\varepsilon} = u^{(\theta)}_{k,\varepsilon\ ttt} - u^{(\theta)}_{k,\varepsilon\ tt} + u^{(\theta)}_{k,\varepsilon\ xx} - u^{(\theta)}_{k,\varepsilon\ t}$. Let us estimate it in the same way as in the case of the first estimate from below and from above. Let us estimate identity (3.21) from below

$$\begin{gathered}\varepsilon e^{-\mu T}\left\|u^{(\theta)}_{k,\varepsilon ttt}\right\|^2_0 + \int\limits_Q\left[\delta_0 u^{2(\theta)}_{k,\varepsilon tt} + a\mu u^{2(\theta)}_{k,\varepsilon xt} + a u^{2(\theta)}_{k,\varepsilon xx} + \delta_0 u^{2(\theta)}_{k,\varepsilon\ t} + a\mu u^{2(\theta)}_{k,\varepsilon\ x} + \mu\left(\frac{\pi}{\ell}\right)^2 u^{2(\theta)}_{k,\varepsilon}\right]e^{-\mu t}dQ\\ -\int\limits_Q\left(\|\alpha_x\|^2_C + \|\alpha_t\|^2_C\right)u^{2(\theta)}_{k,\varepsilon\,t}e^{-\mu t}dQ \leq \left|-2\left(L_\varepsilon u^{(\theta)}_{k,\varepsilon}, Pu^{(\theta)}_{k,\varepsilon}e^{-\mu t}\right)_0\right|.\end{gathered} \tag{3.22}$$

Now let us estimate identity (3.21) from above

$$\left|-2\left(F_\varepsilon(u_{k,\varepsilon}^{(\theta-1)}), Pu_{k,\varepsilon}^{(\theta)}e^{-\mu t}\right)_0\right| \leq 3\sigma\left(1+\mu^2\right)\|g_k\|_1^2 + \left(\|\alpha_x\|_C^2 + \|\alpha_t\|_C^2\right)\int\limits_Q u_{k,\varepsilon\, t}^{2(\theta)}e^{-\mu t}dQ$$

$$+4\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2 + 2\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2$$

$$+24\sigma^{-1}\int\limits_Q u_{k\ tt}^{2(\theta)}e^{-\mu t}dQ + 5\sigma^{-1}\int\limits_Q u_{k\ t}^{2(\theta)}e^{-\mu t}dQ + 5\sigma^{-1}\int\limits_Q u_{k\ xx}^{2(\theta)}e^{-\mu t}dQ. \tag{3.23}$$

Combining (3.22) and (3.23), we get

$$\begin{aligned}
\varepsilon e^{-\mu T}\left\|u_{k,\varepsilon\ ttt}^{(\theta)}\right\|_0^2 + \int\limits_Q &\left[(\delta_0 - 24\sigma^{-1})u_{k,\varepsilon\ tt}^{2(\theta)} + a\mu u_{k,\varepsilon\ xt}^{2(\theta)} + (a-5\sigma^{-1})u_{k,\varepsilon\ xx}^{2(\theta)}\right.\\
&\left.+(\delta_0 - 5\sigma^{-1})u_{k,\varepsilon\ t}^{2(\theta)} + a\mu u_{k,\varepsilon\ x}^{2(\theta)} + \mu\left(\frac{\pi}{\ell}\right)^2 u_{k,\varepsilon}^{2(\theta)}\right]e^{-\mu t}dQ\\
&\leq 3\sigma\left(1+\mu^2\right)\|g_k\|_1^2 + \left(\|\alpha_x\|_C^2 + \|\alpha_t\|_C^2\right)\int\limits_Q u_{k,\varepsilon\, t}^{2(\theta)}dQ\\
&+4\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\
&+2\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right).
\end{aligned} \tag{3.24}$$

Choosing σ equal to $\sigma = 48\delta_0^{-1}$, it follows that $\delta_0 - 24\sigma^{-1} = \frac{\delta_0}{2}$, $\delta_0 - 5\sigma^{-1} > \frac{\delta_0}{2}$, $a - 5\sigma^{-1} > \delta_1 > 1$. Then from inequality (3.24) we obtain

$$\begin{aligned}
\varepsilon e^{-\mu T}\left\|u_{k,\varepsilon\ ttt}^{(\theta)}\right\|_0^2 + \int\limits_Q &\left[\frac{\delta_0}{2}u_{k,\varepsilon\ tt}^{2(\theta)} + a\mu u_{k,\varepsilon\ xt}^{2(\theta)} + \delta_1 u_{k,\varepsilon\ xx}^{2(\theta)} + \delta_0 u_{k,\varepsilon\ t}^{2(\theta)} + a\mu u_{k,\varepsilon\ x}^{2(\theta)} + \mu\left(\frac{\pi}{\ell}\right)^2 u_{k,\varepsilon}^{2(\theta)}\right]e^{-\mu t}dxdt\\
&\leq 184\delta_0^{-1}\left(1+\mu^2\right)\|g_k\|_1^2 + \left(\|\alpha_x\|_C^2 + \|\alpha_t\|_C^2\right)\int\limits_Q u_{k,\varepsilon\, t}^{2(\theta)}\\
&+192\delta_0^{-1}c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\
&+192\delta_0^{-1}c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right).
\end{aligned} \tag{3.25}$$

Using the notation $\delta = min\left\{\frac{\delta_0}{2}, \delta_1, a\mu, \mu\left(\frac{\pi}{\ell}\right)^2\right\}$, we obtain

$$\begin{aligned}
\varepsilon e^{-\mu T}\left\|u_{k,\varepsilon\ ttt}^{(\theta)}\right\|_0^2 + \delta e^{-\mu T}\left\|u_{k,\varepsilon}^{(\theta)}\right\|_2^2 &\leq 184\delta_0^{-1}\left(1+\mu^2\right)\|g_k\|_1^2 + \left(\|\alpha_x\|_C^2 + \|\alpha_t\|_C^2\right)\int\limits_Q u_{k,\varepsilon\, t}^{2(\theta)}dQ\\
&+192\delta_0^{-1}c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\
&+192\delta_0^{-1}c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right).
\end{aligned} \tag{3.26}$$

From (3.20) we have $\left\langle u_{\varepsilon\ t}^{(\theta)}\right\rangle_0^2 < 2\gamma_{00}$. Now multiplying (3.26) by $\left(1+\lambda_k^2\right)^3$ and summing the equations by k from 1 to ∞, we get

$$\frac{\varepsilon}{\delta}\left\langle u_{\varepsilon\ ttt}^{(\theta)}\right\rangle_0^2 + \left\langle u_{\varepsilon}^{(\theta)}\right\rangle_2^2 \le \gamma_0 + \gamma_1\left\langle u_{\varepsilon}^{(\theta-1)}\right\rangle_2^2 + \gamma_2\left\langle u_{\varepsilon}^{(\theta-1)}\right\rangle_2^4, \tag{3.27}$$

where

$$\gamma_0 = 60\delta_0^{-1}\delta^{-1}e^{\mu T}\left(4\left(1+\mu^2\right)+\delta^{-1}e^{\mu T}\left(\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\right)[g]_{1,3}^2,$$
$$\gamma_1 = 192\delta_0^{-1}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)[f]_{2,3}^2,$$
$$\gamma_2 = 192\delta_0^{-1}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right).$$

As in the first estimate, we introduce the notation $I_3^{(\theta)} = \left\langle u_{\varepsilon\ ttt}^{(\theta)}\right\rangle_0^2$, $J_2^{(\theta)} = \left\langle u_{\varepsilon}^{(\theta)}\right\rangle_2^2$, then from (3.27) we obtain the second recurrent formula

$$\frac{\varepsilon}{\delta}I_3^{(\theta)} + J_2^{(\theta)} \le \gamma_0 + \gamma_1 J_2^{(\theta-1)} + \gamma_2 J_2^{2(\theta-1)}. \tag{3.28}$$

Let us take $\left\{u_{k,\varepsilon}^{(-1)}\right\} = 0$ as the initial approximation. Then for the zero approximation we have:

$$\frac{\varepsilon}{\delta}I_3^{(0)} + J_2^{(0)} \le \gamma_0 < 2\gamma_0. \tag{3.29}$$

Hence, taking into account the inequality $\gamma_1 + 2\gamma_0\gamma_2 < q < \frac{1}{2}$, from (3.29) for the first approximation we have

$$\frac{\varepsilon}{\delta}I_3^{(1)} + J_2^{(1)} \le \gamma_0 + 2\gamma_1\gamma_0 + 4\gamma_2\gamma_0^2 \le 2\gamma_0. \tag{3.30}$$

Continuing this process, taking into account the conditions of Theorem 3.1, by iteration, we obtain the following second a priori estimate for $\forall\,\theta \ge 2$:

$$\frac{\varepsilon}{\delta}I_3^{(\theta)} + J_2^{(\theta)} < \gamma_0 + \gamma_1 J_2^{(\theta-1)} + \gamma_2 J_2^{2(\theta-1)} < \gamma_0 + 2\gamma_1\gamma_0 + 4\gamma_2\gamma_0^2 < 2\gamma_0.$$

Thus, the validity of the second estimate of Lemma 3.4 is proven.

□

Now let us introduce a new function from U according to the formula $\vartheta_{k,\varepsilon}^{(\theta)} = u_{k,\varepsilon}^{(\theta)} - u_{k,\varepsilon}^{(\theta-1)}$, $k = 1, 2, \ldots$, $\theta = 0, 1, \ldots$. Then the following lemma holds for it.

Lemma 3.6. *Let the conditions of Theorem 3.1 be satisfied. Then the following estimates are valid for the functions*

III) $\frac{\varepsilon}{\delta}\left\langle \vartheta_{\varepsilon\ \ tt}^{(\theta)}\right\rangle_0^2 + \left\langle \vartheta_{\varepsilon}^{(\theta)}\right\rangle_1^2 < 2\gamma_0 q^\theta$,
VI) $\frac{\varepsilon}{\delta}\left\langle \vartheta_{\varepsilon\ \ ttt}^{(\theta)}\right\rangle_0^2 + \left\langle \vartheta_{\varepsilon}^{(\theta)}\right\rangle_2^2 < 2\gamma_0 q^\theta$
where $q < \frac{1}{2}$.

Proof. First, we prove the third III) estimate. From (3.11)–(3.13) for functions $\left\{\vartheta_{\varepsilon,k}^{(\theta)}\right\} \in U(Q)$ we obtain the following problem:

$$\begin{aligned} L_\varepsilon\vartheta_{k,\varepsilon}^{(\theta)} &= -\varepsilon\vartheta_{k,\varepsilon\,ttt}^{(\theta)} + L_0\vartheta_{k,\varepsilon}^{(\theta)} + \lambda_k^2\vartheta_{k,\varepsilon}^{(\theta)} \\ &= \frac{1}{H}u_{k,\varepsilon}^{(\theta-1)}f_2\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1 - \frac{1}{H}u_{k,\varepsilon}^{(\theta-1)}f_1\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_2 \\ &\quad -\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_2\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-2)}\sin\lambda_m\ell_1 + \frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_1\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-2)}\sin\lambda_m\ell_2 \\ &\quad +\frac{1}{H}f_k\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta_m^{(\theta-1)}\sin\lambda_m\ell_2 - \varphi_2\sum_{m=1}^{\infty}\lambda_m^2\vartheta_m^{(\theta-1)}\sin\lambda_m\ell_1\right] \end{aligned} \tag{3.31}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^q \vartheta_{k,\varepsilon}^{(\theta)}\Big|_{t=0} = D_t^q \vartheta_{k,\varepsilon}^{(\theta)}\Big|_{t=T}, \quad q = 0, 1, 2, \tag{3.32}$$

$$D_x^p \vartheta_{k,\varepsilon}^{(\theta)}\Big|_{x=-1} = D_x^p \vartheta_{k,\varepsilon}^{(\theta)}\Big|_{x=1}, \quad p = 0, 1, \tag{3.33}$$

where $k = 1, 2, ..., \theta = 0, 1, ...$ We change the right-hand side of equation (3.31) adding and subtracting terms $\frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_2 \sum\limits_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1$ and $\frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_1 \sum\limits_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1$ as follows:

$$\begin{gathered} L_\varepsilon u_{k,\varepsilon}^{(\theta)} - L_\varepsilon u_{k,\varepsilon}^{(\theta-1)} = L_\varepsilon \vartheta_k^{(\theta)} = F_\varepsilon u_{k,\varepsilon}^{(\theta-1)} - F_\varepsilon u_{k,\varepsilon}^{(\theta-2)} \\ = \frac{1}{H} u_{k,\varepsilon}^{(\theta-1)} f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1 - \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1 \\ - \frac{1}{H} u_{k,\varepsilon}^{(\theta-1)} f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_2 + \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_2 \\ - \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-2)} \sin \lambda_m \ell_1 + \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1 \\ + \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-2)} \sin \lambda_m \ell_2 - \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_2 \\ + \frac{1}{H} f_k \left[\varphi_1 \sum_{m=1}^{\infty} \lambda_m^2 \vartheta_m^{(\theta-1)} \sin \lambda_m \ell_2 - \varphi_2 \sum_{m=1}^{\infty} \lambda_m^2 \vartheta_m^{(\theta-1)} \sin \lambda_m \ell_1 \right]. \end{gathered} \tag{3.34}$$

Let us consider identity

$$\left| -2 \left(L_\varepsilon \vartheta_k^{(\theta)}, e^{-\mu t} \vartheta_{t\,k,\varepsilon}^{(\theta)} \right)_0 \right| = \left| -2 \left(F_k(u_{k,\varepsilon}^{(\theta-1)}) - F_k(u_{k,\varepsilon}^{(\theta-2)}), e^{-\mu t} \vartheta_{t\,k,\varepsilon}^{(\theta)} \right)_0 \right|. \tag{3.35}$$

Let us show that the identity (3.35) is bounded below. Integrating by parts we obtain:

$$\begin{gathered} \varepsilon e^{-\mu T} \left\| \vartheta_{k,\varepsilon\ tt}^{(\theta)} \right\|_0^2 + \int\limits_Q (2\alpha(x,t) + \mu x) \vartheta_{k,\varepsilon\ t}^{2(\theta)} e^{-\mu t} dQ \\ + \int\limits_Q a\mu \vartheta_{k,\varepsilon\ x}^{2(\theta)} e^{-\mu t} dQ + \mu \left(\frac{\pi}{\ell} \right)^2 \int\limits_Q \vartheta_{k,\varepsilon}^{2(\theta)} e^{-\mu t} dQ \\ + \int\limits_{\partial Q} (2\varepsilon \vartheta_{k,\varepsilon\ t}^{(\theta)} \vartheta_{k,\varepsilon\ tt}^{(\theta)} + x \vartheta_{k,\varepsilon\ t}^{2(\theta)} - \mu \vartheta_{k,\varepsilon\ t}^{2(\theta)} + \lambda_k^2 \vartheta_{k,\varepsilon\ t}^{2(\theta)} + \vartheta_{k,\varepsilon\ x}^{2(\theta)}) e^{-\mu t} e_t dS \\ + \int\limits_{\partial Q} (-2) u_{k,\varepsilon\ t}^{(\theta)} \vartheta_{k,\varepsilon\ x}^{(\theta)} e_x e^{-\mu t} dS \le \left| -2 \left(L_\varepsilon \vartheta_k^{(\theta)}, e^{-\mu t} \vartheta_{t\,k,\varepsilon}^{(\theta)} \right)_0 \right|. \end{gathered}$$

Considering the boundary conditions (3.32), (3.33) and $\gamma^2 = e^{\mu T}$, we get that the boundary integrals are equal to zero:

$$\begin{gathered} \varepsilon e^{-\mu T} \left\| \vartheta_{k,\varepsilon\ tt}^{(\theta)} \right\|_0^2 + \int\limits_Q ((2\alpha(x,t) + \mu x) \vartheta_{k,\varepsilon\ t}^{2(\theta)} + a\mu \vartheta_{k,\varepsilon\ x}^{2(\theta)} + \mu \left(\frac{\pi}{\ell} \right)^2 \vartheta_{k,\varepsilon}^{2(\theta)}) e^{-\mu t} dQ \\ \le \left| -2 \left(L_\varepsilon \vartheta_k^{(\theta)}, e^{-\mu t} \vartheta_{t\,k,\varepsilon}^{(\theta)} \right)_0 \right|. \end{gathered} \tag{3.36}$$

Let us show the upper bound of expression (3.36). Let us write the right part in expanded form:

$$\left|-2\Big(F_k(u_{k,\varepsilon}^{(\theta-1)})-F_k(u_{k,\varepsilon}^{(\theta-2)}),e^{-\mu t}\vartheta_{t\,k,\varepsilon}^{(\theta)}\Big)_0\right|$$

$$\begin{aligned}=\Bigg|-2\int\limits_Q\Bigg(&\frac{1}{H}u_{k,\varepsilon}^{(\theta-1)}f_2\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1-\frac{1}{H}u_{k,\varepsilon}^{(\theta-1)}f_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_2\\ &-\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_2\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-2)}\sin\lambda_m\ell_1+\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_1\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-2)}\sin\lambda_m\ell_2\\ &+\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_2\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1-\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_2\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1\\ &+\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_1\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m l_2-\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_1\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_2\\ +\frac{1}{H}f_k&\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m l_2-\varphi_2\sum_{m=1}^{\infty}\lambda_m^2\vartheta_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m l_1\right]\Bigg)e^{-\mu t}\vartheta_{t\,k,\varepsilon}^{(\theta)}dQ\Bigg|.\end{aligned}\tag{3.37}$$

Applying the Cauchy inequality with σ to (3.37) and using the results of Lemma 3.4 we obtain

$$\begin{aligned}\left|-2\left(F_k(u_{k,\varepsilon}^{(\theta-1)})-F_k(u_{k,\varepsilon}^{(\theta-2)}),e^{-\mu t}\vartheta_{t\,k,\varepsilon}^{(\theta)}\right)_0\right|&\le 4\sigma c_1c_3^2\mathfrak{F}\left\|\vartheta_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right)\\ &+4\sigma c_1c_3^2\mathfrak{F}\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right)\\ +4\sigma c_1c_3^2\mathfrak{F}\left\|f_k\right\|_2^2&\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right)+5\sigma^{-1}\int\limits_Q\vartheta_{k,\varepsilon\,t}^{2(\theta)}e^{-\mu t}dQ.\end{aligned}\tag{3.38}$$

Taking into account the conditions of Theorem 3.1, the boundary conditions and we note that the boundary integrals that appear during the calculations vanish. From below, identity (3.35) is estimated similarly to the first estimate of Lemma 3.4. Now, combining the upper and lower bounds for identity (3.36), we obtain

$$\begin{aligned}\frac{\varepsilon}{\delta}\left\|\vartheta_{k,\varepsilon\;tt}^{(\theta)}\right\|_0^2+\left\|\vartheta_{k,\varepsilon}^{(\theta)}\right\|_1^2\le\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left\|\vartheta_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right)\\ +\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_1^2+\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left\|f_k\right\|_2^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_1^2.\end{aligned}\tag{3.39}$$

Here we took σ equal to $\sigma=\frac{10}{\delta_0}$, from which follows $\delta_0-5\sigma^{-1}=\frac{\delta_0}{2}$. Multiplying inequalities (3.39) by $\left(1+\lambda_k^2\right)^3$ and summing by k from 1 to ∞, we obtain the first recurrent formula

$$\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\;tt}^{(\theta)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(\theta)}\right\rangle_1^2\le\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(2\gamma_0+[f]_{2,3}^2\right)\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_1^2.\tag{3.40}$$

Let us denote the coefficient before $\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_1^2$ by $q_1\equiv\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(2\gamma_0+[f]_{2,3}^2\right)$. It is clear that $q_1<q$. Then (3.40) has the form

$$\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\;tt}^{(\theta)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(\theta)}\right\rangle_1^2\le q_1\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_1^2<q\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_1^2.$$

Let us consider the zero approximation $\vartheta_{\varepsilon,k}^{(0)}=u_{\varepsilon,k}^{(0)}-u_{\varepsilon,k}^{(-1)}=u_{\varepsilon,k}^{(0)}$. Since $\left\langle u_{\varepsilon}^{(0)}\right\rangle_1^2\le 2\gamma_0$ by Lemma 3.4, then $\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\;tt}^{(1)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(1)}\right\rangle_1^2<2\gamma_0q$ holds for it. Let us now consider the second approximation. The following inequality is valid for it $\frac{\varepsilon}{\delta}\left\langle \vartheta^{(1)}_{\varepsilon\ tt}\right\rangle_0^2+\langle\vartheta^{(1)}_{\varepsilon}\rangle_1^2\le 2\gamma_0 q_1<2\gamma_0 q$. Continuing this process, we obtain for an arbitrary following relation

$$\frac{\varepsilon}{\delta}\left\langle \vartheta^{(\theta)}_{\varepsilon\ \ tt}\right\rangle_0^2+\left\langle\vartheta^{(\theta)}_{\varepsilon}\right\rangle_1^2\le 2\gamma_0 q_1^{\theta}<2\gamma_0 q^{\theta}.$$

The inequality $q<\frac{1}{2}$ comes from the condition (3.21). Thus, the third *III)* estimate is proven. Now we will prove the fourth *IV)* estimate. Let us consider the identity

$$\left|-2\left(L_\varepsilon\vartheta^{(\theta)}_{k,\varepsilon},e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\right)_0\right|=\left|-2\left(F_k(u^{(\theta-1)}_{k,\varepsilon})-F_k(u^{(\theta-2)}_{k,\varepsilon}),e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\right)_0\right|, \tag{3.41}$$

where $P\vartheta^{(\theta)}_{k,\varepsilon}=\vartheta^{(\theta)}_{k,\varepsilon\,ttt}-\vartheta^{(\theta)}_{k,\varepsilon\,tt}+\vartheta^{(\theta)}_{k,\varepsilon\,xx}-\vartheta^{(\theta)}_{k,\varepsilon\,t}$. Let us show the lower boundedness of expression (3.40). Let us estimate it in a similar way as in the case of the third estimate from below. Then we have

$$\begin{gathered}\varepsilon e^{-\mu T}\left\|\vartheta^{(\theta)}_{k,\varepsilon ttt}\right\|_0^2+\int\limits_Q\left[\delta_0\vartheta^{2(\theta)}_{k,\varepsilon tt}\ +a\mu\vartheta^{2(\theta)}_{k,\varepsilon xt}+a\vartheta^{2(\theta)}_{k,\varepsilon xx}+\delta_0\vartheta^{2(\theta)}_{k,\varepsilon\ t}+a\mu\vartheta^{2(\theta)}_{k,\varepsilon\ x}+\mu\vartheta^{2(\theta)}_{k,\varepsilon}\right]e^{-\mu t}dQ\\ -\left(\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\int\limits_Q\vartheta^{2(\theta)}_{k,\varepsilon\,t}e^{-\mu t}dQ\le\left|-2\left(L_\varepsilon\vartheta^{(\theta)}_{k},e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\right)_0\right|.\end{gathered} \tag{3.42}$$

Now we will show the upper bound (3.41). Let us write the right part in expanded form

$$\begin{aligned}&\left|-2\Big(F_k(u^{(\theta-1)}_{k,\varepsilon})-F_k(u^{(\theta-2)}_{k,\varepsilon}),e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\Big)_0\right|\\ =&\left|-2\int\limits_Q\left(\frac{1}{H}u^{(\theta-1)}_{k,\varepsilon}f_2\sum_{m=1}^{\infty}\lambda_m^2u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1-\frac{1}{H}u^{(\theta-1)}_{k,\varepsilon}f_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2\right.\right.\\ &-\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_2\sum_{m=1}^{\infty}\lambda_m^2u^{(\theta-2)}_{m,\varepsilon}\sin\lambda_m\ell_1+\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_1\sum_{m=1}^{\infty}\lambda_m^2u^{(\theta-2)}_{m,\varepsilon}\sin\lambda_m\ell_2\\ &+\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_2\sum_{m=1}^{\infty}\lambda_m^2u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1-\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_2\sum_{m=1}^{\infty}\lambda_m^2u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1\\ &+\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_1\sum_{m=1}^{\infty}\lambda_m^2u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2-\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_1\sum_{m=1}^{\infty}\lambda_m^2u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2\\ &+\left.\frac{1}{H}f_k\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2-\varphi_2\sum_{m=1}^{\infty}\lambda_m^2\vartheta^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1\right]\right)\\ &\left.\times\left(\vartheta^{(\theta)}_{k,\varepsilon\ ttt}-\mu\vartheta^{(\theta)}_{k,\varepsilon\ tt}+\vartheta^{(\theta)}_{k,\varepsilon\ xx}-\vartheta^{(\theta)}_{k,\varepsilon\ t}\right)dQ\right|.\end{aligned} \tag{3.43}$$

Let us show the upper bound of expression (3.43). Indeed, applying the Cauchy inequalities with σ and using the previous results we obtain

$$\begin{gathered}\left|-2\left(F_k(u^{(\theta-1)}_{k,\varepsilon})-F_k(u^{(\theta-2)}_{k,\varepsilon}),e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\right)_0\right|\le 4\sigma c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|\vartheta^{(\theta-1)}_{k,\varepsilon}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u^{(\theta-1)}_{m,\varepsilon}\right\|_2^2\right)\\ +4\sigma c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u^{(\theta-1)}_{k,\varepsilon}\right\|_2^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta^{(\theta-1)}_{m,\varepsilon}\right\|_2^2+4\sigma c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta^{(\theta-1)}_{m,\varepsilon}\right\|_2^2\\ +30\sigma^{-1}\int\limits_Q\vartheta^{2(\theta)}_{k\ tt}e^{-\mu t}dQ+6\sigma^{-1}\int\limits_Q\vartheta^{2(\theta)}_{k\ xx}e^{-\mu t}dQ+6\sigma^{-1}\int\limits_Q\vartheta^{2(\theta)}_{k\ t}e^{-\mu t}dQ.\end{gathered} \tag{3.44}$$

Let us choose σ equal to $\sigma=\frac{60}{\delta_0}$, then $\delta_0-30\sigma^{-1}=\frac{\delta_0}{2}$, $a-6\sigma^{-1}>\delta_1$. Taking into account the conditions of Theorem 3.1, the boundary conditions and $\gamma^2=e^{\mu T}$, we note that the boundary integrals that appear during the calculations vanish. From below, identity (3.41) is estimated similarly to the second estimate of Lemma 3.4. Now combining the upper (3.44) and lower (3.42) estimates for identity (3.41) we obtain

$$\begin{gathered}\frac{\varepsilon}{\delta}\left\|\vartheta_{k,\varepsilon\ ttt}^{(\theta)}\right\|_0^2+\left\|\vartheta_{k,\varepsilon}^{(\theta)}\right\|_2^2\leq\left(\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\int\limits_Q\vartheta_{k,\varepsilon\,t}^{2(\theta)}e^{-\mu t}dQ\\+\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|\vartheta_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\+\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\+\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right).\end{gathered}\tag{3.45}$$

Multiplying the inequalities (3.48) by $(1+\lambda_k^2)^3$ and summing by k from 1 to ∞ we obtain the second recurrent formula

$$\begin{aligned}\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ ttt}^{(\theta)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(\theta)}\right\rangle_2^2&\leq\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2+\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\left(2\gamma_0+[f]_{2,3}^2\right)\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_2^2\\&<\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(4(1+\mu^2)+\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\left(2\gamma_0+[f]_{2,3}^2\right)\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_2^2.\end{aligned}\tag{3.46}$$

Here we used inequality (3.40). By the condition of Theorem 3.1, the factor in front of $\langle\vartheta_{\varepsilon}^{(\theta-1)}\rangle_2^2$ is equal to q, i.e.

$$q=\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2+\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\left(2\gamma_0+[f]_{2,3}^2\right).$$

Then (3.46) has the form $\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ ttt}^{(\theta)}\right\rangle_0^2+\langle\vartheta_{\varepsilon}^{(\theta)}\rangle_2^2\leq q\,\langle\vartheta_{\varepsilon}^{(\theta-1)}\rangle_2^2$. Again as in the third estimate, consider the zero approximation $\vartheta_{\varepsilon,k}^{(0)}=u_{\varepsilon,k}^{(0)}-u_{\varepsilon,k}^{(-1)}=u_{\varepsilon,k}^{(0)}$. For him $\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ ttt}^{(1)}\right\rangle_0^2+\langle\vartheta_{\varepsilon}^{(1)}\rangle_2^2\leq 2\gamma_0 q$ holds, since $\left\langle u_{\varepsilon,k}^{(0)}\right\rangle_2^2\leq 2\gamma_0$ by Lemma 3.4. Let us now consider the second approximation. For him the following inequality is true

$$\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ ttt}^{(2)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(2)}\right\rangle_2^2\leq 2\gamma_0q^2.$$

Continuing this process, we obtain for an arbitrary $\theta\geq 2$ the following relation

$$\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ \ ttt}^{(\theta)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(\theta)}\right\rangle_2^2\leq 2\gamma_0q^{\theta}<Rq^{\theta}.$$

Thus, estimate *IV)* is proven. *Lemma 3.6 is proven.* □

Theorem 3.7. *Let all the conditions of Theorem 3.1 be satisfied. Then problem* (2.8)−(2.10) *has a unique solution in* $W(Q)$.

Proof. To prove the theorem, we show that the sequence $\{u_k^{(\theta)}\}\in W(Q)$ is fundamental. We assume that $\left\{u_{\varepsilon}^{(\theta)}\right\}$, $\left\{u_{\varepsilon}^{(\theta-1)}\right\}\in W(Q)$. Now we consider the new functions $\left\{\vartheta_k^{(\theta)}\right\}=\left\{u_k^{(\theta)}-u_k^{(\theta-1)}\right\}$. For these functions the a priori estimate IV) in Lemma 3.6 is appropriate, i.e.

$$\left[\vartheta_{\varepsilon}^{(\theta)}\right]_{W(Q)}^2<2\gamma_0q^{\theta}.\tag{3.47}$$

Now we prove that the sequence of functions $\{u_{\varepsilon}^{(\theta)}\}\in W(Q)$ is fundamental. To do this, using the estimate (3.47) and the triangle inequality, we obtain the following inequality

$$\left[u_{\varepsilon}^{(\theta+p)}-u_{\varepsilon}^{(\theta)}\right]_{W(Q)}^2\leq\left[u_{\varepsilon}^{(\theta+p)}-u_{\varepsilon}^{(\theta+p-1)}\right]_{W(Q)}^2+\left[u_{\varepsilon}^{(\theta+p-1)}-u_{\varepsilon}^{(\theta+p-2)}\right]_{W(Q)}^2+\ldots$$

$$+\left[u_\varepsilon^{(\theta+1)}-u_\varepsilon^{(\theta)}\right]^2_{W(Q)}\le 2\gamma_0\left(q^{\theta+p}+q^{\theta+p-1}+...+q^{\theta+1}\right)=2\gamma_0 q^{\theta+1}(1+q+q^2+...+q^p)\le\frac{2\gamma_0 q^{\theta+1}}{1-q}.$$

It is clear that for sufficiently large θ since $q<\frac{1}{2}$, this quantity is arbitrarily small. From this it follows that the sequence $\left\{ u_\varepsilon^{(\theta)} \right\}$ is fundamental. Hence, the problem (3.8)-(3.10) has a solution in the space $W(Q)$, i.e. $\{u_\varepsilon^{(\theta)}\}\to\{u_\varepsilon\}$. It follows that the functions $\{u_\varepsilon\}\in W(Q)$ are solutions to the problem (3.8)-(3.10).

Now we show that $\{u_{k,\varepsilon}\}$ has a limit in an appropriate space and the limit functional sequence $\{u_k\}$ is a solution to the problem (2.8)$-$(2.10).

We notice that $F\left(u_{k,\varepsilon}\right)$ in (3.8) consist of nonlinear terms. First we estimate this term. Using Theorem A.2 and Cauchy inequality, we have

$$\begin{aligned}
&\int\limits_Q\left(\sum_{m=1}^{\infty}{\lambda_m}^2 u_{m,\varepsilon_j}u_{k,\varepsilon_j}\sin\lambda_m\ell_1\right)^2 dxdt=\int\limits_Q\left(\sum_{m=1}^{\infty}\frac{1}{\lambda_m}{\lambda_m}^3 u_{m,\varepsilon_j}u_{k,\varepsilon_j}\sin\lambda_m\ell_1\right)^2 dxdt\\
&\le\int\limits_Q\sum_{m=1}^{\infty}\frac{1}{{\lambda_m}^2}\sum_{m=1}^{\infty}{\lambda_m}^6 u^2_{m,\varepsilon_j}u^2_{k,\varepsilon_j}dxdt=\sum_{m=1}^{\infty}\frac{1}{\lambda_m^2}\sum_{m=1}^{\infty}{\lambda_m}^6\int\limits_Q u^2_{m,\varepsilon_j}u^2_{k,\varepsilon_j}dxdt\\
&\le c_1\sum_{m=1}^{\infty}{\lambda_m}^6\left(\int\limits_Q u^4_{m,\varepsilon_j}dxdt\right)^{\frac12}\left(\int\limits_Q u^4_{k,\varepsilon_j}dxdt\right)^{\frac12}\le c_1c_2^2\sum_{m=1}^{\infty}{\lambda_m}^6\left\|u_{m,\varepsilon_j}\right\|^2_{W_2^1(Q)}\left\|u_{k,\varepsilon_j}\right\|^2_{W_2^1(Q)}\\
&\le c_1c_2^2\sum_{m=1}^{\infty}\left(1+{\lambda_m}^2\right)^3\left\|u_{m,\varepsilon_j}\right\|^2_{W_2^1(Q)}\sum_{k=1}^{\infty}\left(1+{\lambda_k}^2\right)^3\left\|u_{k,\varepsilon_j}\right\|^2_{W_2^1(Q)}\le c_1c_2^2(2\gamma_0)^2.
\end{aligned}\tag{3.48}$$

Here we have the boundedness of the integrand in $L_2(Q)$, which allows us extract weak convergent subsequence in $L_2(Q)$. So, from estimation of Lemma 3.4 we can pass to the limit in subsequence $\left\{u_{k,\varepsilon_j}\right\}$, such that

$$u_{k,\varepsilon_j}\to u_k \qquad weakly \quad in \quad W_2^2;$$
$$u_{k,\varepsilon_j}\to u_k \qquad weakly \quad in \quad W;$$

considering above results and from (3.48)

$$F\left(u_{k,\varepsilon_j}\right)\to F\left(u_k\right) \qquad weakly \quad in \quad L_2.$$

From Lemma 3.4 we have that $\sqrt{\varepsilon_j}u_{k,\varepsilon_j ttt}$ is bounded, so we have $\sqrt{\varepsilon_j}\sqrt{\varepsilon_j}u_{k,\varepsilon_j ttt}\to 0$ in $L_2(Q)$. Since, convergence in $L_2(Q)$ is stronger than weak convergence, it follows $\sqrt{\varepsilon_j}\left(\sqrt{\varepsilon_j}u_{k,\varepsilon_j ttt},\vartheta\right)\to 0$. Passing to the weak limit in (3.8) as $\varepsilon_j\to 0$, we obtain $Lu_k=F(u_k)$. This means that the function u_k with fixed k will be the unique solution of the problem (2.8)$-$(2.10) from $W_2^2(Q)$ [12, 18]. This proves Theorem 3.7. □

Now we prove Theorem 3.1.

Since all the conditions of Theorem 3.1 are satisfied, then, using the Parseval-Steklov equalities [20] to solve the problem (2.8)$-$(2.10), we obtain a solution to problem (2.1)$-$(2.6) from the specified class U.

Estimation for $c(x,t)$:

$$\|c\|_{C(Q)}=\max_Q|c(x,t)|=\max_Q\left|\frac{1}{H}\left[f_2\sum_{m=1}^{\infty}{\lambda_m}^2 u_m\sin\lambda_m\ell_1-f_1\sum_{m=1}^{\infty}{\lambda_m}^2u_m\sin\lambda_m\ell_2\right]\right|$$
$$\le 2\mathfrak{F}\sum_{m=1}^{\infty}{\lambda_m}^2\max|u_m|<2\mathfrak{F}c_1c_2\sum_{m=1}^{\infty}(1+{\lambda_m}^2)^3\|u_m\|_2^2<4c_1c_2\mathfrak{F}\gamma_0$$

We can choose data such that number $4c_1c_2\mathfrak{F}\gamma_0$ will be lower than r.

Theorem 3.1 is proved.

□

Remark 3.8. For definiteness and ease of exposition, the analysis is carried out in the three-dimensional framework. Nonetheless, it should be noted that the extension to an arbitrary number of spatial dimensions does not entail essential difficulties, since the underlying arguments and techniques remain valid in the general multidimensional setting.

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