Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 100–106
DOI: 10.29229/uzmj.2026-1-10

Spectral properties of the one-dimensional Schrödinger Hamiltonian with non-local $\delta'(x \pm y)$ potentials

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Abstract. We consider a model of one-dimensional Schrödinger Hamiltonian perturbed by two identical non-local interactions of the form $\delta'(x \pm y)$, symmetrically located at the points $\pm y$ from the origin. The Schrödinger operator under consideration is constructed as a self-adjoint extension of the symmetric Laplacian. The essential spectrum is described, and the condition for the existence of the eigenvalue of the Schrödinger operator is investigated. The main results are based on the analysis of the spectral analysis of self-adjoint extension of the operator $\mathbf{h}$.

Keywords: Schrödinger operators, non-local delta prime interactions, eigenvalues, eigenfunctions
MSC (2020): 47A10, 47B25, 81Q10

1. Introduction

The problem of point interactions among two or three identical quantum particles interacting via point potentials (also referred to as contact or singular potentials) has been studied extensively in the literature. Early foundational work was carried out by Berezin and Faddeev [1], as well as R.A. Minlos and L.D. Faddeev [2, 3], who proposed a rigorous mathematical framework for describing point interactions between two and three particles, respectively.

In [2, 3], the Hamiltonian of such systems was analyzed using the theory of self-adjoint extensions of symmetric operators. It was represented as a self-adjoint extension of the symmetric Laplace operator, defined on the domain of functions of three variables $x_1, x_2, x_3 \in \mathbb{R}$ that vanish whenever two coordinates coincide, i.e., $x_j = x_k$ for $j \neq k$, $j, k = 1, 2, 3$.

This extension is known as the Skornyakov–Ter-Martirosyan extension. In [4], building upon the results of [1, 2], the Hamiltonian for a three-particle system (two identical fermions and a third particle of different type, all with equal masses) interacting via point potentials was investigated. It was shown that the Skornyakov–Ter-Martirosyan extensions are self-adjoint and semi-bounded.

In the work, we consider a one-dimensional quantum particle interacting with external fields through non-local Dirac delta prime potentials of the form $\delta'(x \pm y)$, located symmetrically at $\pm y$, with $y > 0$. Formally, the corresponding Schrödinger operator is given by

$$\widehat{\mathbf{h}} := \widehat{\mathbf{h}}_0 - \delta'(x + y) - \delta'(x - y), \tag{1.1}$$

where $\widehat{\mathbf{h}}_0 := -\Delta$ is the Laplace operator, and $\delta'(x)$ denotes the derivative of the Dirac delta function. However, due to the singular nature of the delta function's derivative, the expression in (1.1) does not define a proper operator on the Hilbert space $L^2(\mathbb{R})$.

To give a rigorous meaning to (1.1), we construct the operator as a self-adjoint extension of the symmetric Laplace operator with a suitably restricted domain. This construction ensures that the resulting operator is mathematically well-defined and suitable for physical analysis. The theory of self-adjoint extensions provides the necessary tools for this rigorous formulation.

In the work [5], a non-local Schrödinger operator of the form (1.1) is investigated by introducing a regularized version of the singular interaction and applying a renormalization procedure to the coupling constant λ. This approach leads to a self-adjoint extension of the operator, which is shown to possess two negative eigenvalues. The dependence of these eigenvalues on the coupling constant and the separation distance y is analyzed. Furthermore, the resonance behavior of the corresponding operator is also investigated.

One-dimensional models with point perturbations are particularly advantageous, as they allow the exploration of various qualitative features of quantum systems. For related studies involving both local and non-local delta potentials in one-body problems, see [6, 7, 8, 9, 10, 11, 12].

In the momentum representation, after reduction of variables, we construct the Krein-Višik-Birman extension $\mathbf{h}$ of the associated Hamiltonian. It is proven that the essential spectrum of this extension coincides with the non-negative real axis. We also study the conditions under which the operator admits eigenvalues. The main results of this work are based on the spectral analysis of the operator $\mathbf{h}$. In particular, we describe the essential spectrum (see Theorem 2.1) and explicitly establish the existence of eigenvalues of the operator $\mathbf{h}$ (see Theorem 2.3).

1.1. **Preliminaries.** Now, in order to give meaning to a Schrödinger operator with particle interactions involving non-local delta functions, as defined in (1.1), we define it on the set of functions in the $L^2(\mathbb{R})$ space that satisfy the condition $\phi'(\pm y) = 0$. Thus, the singular contributions arising from the action of the Laplace operator are canceled out by the delta functions in (1.1). It should be emphasized that any operator defined in this way is an extension of the symmetric operator $\widehat{\mathbf{h}}_0$, which is defined on the following manifold:

$$D(\widehat{\mathbf{h}}_0) = \left\{\phi \in L^2(\mathbb{R}) : \Delta\phi \in L^2(\mathbb{R}),\ \phi'(\pm y) = 0,\ y > 0\right\}, \tag{1.2}$$

where the singular contributions related to the delta functions in (1.1) disappear.

After applying the Fourier transform, the operator $\widehat{\mathbf{h}}_0$ transforms into the operator

$$(\mathbf{h}_0 f)(p) = p^2 f(p)$$

defined on the set $D(\mathbf{h}_0) \subset L^2(\mathbb{R})$ consisting of functions $f(p)$ that satisfy:

$$\int\limits_{\mathbb{R}} (1+p^2)^2 |f(p)|^2\, dp < \infty,$$

$$\int\limits_{\mathbb{R}} p e^{\pm iyp} f(p)\, dp = 0. \tag{1.3}$$

According to [13], the defeciency subspace $\mathfrak{R}_z$ of the operator $\mathbf{h}_0$, is defined by

$$\mathfrak{R}_z = (\mathrm{Ran}(\mathbf{h}_0 - zI))^{\perp} = \left\{\hat{\varphi} \in L^2(\mathbb{R}) : (\mathbf{h}_0 - zI) f \perp \hat{\varphi},\quad f \in D(\mathbf{h}_0)\right\}. \tag{1.4}$$

Moreover, from (1.3) and (1.4), for any $z \in \Pi_0 = \mathbb{C} \setminus [0,\infty)$, the deficiency subspace $\mathfrak{R}_z \subset L^2(\mathbb{R})$ of $\mathbf{h}_0$ consists of functions of the form

$$\hat{\varphi}(p) = \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2 - \bar{z}}, \quad \hat{c}_1, \hat{c}_2 \in \mathbb{C}. \tag{1.5}$$

It follows from (1.5) that for any $z \in \Pi_0$, the deficiency subspace $\mathfrak{R}_z$ is two-dimensional. Therefore, $\mathbf{h}_0$ is a symmetric operator with deficiency indices (2,2). Using the general extension theory [14], we conclude that the operator $\mathbf{h}_0$ admits two-parameter family of self-adjoint extensions.

Since the operator $\mathbf{h}_0$ is non-negative, we use the theory of extensions of semibounded operators, as developed in [2, 3]. Furthermore, the deficiency indices of $\mathbf{h}_0$ remain constant for all $z \in \Pi_0$. Therefore, it suffices to analyze the case $z = -1$. The deficiency subspace $\mathfrak{R}_{-1}$ of $\mathbf{h}_0$ consists of functions of the form

$$\hat{\varphi}_{-1}(p) = \frac{\hat{c}_1 p e^{ix_0 p} + \hat{c}_2 p e^{-ix_0 p}}{p^2 + 1},\ (\hat{c}_1, \hat{c}_2) \in \mathbb{C}^2.$$

Following the approach in [2, 3] and [4], the adjoint operator $\mathbf{h}_0^*$ is characterized by the following lemma.

Lemma 1.1. *The domain $D(\mathbf{h}_0^*)$ of the adjoint operator $\mathbf{h}_0^*$ consists of functions of the form*

$$g(p) = f(p) + \frac{\hat{d}_1 p e^{iyp} + \hat{d}_2 p e^{-iyp}}{p^2 + 1} + \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{(p^2+1)^2} \tag{1.6}$$

where $f \in D(\mathbf{h}_0), \hat{c}_1, \hat{c}_2, \hat{d}_1, \hat{d}_2 \in \mathbb{C}$. The operator $\mathbf{h}_0^*$ acts on functions of the form (1.6) *via the rule*

$$\mathbf{h}_0^* g(p) = p^2 g(p) - \hat{d}_1 p e^{ix_0 p} - \hat{d}_2 p e^{-ix_0 p},$$

with the constants $\hat{d}_1, \hat{d}_2$ taken from the decomposition (1.6).

We now construct the extension of the operator $\mathbf{h}_0$. Define the set $D(\mathbf{h}), D(\mathbf{h}_0) \subset D(\mathbf{h}) \subset D(\mathbf{h}_0^*)$, as follows:

$$D(\mathbf{h}) = \left\{ g \in D(\mathbf{h}_0^*) : g(p) = f(p) + \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2+1} + \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{(p^2+1)^2}, f \in D(\mathbf{h}_0) \right\}. \tag{1.7}$$

The restriction of the operator $\mathbf{h}_0^*$ to the domain $D(\mathbf{h})$ is denoted by $\mathbf{h}$, and it acts as

$$(\mathbf{h}g)(p) = p^2 g(p) - \hat{c}_1 p e^{iyp} - \hat{c}_2 p e^{-iyp}, \tag{1.8}$$

where the constants $\hat{c}_1$ and $\hat{c}_2$ come from the decomposition (1.7) of the function g.

By construction, $\mathbf{h}$ is an extension of the operator $\mathbf{h}_0$.

Theorem 1.2. *The extension* $\mathbf{h}$ *is a self-adjoint operator.*

Proof. It is straightforward to check that for any $g_1, g_2 \in D(\mathbf{h})$, the following relation

$$(\mathbf{h}g_1, g_2) = (g_1, \mathbf{h}g_2),$$

holds, showing that $\mathbf{h}$ is symmetric operator. To prove self-adjointness, it suffices to show that the deficiency indices of $\mathbf{h}$ are $(0,0)$.

Let $\hat{\varphi}_b \in \mathfrak{R}_{-1}(\mathbf{h}_0)$. Then

$$\hat{\varphi}_b(p) = \frac{\hat{b}_1 p e^{iyp} + \hat{b}_2 p e^{-iyp}}{p^2+1}, \quad \hat{b}_1, \hat{b}_2 \in \mathbb{C}.$$

For any $g \in D(\mathbf{h})$, the equality

$$((\mathbf{h}+I)g, \hat{\varphi}_b) = 0$$

holds. Considering (1.7), we obtain

$$((\mathbf{h}+I)g, \hat{\varphi}_b) = ((\mathbf{h}_0+I)f, \hat{\varphi}_b) + \int\limits_{\mathbb{R}} \frac{(\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp})\overline{(\hat{b}_1 p e^{iyp} + \hat{b}_2 p e^{-iyp})}}{(p^2+1)^2}\, dp.$$

From the relation

$$\int_{\mathbb{R}^3} (p^2+1) f(p) \overline{\hat{\varphi}_b(p)}\, dp = 0,$$

and choosing $\hat{c}_1 = \hat{b}_1, \hat{c}_2 = \hat{b}_2$, we get

$$\int_{\mathbb{R}} \frac{|\hat{b}_1 p e^{iyp} + \hat{b}_2 p e^{-iyp}|^2}{(p^2+1)^2}\, dp = 0,$$

which implies

$$\hat{b}_1 p e^{iyp} + \hat{b}_2 p e^{-iyp} = 0.$$

Hence $\hat{b}_1 = \hat{b}_2 = 0$, so $\hat{\varphi}_b(p) = 0$, and the deficiency indices of $\mathbf{h}$ are $(0,0)$.

2. Spectral properties of the operator $\mathbf{h}$

The main results of the paper are stated in the following theorems.

Theorem 2.1. *The essential spectrum of the operator* $\mathbf{h}$ *coincides with the nonnegative real semiaxis* $[0, \infty)$.

Proof. For each $z \geq 0$ consider the sequence of cut-off layers:

$$\mathcal{G}_n(z) = \left\{ p \in \mathbb{R} : \sqrt{z} + \frac{1}{n+1} < |p| < \sqrt{z} + \frac{1}{n} \right\}, \quad n = 1, 2, 3, \ldots$$

Each layer $\mathcal{G}_n(z)$ is divided into two symmetric parts:

$$\mathcal{G}_n^+(z) = \{p \in \mathcal{G}_n(z) : p \geq 0\}, \quad \mathcal{G}_n^-(z) = \{p \in \mathcal{G}_n(z) : p < 0\}.$$

By construction, these parts are equal in measure:

$$\mu(\mathcal{G}_n^+(z)) = \mu(\mathcal{G}_n^-(z)) = \frac{1}{2}\mu(\mathcal{G}_n(z)).$$

A simple calculation shows that

$$V_n = \mu\left(\mathcal{G}_n(z)\right) = \frac{2}{n\left(n+1\right)}.$$

Define a sequence of test functions $f_n^{(z)}$, $n \in \mathbb{N}$, as follows:

$$f_n^{(z)}(p) = \begin{cases} \dfrac{p\cos\left(yp\right)}{\sqrt{V_n}}, \text{if } p \in \mathcal{G}_n^+(z) \\ -\dfrac{p\cos\left(yp\right)}{\sqrt{V_n}}, \text{if } p \in \mathcal{G}_n^-(z) \\ 0, \text{if } p \in \mathbb{R} \setminus \mathcal{G}_n(z). \end{cases}$$

It is easy to verify that $f_n^{(z)} \in L^2(\mathbb{R})$, $\|f_n^{(z)}\| = 1$, and $f_n^{(z)} \perp f_m^{(z)}$ for $n \neq m$. Moreover, $f_n^{(z)} \in D(\mathbf{h}_0)$, i.e.,

$$\int\limits_{\mathbb{R}} pe^{\pm iyp} f_n^{(z)}(p)dp = 0, \quad \forall n \in \mathbb{N}.$$

Furthermore,

$$\left\|(\mathbf{h} - zI)f_n^{(z)}\right\|^2 = \int\limits_{\mathbb{R}} |(p^2 - z)f_n^{(z)}(p)|^2 dp = \frac{1}{V_n} \int\limits_{\mathcal{G}_n(z)} |(p^2 - z)p\cos(yp)|^2\, dp \leq$$

$$\leq \frac{1}{V_n} \int\limits_{\mathcal{G}_n(z)} |(p^2 - z)p|^2 dp = \frac{2}{V_n} \int\limits_{\sqrt{z}+\frac{1}{n+1}}^{\sqrt{z}+\frac{1}{n}} \left(p^2 - z\right)^2 p^2 dp <$$

$$< \frac{2}{V_n}\left(2\sqrt{z} + \frac{1}{n}\right)^2 \left(\sqrt{z} + \frac{1}{n}\right)^2 \frac{1}{n^2} \cdot \frac{1}{n(n+1)} = \frac{1}{n^2}\left(2\sqrt{z} + \frac{1}{n}\right)^2 \left(\sqrt{z} + \frac{1}{n}\right)^2.$$

Thus,

$$\lim_{n\to\infty} \|(\mathbf{h} - zI)f_n^{(z)}\| = 0.$$

This implies that for every $z \geq 0$, we have $z \in \sigma_{ess}(\mathbf{h})$, so $[0;\infty) \subset \sigma_{ess}(\mathbf{h})$. To prove the reverse inclusion $\sigma_{ess}(\mathbf{h}) \subset [0;\infty)$, we construct the resolvent of $\mathbf{h}$.

Let $\psi \in L^2(\mathbb{R})$. Then, $(\mathbf{h} - zI)g = \psi$. Moreover,

$$(p^2 - z)g(p) - \hat{c}_1 pe^{iyp} - \hat{c}_2 pe^{-iyp} = \psi(p)$$

or

$$g(p) = \frac{\psi(p)}{p^2 - z} + \frac{\hat{c}_1 pe^{iyp} + \hat{c}_2 pe^{-iyp}}{p^2 - z}. \tag{2.1}$$

Comparing this with the know extension of functions in the extended domain (e.g.(1.7)), (2.1) we derive an equation for the unknown coefficients $\hat{c}_1$ and $\hat{c}_2$.

Multiplying both sides by $pe^{\pm iyp}$ and integrating over $\mathbb{R}$ gives:

$$\begin{cases} \hat{a}(z)c_1 + \hat{b}(z)c_2 = \dfrac{2}{\pi}\displaystyle\int\limits_{\mathbb{R}} \frac{se^{iys}\psi(s)}{s^2 - z}\, ds, \\ \hat{b}(z)c_1 + \hat{a}(z)c_2 = \dfrac{2}{\pi}\displaystyle\int\limits_{\mathbb{R}} \frac{se^{-iys}\psi(s)}{s^2 - z}\, ds, \end{cases} \tag{2.2}$$

where

$$\hat{a}(z) = 2\sqrt{-z}e^{-2y\sqrt{-z}} - (1+2y)e^{-2y},\ \hat{b}(z) = 2\sqrt{-z} - 1. \tag{2.3}$$

Here the following elementary integrals are used

$$\int_{\mathbb{R}} \left(\frac{p^2 e^{iyp}}{p^2+1} - \frac{p^2 e^{iyp}}{p^2 - z} \right) dp = -\pi e^{-y} + \pi\sqrt{-z}e^{-y\sqrt{-z}},\ z<0,$$

$$\int_{\mathbb{R}} \frac{p^2 e^{iyp}}{(p^2+1)^2} dp = \frac{\pi}{2}(1-y)e^{-y}.$$

Using these equations, we solve for $\hat{c}_1$ and $\hat{c}_2$ and obtain the following representation of the resolvent $R_z(\mathbf{h})$:

$$(R_z\psi)(p) = \frac{\psi(p)}{p^2-z} + \frac{4}{\pi(p^2-z)} \left(\frac{p\cos(yp)}{\hat{u}(z)} \int_{\mathbb{R}} \frac{s\cos(ys)\psi(s)}{s^2-z} ds - \frac{p\sin(yp)}{\hat{v}(z)} \int_{\mathbb{R}} \frac{s\sin(ys)\psi(s)}{s^2-z} ds \right).$$

Here

$$\begin{aligned} \hat{u}(z) :=& \hat{u}(y;z) = 2\sqrt{-z}(e^{-2y\sqrt{-z}}+1) - (1+2y)e^{-2y} - 1, \\ \hat{v}(z) :=& \hat{v}(y;z) = 2\sqrt{-z}(e^{-2y\sqrt{-z}}-1) - (1+2y)e^{-2y} + 1. \end{aligned}$$

The condition $p^2 - z \neq 0$ for $z<0$ ensures that the resolvent of the operator $\mathbf{h}$ exists and is bounded within the domain $\mathbb{C} \setminus ([0,\infty) \cup \{z \in (-\infty,0) : \hat{u}(z) = 0 \,\text{or}\, \hat{v}(z) = 0\})$.

In light of Lemma 2.2, which states that $\hat{u}(z)$ and $\hat{v}(z)$ possess at most one simple zero, we conclude that $\sigma_d(h) = \{z \in (-\infty,0) : \hat{u}(z) = 0 \,\text{or}\, \hat{v}(z) = 0\}$ and $\sigma_{ess}(\mathbf{h}) = [0,\infty)$. □

The number $z, z<0$ is an eigenvalue of the operator $\mathbf{h}$ if and only if the number z is the zeros of the function $\hat{u}(z)$ or $\hat{v}(z)$.

Lemma 2.2. *For any $y \in (0,\infty)$, the functions $\hat{u}(z)$ and $\hat{v}(z)$ have only simple zeros in $(-\infty,0)$.*

Proof. First, we show the monotonicity of the functions $\hat{u}(z)$ and $\hat{v}(z)$. Evaluating the derivatives, we obtain:

$$\hat{u}'(z) = \frac{1-(1-2y\sqrt{-z})e^{-2y\sqrt{-z}}}{\sqrt{-z}},\ \hat{v}'(z) = -\frac{1+e^{-2y\sqrt{-z}}(1-2y\sqrt{-z})}{\sqrt{-z}}.$$

Note that for all $y \neq 0$

$$e^{2y} > 1+2y. \tag{2.4}$$

According to (2.4), the function $\hat{u}(z)$ is strictly increasing and the function $\hat{v}(z)$ is strictly decreasing.

Next, to examine the zeros of the functions $\hat{u}(z)$ and $\hat{v}(z)$, we establish the following limits:

$$\begin{aligned} \hat{u}(-0) &= \lim_{z\nearrow 0} \hat{u}(z) = -\left(1+(1+2y)e^{-2y}\right), \\ \hat{u}(-\infty) &= \lim_{z\to-\infty} \hat{u}(z) = +\infty \\ \hat{v}(-0) &= \lim_{z\nearrow 0} v(z) = 1-(1+2y)e^{-2y}, \\ \hat{v}(-\infty) &= \lim_{z\to-\infty} \hat{v}(z) = -\infty. \end{aligned}$$

Thus, for any $y>0$, the inequality $\hat{u}(-0) < 0 < \hat{u}(-\infty)$ holds. Additionally, since $\hat{u}$ is strictly increasing, for each y there exists a value $z_u(y)$ such that $\hat{u}(z_u(y)) = 0$. Similarly, for each y, we can show that $\hat{v}(z)$ $(-\infty,0)$ has a unique simple zero $z_v(y)$ in the interval.

Moreover, the following graph illustrates the behavior of the functions $\hat{u}(z)$ and $\hat{v}(z)$ for a fixed value of y.

□

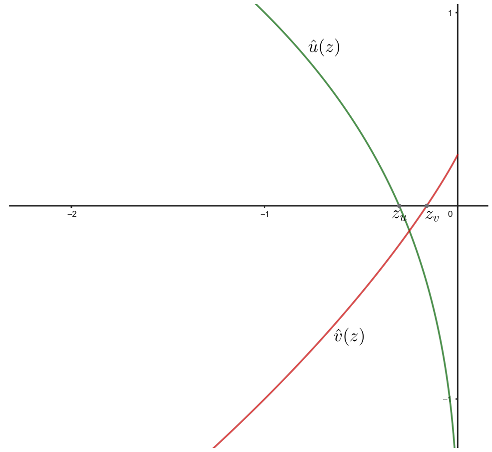

FIGURE 1. Graphs of the functions $\hat{u}(z)$ and $\hat{v}(z)$.

Theorem 2.3. *For any $y \in (0, \infty)$, the operator $\mathbf{h}$ has exactly two negative eigenvalues, z_u and z_v, corresponding to the eigenfunctions (up to a constant factor) of the form*

$$g_1(y;p) = \frac{p\cos(yp)}{p^2 - z_u} \quad \text{and} \quad g_2(y;p) = \frac{p\sin(yp)}{p^2 - z_v},$$

where z_u and z_v are zeros of the functions $\hat{u}(z)$ and $\hat{v}(z)$, respectively.

Proof. The proof of the existence of the eigenvalues of $\mathbf{h}$ follows from Lemma 2.2.

Now we prove that the eigenfunctions of the operator $\mathbf{h}$ have the form

$$g_1(p) = \frac{p\cos(yp)}{p^2 - z_1(y)} \quad \text{and} \quad g_2(p) = \frac{p\sin(yp)}{p^2 - z_2(y)}.$$

From equation $(\mathbf{h} - zI)g(p) = 0$ we receive

$$g(p) = \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2 - z}. \tag{2.5}$$

Comparing (1.7) and (2.5) we take the equality

$$f(p) + \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2 + 1} + \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{(p^2+1)^2} = \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2 - z}.$$

Multiplying by the factors $pe^{\pm iyp}$, we have

$$f(p)pe^{iyp} + \frac{\hat{c}_1 p^2 e^{2iyp} + \hat{c}_2 p^2}{p^2+1} + \frac{\hat{c}_1 p^2 e^{2iyp} + \hat{c}_2 p^2}{(p^2+1)^2} = \frac{\hat{c}_1 p^2 e^{2iyp} + \hat{c}_2 p^2}{p^2 - z},$$
$$f(p)pe^{-iyp} + \frac{\hat{c}_1 p^2 + \hat{c}_2 p^2 e^{-2iyp}}{p^2+1} + \frac{\hat{c}_1 p^2 + \hat{c}_2 p^2 e^{-2iyp}}{(p^2+1)^2} = \frac{\hat{c}_1 p^2 + \hat{c}_2 p^2 e^{-2iyp}}{p^2 - z}.$$

Integrating the last equalities over $\mathbb{R}$ we obtain a system of equations for determining $\hat{c}_1$ and $\hat{c}_2$,

$$\begin{cases} \hat{a}(z)\hat{c}_1 + \hat{b}(z)\hat{c}_2 = 0, \\ \hat{b}(z)\hat{c}_1 + \hat{a}(z)\hat{c}_2 = 0, \end{cases}$$

where $\hat{a}(z)$ and $\hat{b}(z)$ are defined as in equation (2.3). Hence

$$\left(\hat{a}^2(z) - \hat{b}^2(z)\right) c_i = 0, \quad i = 1, 2.$$

If $\hat{a}(z) = \hat{b}(z)$ (resp. $\hat{a}(z) = -\hat{b}(z)$), then as c_i we can take any number, in particular, $c_i = 1$.

Thus, if the number z satisfies the equation $\hat{u}(z) = \hat{a}(z) + \hat{b}(z) = 0$ (resp. $\hat{v}(z) = \hat{a}(z) - \hat{b}(z) = 0$), then z is an eigenvalue of the operator $\mathbf{h}$ and of the functions

$$g_1(p) = \frac{p\cos(yp)}{p^2 - z} \quad \left(\text{resp.} \quad g_2(p) = \frac{p\sin(yp)}{p^2 - z}\right)$$

corresponding eigenfunctions of the operator $\mathbf{h}$. □

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