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Cauchy problem for fractional high order equation with singular coefficient

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Abstract. In this paper, we aim to study the Cauchy problem posed for a fractional order equation. General solution of the time-fractional equation is found by the Fourier method. The solution to the given problem is shown to be unique using the generalized Hankel transform. The solution consists of the Bessel function and the Mittag-Leffler function. Unknown coefficients are found by Hankel transformation. It is shown that the constructed solution satisfies the initial condition and equation.

Keywords: Cauchy problem, Riemann–Liouville fractional operator, Hankel transform, Mittag-Leffler function, Bessel function
MSC (2020): 33C10, 33E12, 35A22, 35B30, 35R11

1. Introduction

Third-order partial differential equations are considered when solving problems in the theory of nonlinear acoustics and in the hydrodynamic theory of space plasma and fluid filtration in porous media [1]. In total, all third-order equations occupy a special place due to their specific nature, equations with multiple characteristics. In [2], [3], taking into account the properties of viscosity and thermal conductivity of the gas, a third-order equation with multiple characteristics was obtained from the Navier-Stokes system, containing the second derivative with respect to time

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \;\; \nu = const.$$

This equation at $\nu = 1$ describes an axisymmetric flow, and at $\nu = 0$ describes a plane-parallel flow [4].

The first results on a third-order equation with multiple characteristics were obtained in the works of H. Block [5] and E. DelVecehio [6]. In [7] and [8], fundamental solutions of third-order equations with multiple characteristics were constructed, containing second derivatives with respect to time, expressed through degenerate hypergeometric functions, their properties were studied, and estimates for $|t| \to \infty$ were found. In works [9] and [10], boundary value problems for third-order equations with multiple characteristics are considered using the constructed Green function. In recent years, interest in degenerate and singular equations has grown significantly, including equations containing the Bessel differential operator. These equations are often encountered in applications, for example, in problems with axial symmetry in continuum mechanics. Interest in problems related to the Bessel operator is also known from fundamental physics. This is due to its numerous applications in gas dynamics, shell theory, magnetohydrodynamics, and other fields of science and technology [11]. A special place in the theory of degenerate and singular equations is occupied by equations containing the Bessel differential operator

$$B_\nu = x^{-\nu} \frac{d}{dx} \left(x^\nu \frac{d}{dx} \right) = \frac{d^2}{dx^2} + \frac{\nu}{x} \frac{d}{dx}$$

According to the terminology by the Voronezh mathematician Ivan Aleksandrovich Kipriyanov, equations of three main classes containing the Bessel operator are called B-elliptic, B-hyperbolic, and B-parabolic, respectively. The monograph [12] studies boundary value problems for B-elliptic equations, in addition to this, the account of multi-dimension integral Fourier-Bessel-Hankel transformation theory is given in the monograph. The final chapters contain new results on general weight boundary value problems for singular B-elliptic and B-parabolic equations where parameter may be complex. There it is shown how spectral characteristics of B-elliptic operators including kernels of fractional powers are produced. The theory of boundary value problems for the equations with peculiarity has been reflected there, while the study of B-hyperbolic equations is presented in the monograph by R. Carroll and R. Showalter [13] and of B-parabolic ones, in the monograph by M.I. Matiichuk [14]. A wide range of questions for equations with Bessel operators was studied by I.A. Kipriyanov [15], [12] and his students L.A. Ivanov [16], V.V. Katrakhov [17], [18], [19], [20], V.I. Kononenko [21], L.N. Lyakhov [22], A.B. Muravnik [23], I.P. Polovinkin [24], S.M. Sitnik [25], [26], E.L. Shishkina [27], [28] and others. Mixed problems with integral conditions for hyperbolic equations with the Bessel operator were studied by N. V. Zaitseva [11].

Until now, many mathematicians have conducted many scientific researches on third-order partial differential equations. In contrast to them, in this article, we solve the Cauchy problem for the high order fractional differential equation of the composed type using the Hankel transformation method.

2. Definition of the Hankel Transform

Hermann Hankel is remembered for his numerous contributions to mathematical analysis including the Hankel transformation, which occurs in the study of functions, which depend only on the distance from the origin. The Hankel transform involving Bessel functions as the kernel arises naturally in axisymmetric problems formulated in cylindrical polar coordinates.

By authors Lokenath Debnath and Dambaru Bhatta in [29] engaged with the definition and basic operational properties of the Hankel transform. A large number of axisymmetric problems in cylindrical polar coordinates are solved with the aid of the Hankel transform. The use of the joint Laplace and Hankel transforms is illustrated by several examples of applications to partial differential equations [29]-[30].

Hankel transformation and other transformations are used to solve problems in mechanics, elasticity theory, thermal conductivity, electrodynamics and other branches of theoretical physics. More detailed information about the Hankel transform can be found in [31]-[32].

The Hankel transform is an integral transform and was first developed by the German mathematician Hermann Hankel (1839–1873). It is also known as the Fourier–Bessel transform. Just as the Fourier transform for an infinite interval is related to the Fourier series over a finite interval, so the Hankel transform over an infinite interval is related to the Fourier–Bessel series over a finite interval. The Hankel transform expresses any given function $f(\xi)$ as the weighted sum of an infinite number of Bessel functions of the first kind $J_\nu(\eta\xi)$. The Bessel functions in the sum are all of the same order ν, but differ in a scaling factor η along the ξ axis. The necessary coefficient F_ν of each Bessel function in the sum, as a function of the scaling factor η constitutes the transformed function.

Definition 2.1. Let $f(r)$ be a function defined for $r \geq 0$. The ν^{th} order Hankel transform of $f(r)$ is defined as

$$F_\nu(\eta) = \int_0^\infty r f(r) J_\nu(\eta r)\,dr, \left(\nu > -\frac{1}{2}\right), \tag{2.1}$$

where $J_\nu(x)$ is well known Bessel function of the first kind and defined as

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\nu+1)\,n!}\left(\frac{x}{2}\right)^{2n+\nu}. \tag{2.2}$$

Theorem 2.2. *If the function $f(x)$ piecewise continuous in any finite interval (has bounded variation) belonging to the interval $(0,\ \infty)$ and integral converges*

$$\int_0^\infty |f(\xi)|\sqrt{\xi}d\xi,$$

then the Hankel transform exists and the inversion of the Hankel transform is given by the following formula

$$f(r) = \int_0^\infty \eta F_\nu(\eta) J_\nu(\eta r)\,d\eta. \tag{2.3}$$

Formulas (2.1) and (2.3) can be written in the following form

$$f(r) = \int_0^\infty J_\nu(r\xi)\,\xi d\xi \int_0^\infty f(\zeta) J_\nu(\xi\zeta)\,\zeta d\zeta = \int_0^\infty f(\zeta)\,\zeta d\zeta \int_0^\infty J_\nu(\xi\zeta) J_\nu(r\xi)\,\xi d\xi. \tag{2.4}$$

From relation (2.4), it follows

$$f(\eta) = \int_0^\infty a(\xi)\,\xi J_\nu(\eta\xi)\,d\xi, \tag{2.5}$$

where

$$a(\xi) = \int_0^\infty f(\rho)\,J_\nu(\rho\xi)\,\rho d\rho, \tag{2.6}$$

formulas (2.1), (2.3) and (2.4) are given in monographs [29]-[33].

From the properties of the Dirac delta function [34]-[35], it follows

$$\int_0^\infty \delta(x-a)\,\Phi(x)\,dx = \Phi(a), \tag{2.7}$$

$$\delta(x-a) = x\int_0^\infty tJ_\mu(xt)\,J_\mu(at)\,dt,\ \ |\mu| < \frac{1}{2}, \tag{2.8}$$

As can be seen from the above properties, we can conclude that the function $f(r)$ is compatible with Dirac delta function.

3. Cauchy problem for high order fractional differential equation

A real valued C^∞-smooth function on an open subset of $\mathbb{R}^n$ is called a Schwartz function, if it and all of its partial derivatives rapidly decay when approaching any boundary point of the subset, including ∞ if the subset is unbounded. The space of all Schwartz functions on a given subset $U \subset \mathbb{R}^n$ is a Fréchet space denoted by $\mathcal{S}(U)$, and is called the Schwartz space of U. Schwartz spaces were first introduced on $\mathbb{R}^n$ by Laurent Schwartz [36] and throughout the years were defined and studied in various contexts on various objects. First introduced in the first half of the 20^{th} century, Schwartz spaces still play an important role in many fields of mathematics, such as harmonic analysis, representation theory and number theory.

Let $\mathbb{N}$ be the set of non-negative integers, and for any $n \in \mathbb{N}$, let $\mathbb{N}^n$ be the n-fold Cartesian product.

Definition 3.1. The Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^n$ is the function space

$$\mathcal{S}(\mathbb{R}^n, \mathbf{C}) = \{f \in C^\infty(\mathbb{R}^n, \mathbf{C}) | \forall p, q \in \mathbb{N}^n, ||f||_{p,q} < \infty\},$$

where $C^\infty(\mathbb{R}^n, \mathbf{C})$ is the function space of smooth functions from $\mathbb{R}^n$ into $\mathbf{C}$, and

$$||f||_{p,q} = \sup_{x\in\mathbb{R}^n} |x^p (D^q f)(x)|.$$

Here, sup denotes the supremum, and we used multi-index notation $x^p = x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$ and $D^q = \partial_1^{q_1}\partial_2^{q_2}...\partial_n^{q_n}$. To put common language to this definition, one could consider a rapidly decreasing function as essentially a function $f(x)$ such that $f(x), f'(x), f''(x), ...$ all exist everywhere on $\mathbb{R}$ and go to zero as $x \to \pm\infty$ faster than any reciprocal power of x. In particular, $\mathcal{S}(\mathbb{R}^n, \mathbf{C})$ is a subspace of the function space $C^\infty(\mathbb{R}^n, \mathbf{C})$ of smooth functions from $\mathbb{R}^n$ into $\mathbf{C}$.

3.1. **Statement of problem.** Let us look for a solution in the space $\mathcal{S}(\Omega)$ of the time-fractional order equation given in the domain $\Omega = \{(x,t):\ x > 0,\ 0 < t < t_0\}$

$${}^{RL}D_{0t}^{\alpha} u(x,t) = u_{xx}(x,t) + \frac{\nu}{x} u_x(x,t),\quad 0 < \nu < 1,\quad 2 < \alpha < 3, \tag{3.1}$$

satisfying initial conditions

$${}^{RL}D_{0t}^{\alpha-1} u(x,t)\big|_{t=0} = \psi(x),\quad 0 \le x < \infty, \tag{3.2}$$

$${}^{RL}D_{0t}^{\alpha-2} u(x,t)\big|_{t=0} = \varphi(x),\quad 0 \le x < \infty, \tag{3.3}$$

$${}^{RL}D_{0t}^{\alpha-3} u(x,t)\big|_{t=0} = \tau(x),\quad 0 \le x < \infty, \tag{3.4}$$

where $\psi(x), \varphi(x)\ \tau(x) \in \mathbf{C}^2(0,\infty)$ and

$$\int\limits_0^\infty |\psi(x)|\, x^{\frac{\nu}{2}} dx < c = const, \quad \int\limits_0^\infty |\varphi(x)|\, x^{\frac{\nu}{2}} dx < c = const, \quad \int\limits_0^\infty |\tau(x)|\, x^{\frac{\nu}{2}} dx < c = const$$

in addition to this

$$\psi(0) = \varphi(0) = \tau(0) = 0, \quad \lim_{x\to\infty} \psi(x) = 0, \ \lim_{x\to\infty} \varphi(x) = 0, \ \lim_{x\to\infty} \tau(x) = 0, \tag{3.5}$$

and for any fixed t we have

$$\lim_{x\to\infty} u(x,t) = 0, \tag{3.6}$$

$$\lim_{x\to 0} u(x,t) = 0, \tag{3.7}$$

where ${}^{RL}D_{0t}^{\alpha}$ is the Riemann–Liouville fractional derivative operator of order α defined by

$${}^{RL}D_{0t}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n \left\{ {}^{RL}I_{0t}^{n-\alpha} f(t) \right\}, \quad \mathrm{Re}(\alpha) \geq 0, \ n = [\mathrm{Re}(\alpha)] + 1, \tag{3.8}$$

and

$${}^{RL}I_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \quad t > 0, \ \mathrm{Re}(\alpha) \geq 0, \tag{3.9}$$

represents Riemann–Liouville fractional integral [32].

Using the generalized Hankel transform, we can show that the solution to the given problem is unique.

Definition 3.2. The generalized Hankel transform [37] of the function $f(r)$ is introduced as follows

$$\mathbf{J}_{a,b;\nu}[f(r)](\eta) = \eta^a \int_0^\infty f(r)\, r^a J_\nu\left(\eta^b r^b\right) dr = F_{a,b;\nu}(\eta), \left(\nu > -\frac{1}{2}\right), \tag{3.10}$$

where $a, b \in \mathbf{R}$, $b \neq 0$, $\eta \in \mathbf{R}_+$.

The generalized Hankel transformation (3.10) is related to the Hankel transformation (2.1) by the equality

$$F_{a,b;\nu}(\eta) = \frac{1}{|b|} \eta^a H_\nu[f(r^{1/b}) r^{-2+(a+1)/b}](\eta^b). \tag{3.11}$$

Relation (3.11) also allows us to obtain the inversion formula

$$f(r) = b^2 r^{2b-a-1} \int_0^\infty F_{a,b;\nu}(\eta)\, \eta^{2b-a-1} J_\nu\left(\eta^b r^b\right) d\eta, \tag{3.12}$$

for the transformation (3.10) of the function $f(\tau)$ from the class $\mathcal{S}(\Omega)$ with weight $\tau^{a-b/2}$, which has bounded variation in the near of the point r.

3.2. **Uniqueness of solution.** Suppose that there are two solution $u_1(x,t)$ and $u_2(x,t)$ to the Cauchy problem (3.1)-(3.4), denote

$$u(x,t) = u_1(x,t) - u_2(x,t).$$

Then, the function $u(x,t)$ clearly satisfy equation (3.1), the conditions in (3.6)-(3.7) and the homogeneous conditions

$${}^{RL}D_{0t}^{\alpha-1} u(x,t)\Big|_{t=0} = 0, \quad {}^{RL}D_{0t}^{\alpha-2} u(x,t)\Big|_{t=0} = 0, \quad {}^{RL}D_{0t}^{\alpha-3} u(x,t)\Big|_{t=0} = 0, \ \ 0 \leq x < \infty. \tag{3.13}$$

Let us the generalized Hankel transform of the function $u(x,t)$ introduce the following

$$U(t,\mu) = \mu^{\frac{1+\nu}{2}} \int_0^\infty u(x,t) x^{\frac{1+\nu}{2}} J_{\frac{1-\nu}{2}}(\mu x)\, dx, \ \ (\mu \in \mathbf{R}\backslash\{0\}). \tag{3.14}$$

Note that the homogeneous conditions in (3.13) lead to

$$ {}^{RL}D_{0t}^{\alpha-1}U(t,\mu)\big|_{t=0} = {}^{RL}D_{0t}^{\alpha-2}U(t,\mu)\big|_{t=0} = {}^{RL}D_{0t}^{\alpha-3}u(t,\mu)\big|_{t=0} = 0 \tag{3.15} $$

and Riemann–Liouville fractional derivative of the expression (3.14) gives

$$ {}^{RL}D_{0t}^{\alpha}U(t,\mu) = \mu^{\frac{1+\nu}{2}} \int_0^\infty x^{-\nu}\frac{\partial}{\partial x}(x^\nu u_x)x^{\frac{1+\nu}{2}} J_{\frac{1-\nu}{2}}(\mu x)\,dx, $$

which on integrating by parts and using the conditions in (3.6)-(3.7) and Bessel equation reduces to

$$ {}^{RL}D_{0t}^{\alpha}U(t,\mu) + \mu^2 U(t,\mu) = 0, \tag{3.16} $$

In [38], the solution of equation (3.16) is represented

$$ U(t,\mu) = a_1(\mu)\,t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) + a_2(\mu)\,t^{\alpha-2}E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) + a_3(\mu)\,t^{\alpha-3}E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right) \tag{3.17} $$

where $a_i(\lambda)$, $i = 1,2,3$ are unknown coefficients and $E_{\alpha,\beta}(z)$ Mittag-Leffler function with two parameters

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ (\alpha,\beta\in\mathbb{C},\ \operatorname{Re}(\alpha)>0,\ \operatorname{Re}(\beta)>0). $$

By (3.17) we determine the unknown function $a_i(\mu)$, $i = 1,2,3$. We obtain Riemann–Liouville fractional derivative of order $\alpha-1$ of the function (3.17)

$$ {}^{RL}D_{0t}^{\alpha-1}U(t,\mu) = a_1(\mu)\ E_{\alpha,1}\left(-\mu^2 t^\alpha\right) - \mu^2 a_2(\mu)\,t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) - \mu^2 a_3(\mu)\,t^{\alpha-2}E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) $$

Using the condition (3.15), we get ${}^{RL}D_{0t}^{\alpha-1}U(t,\mu)\big|_{t=0} = a_1(\mu) = 0$ The Riemann-Liouville fractional derivative of order $\alpha-2$ $(2<\alpha<3)$ with respect to the variable from the function (3.17) has the following form

$$ {}^{RL}D_{0t}^{\alpha-2}U(t,\mu) = a_1(\mu)\,t\,E_{\alpha,2}\left(-\mu^2 t^\alpha\right) + a_2(\mu)\,E_{\alpha,1}\left(-\mu^2 t^\alpha\right) - \mu^2 a_3(\mu)\,t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right). $$

Using the condition (3.15), we get ${}^{RL}D_{0t}^{\alpha-2}U(t,\mu)\big|_{t=0} = a_2(\mu) = 0$. The Riemann-Liouville fractional derivative of order $\alpha-3$ $(2<\alpha<3)$ with respect to the variable from the function (3.17) has the following form

$$ {}^{RL}D_{0t}^{\alpha-3}U(t,\mu) = a_1(\mu)\,t^2E_{\alpha,3}\left(-\mu^2 t^\alpha\right) + a_2(\mu)\,tE_{\alpha,2}\left(-\mu^2 t^\alpha\right) + a_3(\mu)\,E_{\alpha,1}\left(-\mu^2 t^\alpha\right). $$

Using the condition (3.15), we get ${}^{RL}D_{0t}^{\alpha-3}U(t,\mu)\big|_{t=0} = a_3(\mu) = 0$ imply that $U(t,\mu)\equiv 0$. Therefore, due to the inversion of the generalized Hankel transform (3.12), we get $u(x,t)\equiv 0$, $u(x,t)\in\mathcal{S}(\Omega)$. This ends the proof of uniqueness of solution to the Cauchy problem (3.1)-(3.4).

3.3. **Existence of Solution.**

Theorem 3.3. *If* $2<\alpha<3$, $0<\nu<1$, *then the Cauchy problem* (3.1)-(3.4) *has the following solution defined in Schwartz space*

$$ u(x,t) = x^{\frac{1-\nu}{2}}t^{\alpha-1}\int_0^\infty\int_0^\infty \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu $$

$$ \times\left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) + \varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) + \tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right)\right]d\rho d\mu \tag{3.18} $$

and this solution satisfied conditions (3.5)-(3.7).

Proof. We look for the solution of equation (3.1) using the Fourier method

$$u(x,t) = X(x)\,T(t). \tag{3.19}$$

Substituting (3.19) in equation (3.1), we have two equations

$$X_{xx} + \frac{\nu}{x}X_x + \mu^2 X = 0, \tag{3.20}$$

$$ {}^{RL}D_{0t}^{\alpha}T(t) + \mu^2 T(t) = 0, \;\; \mu \in \mathbb{R}\setminus\{0\} \tag{3.21}$$

By substituting the product $X = x^{\frac{1-\nu}{2}}\theta(x\mu)$ into (3.20), we get the following Bessel equation

$$(x\mu)^2\theta_{xx}(x\mu) + x\mu\theta_x(x\mu) + \left((x\mu)^2 - \frac{(\nu-1)^2}{4}\right)\theta(x\mu) = 0.$$

We know that, when $\frac{\nu-1}{2}$ is not integer the functions $J_{\frac{\nu-1}{2}}(\mu x)$ and $J_{\frac{1-\nu}{2}}(\mu x)$ are linear independent solutions of above Bessel equation [39]. From conditions (3.5), we use only the function $J_{\frac{1-\nu}{2}}(\mu x)$. Then the solution of equation (3.20) has the following form

$$X = c(\mu)\,x^{\frac{1-\nu}{2}}J_{\frac{1-\nu}{2}}(\mu x), \tag{3.22}$$

where $J_\mu(z)$-Bessel function of the first kind [34] defined as (2.2).

Using the solutions (3.22) and (3.17), we can write the general solution of the equation (3.1) as follows

$$\begin{aligned} u(x,t) = {} & x^{\frac{1-\nu}{2}}t^{\alpha-1}\int_0^\infty C_1(\mu)\,E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}(\mu x)\,d\mu \\ & + x^{\frac{1-\nu}{2}}t^{\alpha-2}\int_0^\infty C_2(\mu)\,E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}(\mu x)\,d\mu \\ & + x^{\frac{1-\nu}{2}}t^{\alpha-3}\int_0^\infty C_3(\mu)\,E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}(\mu x)\,d\mu, \end{aligned}$$

or

$$u(x,t) = x^{\frac{1-\nu}{2}}t^{\alpha-1}\int_0^\infty J_{\frac{1-\nu}{2}}(\mu x)\times$$

$$\times\left[C_1(\mu)\,E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) + C_2(\mu)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) + C_3(\mu)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\mu. \tag{3.23}$$

By (3.23) we determine the unknown function $C_i(\mu)$, $i = 1,2,3$. We obtain Riemann–Liouville fractional derivative of order $\alpha-1$ of the function (3.23)

$$\begin{aligned} {}^{RL}D_{0t}^{\alpha-1}u(x,t) = {} & x^{\frac{1-\nu}{2}}\int_0^\infty C_1(\mu)\,E_{\alpha,1}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}(\mu x)\,d\mu \\ & -\mu^2x^{\frac{1-\nu}{2}}t^{\alpha-1}\int_0^\infty C_2(\mu)\,E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}(\mu x)\,d\mu \\ & -\mu^2x^{\frac{1-\nu}{2}}t^{\alpha-2}\int_0^\infty C_3(\mu)\,E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}(\mu x)\,d\mu \end{aligned}$$

Considering the initial condition (3.2), we determine the unknown function $C_1(\mu)$.

$$\left.{}^{RL}D_{0t}^{\alpha-1}u\left(x,t\right)\right|_{t=0}=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}C_{1}\left(\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu=\psi\left(x\right),$$

$$\psi\left(x\right)x^{\frac{\nu-1}{2}}=\int\limits_{0}^{\infty}\frac{C_{1}\left(\mu\right)}{\mu}\mu J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu\ ,$$

$$C_{1}\left(\mu\right)=\mu\int\limits_{0}^{\infty}\psi\left(\rho\right)\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)\rho d\rho. \tag{3.24}$$

We obtain Riemann–Liouville fractional derivative of order $\alpha-2$ of the function (3.23)

$${}^{RL}D_{0t}^{\alpha-2}u\left(x,t\right)=x^{\frac{1-\nu}{2}}t\int\limits_{0}^{\infty}C_{1}\left(\mu\right)E_{\alpha,2}\left(-\mu^{2}t^{\alpha}\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu$$

$$+x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}C_{2}\left(\mu\right)E_{\alpha,1}\left(-\mu^{2}t^{\alpha}\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu$$

$$-\mu^{2}x^{\frac{1-\nu}{2}}t^{\alpha-1}\int\limits_{0}^{\infty}C_{3}\left(\mu\right)E_{\alpha,\alpha}\left(-\mu^{2}t^{\alpha}\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu$$

Considering the initial condition (3.3), we determine the unknown function $C_2\left(\mu\right)$.

$$\left.{}^{RL}D_{0t}^{\alpha-2}u\left(x,t\right)\right|_{t=0}=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}C_{2}\left(\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu=\varphi\left(x\right),$$

$$\varphi\left(x\right)x^{\frac{\nu-1}{2}}=\int\limits_{0}^{\infty}\frac{C_{2}\left(\mu\right)}{\mu}\mu J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu,$$

$$C_{2}\left(\mu\right)=\mu\int\limits_{0}^{\infty}\varphi\left(\rho\right)\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)\rho d\rho, \tag{3.25}$$

We obtain Riemann–Liouville fractional derivative of order $\alpha-3$ of the function (3.23)

$${}^{RL}D_{0t}^{\alpha-3}u\left(x,t\right)=x^{\frac{1-\nu}{2}}t^{2}\int\limits_{0}^{\infty}C_{1}\left(\mu\right)E_{\alpha,3}\left(-\mu^{2}t^{\alpha}\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu$$

$$+x^{\frac{1-\nu}{2}}t\int\limits_{0}^{\infty}C_{2}\left(\mu\right)E_{\alpha,2}\left(-\mu^{2}t^{\alpha}\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu$$

$$+x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}C_{3}\left(\mu\right)E_{\alpha,1}\left(-\mu^{2}t^{\alpha}\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu.$$

Considering the initial condition (3.4), we determine the unknown function $C_3\left(\mu\right)$.

$$\left.{}^{RL}D_{0t}^{\alpha-3}u\left(x,t\right)\right|_{t=0}=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}C_{3}\left(\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu=\tau\left(x\right),$$

$$\tau(x)\,x^{\frac{\nu-1}{2}} = \int\limits_0^\infty \frac{C_3(\mu)}{\mu}\mu J_{\frac{1-\nu}{2}}(\mu x)\,d\mu,$$

$$C_3(\mu) = \mu\int\limits_0^\infty \tau(\rho)\,\rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\,\rho d\rho. \tag{3.26}$$

Substituting (3.24), (3.25) and (3.26) in (3.23), we get the solution (3.18) of the Cauchy problem. First, let us show that the constructed function (3.18) satisfies equation (3.1). From the definition of Riemann–Liouville fractional derivative operator of order α (3.8), it is clear that

$$^{RL}D_{0t}^{\alpha} u(x,t) = -x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu^3$$

$$\times\left[\psi(\rho)\,E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) + \varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) + \tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu \tag{3.27}$$

We calculate the first derivative of the constructed function (3.18) with respect to the variable x.

$$u_x(x,t) = x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\left[\frac{1-\nu}{2x}J_{\frac{1-\nu}{2}}(\mu x) + \mu J'_{\frac{1-\nu}{2}}(\mu x)\right]\rho\mu$$

$$\times\left[\psi(\rho)\,E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) + \varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) + \tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu \tag{3.28}$$

Then, we calculate second derivative of the function (3.18) with respect to the variable x . We can make a slight simplification by using the fact that the function $J_{\frac{1-\nu}{2}}(\lambda x)$ are linear independent solution of Bessel equation

$$\lambda^2 J''_{\frac{1-\nu}{2}}(\lambda x) + x^{-1}\lambda J'_{\frac{1-\nu}{2}}(\lambda x) = \left(\frac{(1-\nu)^2}{4x^2} - \lambda^2\right) J_{\frac{1-\nu}{2}}(\lambda x)$$

Then, the second derivative of the function (3.18) with respect to the variable x is as follows

$$u_{xx}(x,t) = x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\,\rho\mu\times$$

$$\times\left[\left(\frac{\nu^2-\nu}{2x^2} - \mu^2\right) J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x) - \frac{\nu\mu}{x} J_{\frac{1-\nu}{2}}(\rho\mu)\,J'_{\frac{1-\nu}{2}}(\mu x)\right]$$

$$\times\left[\psi(\rho)\,E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) + \varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) + \tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu. \tag{3.29}$$

Further we can substitute the obtained derivatives (3.27), (3.28) and (3.29) in equation (3.1).

$$-x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu^3$$

$$\times\left[\psi(\rho)\,E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) + \varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) + \tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu$$

$$= x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\left[\left(\frac{\nu^2-\nu}{2x^2} - \mu^2\right) J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x) - \frac{\nu\mu}{x} J_{\frac{1-\nu}{2}}(\rho\mu)\,J'_{\frac{1-\nu}{2}}(\mu x)\right]\rho\mu$$

$$\times\left[\psi(\rho)\,E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) + \varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) + \tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu$$

$$+\frac{\nu}{x} x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\left[\frac{1-\nu}{2x}J_{\frac{1-\nu}{2}}(\mu x) + \mu J'_{\frac{1-\nu}{2}}(\mu x)\right]\rho\mu$$

$$\times\left[\psi\left(\rho\right)\,E_{\alpha,\alpha}\left(-\mu^{2}t^{\alpha}\right)+\varphi\left(\rho\right)t^{-1}E_{\alpha,\alpha-1}\left(-\mu^{2}t^{\alpha}\right)+\tau\left(\rho\right)t^{-2}E_{\alpha,\alpha-2}\left(-\mu^{2}t^{\alpha}\right)\right]d\rho d\mu.$$

After some simplifications, we get

$$-\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu^{3}$$

$$\times\left[\psi\left(\rho\right)\,E_{\alpha,\alpha}\left(-\mu^{2}t^{\alpha}\right)+\varphi\left(\rho\right)t^{-1}E_{\alpha,\alpha-1}\left(-\mu^{2}t^{\alpha}\right)+\tau\left(\rho\right)t^{-2}E_{\alpha,\alpha-2}\left(-\mu^{2}t^{\alpha}\right)\right]d\rho d\mu$$

$$=-\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu\mu^{2}$$

$$\times\left[\psi\left(\rho\right)\,E_{\alpha,\alpha}\left(-\mu^{2}t^{\alpha}\right)+\varphi\left(\rho\right)t^{-1}E_{\alpha,\alpha-1}\left(-\mu^{2}t^{\alpha}\right)+\tau\left(\rho\right)t^{-2}E_{\alpha,\alpha-2}\left(-\mu^{2}t^{\alpha}\right)\right]d\rho d\mu.$$

We can conclude that the solution (3.18) satisfies equation (3.1).

Let us show that the constructed function (3.18) satisfies the initial conditions (3.2), (3.3) and (3.4). The Riemann-Liouville fractional derivative of order $\alpha-1$ $(2<\alpha<3)$ with respect to the variable from the constructed function (3.18) has the following form.

$${}^{RL}D_{0t}^{\alpha-1}u\left(x,t\right)=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu$$

$$\times\left[\psi\left(\rho\right)\,E_{\alpha,1}\left(-\mu^{2}t^{\alpha}\right)-\mu^{2}\varphi\left(\rho\right)t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^{2}t^{\alpha}\right)-\mu^{2}\tau\left(\rho\right)t^{\alpha-2}E_{\alpha,\alpha-1}\left(-\mu^{2}t^{\alpha}\right)\right]d\rho d\mu$$

Using the condition (3.2) and the properties of the Dirac delta function (2.7), (2.8), we obtain

$${}^{RL}D_{0t}^{\alpha-1}u\left(x,t\right)\Big|_{t=0}=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\psi\left(\rho\right)\,\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu d\rho d\mu$$

$$=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\rho^{\frac{\nu-1}{2}}\psi\left(\rho\right)\delta\left(x-\rho\right)d\rho=x^{\frac{1-\nu}{2}}x^{\frac{\nu-1}{2}}\psi\left(x\right)=\psi\left(x\right).$$

The Riemann-Liouville fractional derivative of order $\alpha-2$ $(2<\alpha<3)$ with respect to the variable from the constructed function (3.18) has the following form

$${}^{RL}D_{0t}^{\alpha-2}u\left(x,t\right)=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu$$

$$\times\left[\psi\left(\rho\right)t\,E_{\alpha,2}\left(-\mu^{2}t^{\alpha}\right)+\varphi\left(\rho\right)E_{\alpha,1}\left(-\mu^{2}t^{\alpha}\right)-\mu^{2}\tau\left(\rho\right)t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^{2}t^{\alpha}\right)\right]d\rho d\mu.$$

Using the condition (3.3) and the properties of the Dirac delta function (2.7), (2.8), we obtain

$${}^{RL}D_{0t}^{\alpha-2}u\left(x,t\right)\Big|_{t=0}=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\varphi\left(\rho\right)\,\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu d\rho d\mu$$

$$=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\rho^{\frac{\nu-1}{2}}\varphi\left(\rho\right)\delta\left(x-\rho\right)d\rho=x^{\frac{1-\nu}{2}}x^{\frac{\nu-1}{2}}\varphi\left(x\right)=\varphi\left(x\right).$$

The Riemann-Liouville fractional derivative of order $\alpha-3$ $(2<\alpha<3)$ with respect to the variable from the constructed function (3.18) has the following form

$${}^{RL}D_{0t}^{\alpha-3}u\left(x,t\right)=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu$$

$$\times\left[\psi\left(\rho\right)t^{2}E_{\alpha,3}\left(-\mu^{2}t^{\alpha}\right)+\varphi\left(\rho\right)tE_{\alpha,2}\left(-\mu^{2}t^{\alpha}\right)+\tau\left(\rho\right)E_{\alpha,1}\left(-\mu^{2}t^{\alpha}\right)\right]d\rho d\mu.$$

Using the condition (3.4) and the properties of the Dirac delta function (2.7), (2.8), we obtain

$${}^{RL}D_{0t}^{\alpha-3}u\left(x,t\right)\big|_{t=0}=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\tau\left(\rho\right)\,\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu d\rho d\mu$$

$$=x^{\frac{1-\nu}{2}}\int\limits_{0}^{\infty}\rho^{\frac{\nu-1}{2}}\tau\left(\rho\right)\delta\left(x-\rho\right)d\rho=x^{\frac{1-\nu}{2}}x^{\frac{\nu-1}{2}}\tau\left(x\right)=\tau\left(x\right).$$

Consequently, the solution (3.18) satisfies the initial conditions (3.2), (3.3) and (3.4). Now we show that function (3.18) satisfies condition (3.6). For any fixed t, function $T\left(t\right)$ is bounded. Further, taking into account the asymptotic representation of the Bessel functions [39] for $z\to\infty$, with $\left|\arg\left(z\right)\right|<\pi$

$$J_{\mu}\left(z\right)\sim\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\pi\mu}{2}-\frac{\pi}{4}\right),$$

then $X\sim c\left(\lambda\right)x^{-\frac{\nu}{2}}$ and in addition to this, it can be shown that $u(x,t)\in\mathcal{S}(\Omega)$. And this means that on the basis of (3.19) we are convinced that the condition (3.6) is satisfied. From this we can conclude that the integral (3.18) is convergent.This ends the proof of existence of solution to the Cauchy problem (3.1)-(3.4). Theorem 3.3 has been proved.

□

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