

Extension of separately analytic functions with thin singularities on one-dimensional parallel sections

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Abstract. Let two bounded simply connected domains $D \subset \mathbb{C}_z$, $G \subset \mathbb{C}_w$, and two locally regular compact subsets $E \subset D$, $F \subset G$ be given. If $f(z, w)$ is a separately analytic function with finitely many singular points in each section of $D \times \{w^0\}$ and $\{z^0\} \times G$ for any $(z^0, w^0) \in E \times F$ on the set $X = (D \times F) \cup (E \times G)$, then it extends holomorphically to the domain

$$\hat{X} = \{(z, w) \in D \times G : \omega^*(z, E, D) + \omega^*(w, F, G) < -1\},$$

except on an analytic set S . Where $\omega^*(z, E, D)$ is the harmonic measure of the set $E \subset D \subset \mathbb{C}$ relative to the domain D .

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1. INTRODUCTION

A function of several complex variables $w = f(z_1, z_2, \dots, z_n)$, $z = (z_1, z_2, \dots, z_n) \in D \subset \mathbb{C}_z^n$ is called holomorphic in the domain D if it is real differentiable with respect to the set of all variables and satisfies the following equalities

$$\frac{\partial f}{\partial \bar{z}_1} = \frac{\partial f}{\partial \bar{z}_2} = \dots = \frac{\partial f}{\partial \bar{z}_n} = 0$$

for any $z \in D$.

The following question naturally arises: if a function $w = f(z_1, z_2, \dots, z_n)$ of several complex variables is holomorphic in each variable when the other variables are fixed, then is it holomorphic with respect to the set of all variables? Hartogs [1] gives a positive answer to this question: *if a function $w = f(z_1, z_2, \dots, z_n)$ is holomorphic in the domain $D \subset \mathbb{C}^n$ with respect to each of the variables z_j , $j = \overline{1, n}$, then it is holomorphic in D with respect to the set of all variables.*

The proof of this theorem is based on the important, so-called Hartogs lemma "On the extension of functions of several complex variables along a fixed direction": *let a function $f(z, w)$ be holomorphic in a polydisk $U \times V_r = \{z \in \mathbb{C}^n : |z| < 1\} \times \{w \in \mathbb{C} : |w| < r\}$, $r > 0$, and for each fixed $z^0 \in U$, the function $f(z^0, w)$ extends holomorphically to a larger disk $V_R = \{w \in \mathbb{C} : |w| < R\}$, $R > r$ with respect to w . Then the function $f(z, w)$ extends holomorphically to a larger polydisk $U \times V_R$ with respect to the set of all variables.*

These results of Hartogs play a fundamental role in the theory of analytic continuation of functions of several complex variables.

The following problem was posed by M.Hukuhara [2]: let two domains $D \subset \mathbb{C}_z^n$, $G \subset \mathbb{C}_w^m$, and two sets $E \subset D$, $F \subset G$ be given. Suppose that the function $f(z, w)$ originally defined on the set $E \times F$ has the following properties:

- 1) for each fixed $w^0 \in F$, the function $f(z, w^0)$ extends holomorphically to D ,
- 2) for each fixed $z^0 \in E$, the function $f(z^0, w)$ extends holomorphically to G .

Then we say that the function $f(z, w)$ defines some separately analytic function on

$$X = (D \times F) \cup (E \times G).$$

The task is to determine the domain \hat{X} , ($\hat{X} \supset X$) to which the function $f(z, w)$ admits a holomorphic extension with respect to the set of all variables.

In case $n = m = 1$, this problem was solved by J.Siciak [3], and in the general case by V.P.Zakharyuta [4]: let $D \subset \mathbb{C}_z^n$ and $G \subset \mathbb{C}_w^m$ be strongly pseudoconvex domains, and let E and F be closed subsets of D and G respectively. If $f(z, w)$ is a separately analytic function on the set

$$X = (D \times F) \cup (E \times G),$$

then it extends holomorphically to the domain

$$\hat{X} = \{(z, w) \in D \times G : \omega^*(z, E, D) + \omega^*(w, F, G) < -1\}.$$

Here, $\omega^*(z, E, D)$ denotes the basic quantity of complex potential theory, known as the P -measure of the set $E \subset D \subset \mathbb{C}^n$ relative to the domain D (see. [5], [6]): let $\mathcal{U}(E, D)$ be a class of functions $u(z) \in psh(D)$, such that $u|_E \leq -1$, $u|_D \leq 0$. We put

$$\omega(z, E, D) = \sup \{u(z) : u \in \mathcal{U}(E, D)\},$$

then $\omega^*(z, E, D) = \overline{\lim}_{\xi \rightarrow z} \omega(\xi, E, D)$ is a maximal plurisubharmonic function, and it is called the P -measure of the set E relative to the domain D . In the one-dimensional case, $\omega^*(z, E, D)$ is called a harmonic measure.

An analogue of Hartogs' theorem in the class of meromorphic functions was given by E.Sakai [7]: let $S \subset D \times G$ be a relatively compact set such that $\text{int}S = \emptyset$ and that it does not split the domain $D \times G$. Let A (respectively B) be the set of all points $z \in D$ ($w \in G$) such that $\text{int}\{w \in G : (z, w) \in S\} = \emptyset$ ($\text{int}\{z \in D : (z, w) \in S\} = \emptyset$). Let the function $f(z, w)$ be defined on $(D \times G) \setminus S$ and be separately meromorphic on $X = (A \times G) \cup (D \times B)$. Then there exists a function \hat{f} meromorphic in the domain $D \times G$, such that $\hat{f}|_{X \setminus S} = f$.

The works of V.Rothstein [8], M.V.Kazaryan [9, 10], S.M.Ivashkovich [11], J.Ruppenthal [12], A.S.Sadullaev and E.M.Chirka [13], A.S.Sadullaev and S.A.Imomkulov [14], S.A.Imomkulov [6], T.T.Tuichiev [15], M.Yarnicki and P.Pflug [16, 17], A.A.Gonchar [18], P.Pflug and V.A.Nguyen [19], A.A.Atamuratov [20] are also directly related to the continuation of this topic.

The following theorem of A.A.Atamuratov [20] on the continuation of separately meromorphic functions is distinctive in that there is no condition on the set of poles of the given separately meromorphic function: let $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ be strongly pseudoconvex domains, and $E \subset D$, $F \subset G$ be nonpluripolar compact sets. Suppose that the function $f(z, w)$ is continuous on the set $E \times F$ and has the following properties:

- 1) for each $w^0 \in F$, there exists a function $\phi_{w^0}(z)$, meromorphic in D , such that for every $\xi \in E$,

$$\phi_{w^0}(\xi) = f(\xi, w^0);$$

- 2) for each $z^0 \in E$, there exists a function $\psi_{z^0}(w)$, meromorphic in G , such that for every $\eta \in F$,

$$\psi_{z^0}(\eta) = f(z^0, \eta).$$

Then $f(z, w)$ defines a separately meromorphic function on the set $X = (D \times F) \cup (E \times G)$ and it extends meromorphically to the domain

$$\hat{X} = \{(z, w) \in D \times G : \omega^*(z, E, D) + \omega^*(w, F, G) < -1\}.$$

In this paper, we continue our study of holomorphic extension with singularities on sections and establish an important theorem concerning separately holomorphic functions.

2. MAIN RESULT

The main theorem of the present paper is the following:

Theorem 2.1. *Let two bounded simply connected domains $D \subset \mathbb{C}_z$, $G \subset \mathbb{C}_w$ and two locally regular compact subsets $E \subset D$, $F \subset G$ be given. Suppose that a function $f(z, w)$, originally defined on the set $E \times F$, satisfies the following conditions of separate analyticity:*

1) for each fixed $w^0 \in F$, the function $f(z, w^0)$, defined on E , extends holomorphically to D , except on a finite set of singular points $S_{w^0} \subset D \setminus E$;

2) for each fixed $z^0 \in E$, the function $f(z^0, w)$, defined on F , extends holomorphically to G , except on a finite set of singular points $S_{z^0} \subset G \setminus F$.

Then the function $f(z, w)$ extends holomorphically to the domain

$$\hat{X} = \{(z, w) \in D \times G : \omega^*(z, E, D) + \omega^*(w, F, G) < -1\},$$

except on an analytic set of singularities $S \subset \hat{X}$.

A set $E \subset D \subset \mathbb{C}_z$ is called (globally) regular at a point $z^0 \in \bar{E}$ relative to the domain D if $\omega^*(z^0, E, D) = -1$. It is called locally regular at the point z^0 if for any circle $U(z^0, r)$, $r > 0$, the intersection $\bar{U}(z^0, r) \cap \bar{E}$ is regular at the point z^0 .

A set $E \subset D$ is called locally regular if it is locally regular at every point of $z \in E$.

3. THE RADIUS OF THE MAXIMAL CIRCLE OF HOLOMORPHY OF THE FUNCTIONS, EXCEPT ON A DISCRETE SET OF SINGULARITIES

In the work of A.S.Sadullaev [21], the following characteristic of holomorphic functions was introduced: let $f(z, w)$ be a holomorphic function in the polydisk $U \times V_r = \{z \in \mathbb{C} : |z| < 1\} \times \{w \in \mathbb{C} : |w| < r\}$, and let $\mathcal{R}(z)$ denote the radius of the maximal circle on which the function $f(z, w)$, with respect to w , extends as a holomorphic function with discrete singularities.

The following holds:

Lemma 3.1. [21] *A function $-\ln \mathcal{R}_*(z)$ is subharmonic in $U \subset \mathbb{C}$, where $\mathcal{R}_*(z) = \lim_{\zeta \rightarrow z} \mathcal{R}(\zeta)$ is the lower regularization. Moreover, the set $\{z \in U : \mathcal{R}_*(z) < \mathcal{R}(z)\}$ is polar in U and the function $f(z, w)$ extends holomorphically to $\Omega \setminus S$, where $\Omega = \{(z, w) : |w| < \mathcal{R}_*(z), z \in U\}$ and S is an analytic subset of Ω .*

Remark 3.2. *The analyticity of the exceptional set S in Lemma 3.1 is established using methods developed by A.S.Sadullaev and E.M.Chirka for extending holomorphic functions with fine singularities along a given direction, as detailed in [13].*

Lemma 3.3. *Let the function $f(z, w)$ be holomorphic in the domain $U \times V_r$. For each fixed point z^0 from some nonpolar compact set $E \subset U$, the function $f(z^0, w)$, with respect to w , extends holomorphically to the larger circle $V_R = \{w \in \mathbb{C} : |w| < R\}$, $R > r$, except on finitely many singular points. Then the function $f(z, w)$ extends holomorphically to the domain*

$$\{(z, w) : |w| < R^{-\omega^*(z, E, U)} \cdot r^{1+\omega^*(z, E, U)}, z \in U\},$$

except on an analytic set of singularities.

Proof. In the proof, we use Lemma 3.1. Let the function $f(z, w)$ satisfy all the conditions of Lemma 3.3; that is, the function $f(z, w)$ is holomorphic in $U \times V_r$ and for each $z^0 \in E \subset U$, the function $f(z^0, w)$, with respect to w , extends holomorphically to $V_R = \{w : |w| < R\}$, except on a finite set of points S_{z^0} .

Hence, by Lemma 3.1, it follows that the function $f(z, w)$ extends holomorphically to the domain

$$\{(z, w) : |w| < \mathcal{R}_*(z), z \in U\},$$

except on some analytic set S , where $\mathcal{R}_*(z)|_E \geq R$, $\mathcal{R}_*(z)|_U \geq r$ holds.

It follows that the subharmonic function

$$u(z) = -\frac{\ln \mathcal{R}_*(z) - \ln r}{\ln R - \ln r}$$

belongs to the class $\mathcal{U}(E, U)$, i.e., $u|_E \leq -1$, $u|_U \leq 0$ and by definition $u(z) \leq \omega^*(z, E, U)$. Hence, $\mathcal{R}_*(z) \geq R^{-\omega^*(z, E, U)} \cdot r^{1+\omega^*(z, E, U)}$, $z \in U$, i.e.,

$$\{(z, w) : |w| < \mathcal{R}_*(z), z \in U\} \supset \{(z, w) : |w| < R^{-\omega^*(z, E, U)} \cdot r^{1+\omega^*(z, E, U)}, z \in U\}.$$

Thus, any function $f(z, w)$ satisfying the conditions of Lemma 3.3 can be holomorphically continued to the domain $\{(z, w) : |w| < R^{-\omega^*(z, E, U)} \cdot r^{1+\omega^*(z, E, U)}\}$, except on the analytic set S . \square

Corollary 3.4. *Let $U \subset \mathbb{C}_z$ be a bounded open set, $G \subset \mathbb{C}_w$ a bounded simply connected domain, $E \subset U$ a nonpolar set and $W \subset G$ an open set. If the function $f(z, w)$ is holomorphic in $U \times W$, and for every fixed $z^0 \in E$, the function $f(z^0, w)$, with respect to w , extends holomorphically to G , except on finitely many singular points, then the function $f(z, w)$ extends holomorphically to the domain*

$$\{(z, w) : \omega^*(z, E, U) + \omega^*(w, W, G) < -1\},$$

except on an analytic set S .

Lemma 3.5. [22] *Let $D \subset \mathbb{C}^n$, ($n > 1$) be a bounded domain, and let \hat{D} be the holomorphic hull of D . Then every holomorphic function defined in the domain $D \setminus A$, where $A \subset D$ is an analytic set of full dimension ($n - 1$), extends holomorphically to the domain \hat{D} , except on an analytic set $\hat{A} \subset \hat{D}$, satisfying $\hat{A} \cap D = A$.*

4. PROOF OF THEOREM 2.1

Let $D \subset \mathbb{C}_z$, $G \subset \mathbb{C}_w$ be bounded simply connected domains and $E \subset D$, $F \subset G$ are locally regular compact sets. Let the function $f(z, w)$ satisfy all the conditions of Theorem 2.1. We consider the following open neighborhoods of the regular compact sets

$$U_j = \left\{ z \in D : \omega^*(z, E, D) < -1 + \frac{1}{j} \right\}, W_j = \left\{ w \in G : \omega^*(w, F, G) < -1 + \frac{1}{j} \right\}$$

and

$$F_j = \{w^0 \in F : U_j \cap S_{w^0} = \emptyset\}, E_j = \{z^0 \in E : W_j \cap S_{z^0} = \emptyset\}, j = 1, 2, \dots$$

Note that the open sets U_j and W_j consist of the union of a finite number of simply connected domains. It is clear that $E_j \subset E_{j+1}$, $F_j \subset F_{j+1}$ and $E = \bigcup_{j=1}^{\infty} E_j$, $F = \bigcup_{j=1}^{\infty} F_j$. Since E and F are nonpolar, there exists a number $j_0 \in \mathbb{N}$ such that the sets E_j and F_j are nonpolar for all $j \geq j_0$. According to the Sichak-Zakharyuta theorem, the function $f(z, w)$ extends holomorphically to the open set

$$\hat{\Omega}_j = \{(z, w) \in U_j \times W_j : \omega^*(z, E_j^0, U_j) + \omega^*(w, F_j^0, W_j) < -1\}.$$

Now, we consider the following open sets:

$$U_j^{(\alpha)} = \{z \in U_j : \omega^*(z, E_j^0, U_j) < -\alpha\}$$

$$W_j^{(\alpha)} = \{w \in W_j : \omega^*(w, F_j^0, W_j) < -1 + \alpha\}$$

for which $\alpha \in (0, 1)$ is a fixed number. It is clear that $U_j^{(\alpha)} \times W_j^{(\alpha)} \subset \hat{\Omega}_j$.

It follows that the function $f(z, w)$ is holomorphic in $U_j^{(\alpha)} \times W_j^{(\alpha)}$. For each fixed $z^0 \in E_j^0$, the function $f(z^0, w)$, with respect to w , is holomorphic in $G \setminus S_{z^0}$. Hence, the restriction of the function $f(z, w)$ to the set $Y_j^{(\alpha)} = (U_j^{(\alpha)} \times W_j^{(\alpha)}) \cup (E_j^0 \times G)$ satisfies all the conditions of Corollary 3.4 and admits a holomorphic continuation to the open set

$$\hat{Y}_j^{(\alpha)} = \left\{ (z, w) \in U_j^{(\alpha)} \times G : \omega^*(z, E_j^0, U_j^{(\alpha)}) + \omega^*(w, W_j^{(\alpha)}, G) < -1 \right\},$$

except on an analytic set $M_j^{(\alpha)} : M_j^{(\alpha)} \cap (\{z^0\} \cap G) = S_{z^0}$, for almost all $z^0 \in E_j$ with respect to the harmonic measure.

Similarly, the restriction of the function $f(z, w)$ to the set $Z_j^{(\alpha)} = (D \times F_j^0) \cup (U_j^{(\alpha)} \times W_j^{(\alpha)})$ satisfies all the conditions of Corollary 3.4 and admits holomorphic continuation to the open set

$$\hat{Z}_j^{(\alpha)} = \left((z, w) \in D \times W_j^{(\alpha)} : \omega^*(z, U_j^{(\alpha)}, D) + \omega^*(w, F_j^0, W_j^{(\alpha)}) < -1 \right),$$

except on an analytic set $L_j^{(\alpha)} : L_j^{(\alpha)} \cap (D \cap \{w^0\}) = S_{w^0}$, for almost all $w^0 \in F_j$ with respect to the harmonic measure.

On the other hand, $Y_j^{(\alpha)} \subset \hat{X}_j$ and $Z_j^{(\alpha)} \subset \hat{X}_j$, where

$$\hat{X}_j = \{(z, w) \in D \times G : \omega^*(z, E_j^0, D) + \omega^*(w, F_j^0, G) < -1\}.$$

It follows that the holomorphic hulls $\hat{Y}_j^{(\alpha)}$ and $\hat{Z}_j^{(\alpha)}$ also belong to \hat{X}_j . Moreover $(D \times F_j^0) \cup (E_j^0 \times G) \subset \hat{Y}_j^{(\alpha)} \cup \hat{Z}_j^{(\alpha)} \subset \hat{X}_j$ holds.

Consequently, by Lemma 3.5, the function $f(z, w)$ extends holomorphically to the domain \hat{X}_j , except on some analytic set S_j satisfying $\hat{Y}_j^{(\alpha)} \cap S_j = M_j^{(\alpha)}$, $\hat{Z}_j^{(\alpha)} \cap S_j = L_j^{(\alpha)}$.

Finally, tending to $j \rightarrow +\infty$, and using the continuity of the harmonic measure, we obtain

$$\lim_{j \rightarrow +\infty} \hat{X}_j = \hat{X},$$

where

$$\hat{X} = \{(z, w) \in D \times G : \omega^*(z, E, D) + \omega^*(w, F, G) < -1\}.$$

Once again, by applying Lemma 3.5, we obtain that the function $f(z, w)$ extends holomorphically to the domain \hat{X} , except on an analytic set S satisfying $\hat{X} \cap S = S_j$. *This completes the proof of Theorem 2.1.*

The following example shows that this theorem does not exclude the case $S \cap (E \times F) \neq \emptyset$.

Example. Let us consider the set $X = (U \times F) \cup (E \times V)$, where $U = \{z \in \mathbb{C} : |z| < 1\}$, $V = \{w \in \mathbb{C} : |w| < 1\}$, $E = [0, \frac{1}{2}]$ and $F = [0, \frac{1}{2}]$. We put

$$f(z, w) = \begin{cases} \frac{z \cdot w}{z + w}, & z \cdot w \neq 0 \\ 0, & z \cdot w = 0. \end{cases}$$

Then the function $f(z, w)$ satisfies all the conditions of Theorem 2.1. The extension of the function has the set of singularities $S = \{(z, w) : z + w = 0\}$, and the intersection $S \cap (E \times F) = \{(0, 0)\}$ is nonempty.

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