Essential and discrete spectrum of the Schrödinger operator of a system of two particles on a lattice

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Abstract. We consider the Hamiltonian of a system of two fermions on a two-dimensional lattice $\mathbb{Z}^2$ with a certain type potential. It is proved that the subspace of odd functions $L_2^o(\mathbb{T}^2)$ is represented as a direct sum of the subspaces $L_2^{eo}(\mathbb{T}^2)$ and $L_2^{oe}(\mathbb{T}^2)$, which are invariant under the operator $H(\mathbf{k})$, $\mathbf{k} = (k_1, k_2) \in \mathbb{T}^2$, associated with this Hamiltonian. For any $k_1 \in (-\pi, \pi]$, it is proved that the operator $H^{eo}(k_1, \pi) = H(k_1, \pi)\big|_{L_2^{eo}(\mathbb{T}^2)}$ has an infinite number of eigenvalues and for any $k_1 \in (-\pi, \pi)$, the operator $H^{oe}(k_1, \pi) = H(k_1, \pi)\big|_{L_2^{oe}(\mathbb{T}^2)}$ has a finite number eigenvalues lying to the left of the essential spectrum. An asymptotic formula is obtained for the number of eigenvalues of the operator $H^{oe}(k_1, \pi)$ as $k_1 \to \pi$.

Keywords: Schrödinger operator, lattice, fermion, quasi-momentum, invariant subspaces, essential spectrum, eigenvalue

1. Introduction and formulation of main results

The discrete spectrum of the two-particle continuous Schrödinger operator $h_\lambda = -\Delta + \lambda V$ has been studied with various assumptions imposed on the potential V. Conditions ensuring the finiteness of the negative spectrum and the absence of positive eigenvalues of h_λ are presented in [1]. When $V \leq 0$, the number of negative eigenvalues $N(\lambda)$ is a non-decreasing function of $\lambda \in (0, \infty)$, and each eigenvalue $z_n(\lambda)$ is monotonically decreasing on $(0, \infty)$. As the coupling constant λ decreases, the bound-state energies of h_λ approach the edge of the continuous spectrum [1], and for certain finite values of λ, they may lie exactly on the boundary. These considerations naturally lead to two questions. Does a threshold state correspond to a bound or virtual state, meaning, is its wave function square-integrable? And as λ decreases further, where do the bound states "disappear to"? The first of these questions has been studied in [2], [3], and [4].

The Hamiltonian of a two-particle system on a lattice is expanded in the momentum representation into the following direct von Neumann integral [5]:

$$H \simeq \int_{\mathbf{k} \in \mathbb{T}^2} \oplus H(\mathbf{k}) \, d\mathbf{k},$$

where $\mathbb{T}^2$ is a two-dimensional torus.

It turns out that the spectrum of the fiber operator is $H(\mathbf{k})$ quite sensitive to variations in the quasi-momentum $\mathbf{k} \in \mathbb{T}^d$. The two-particle Schrödinger operator $H(\mathbf{k})$, $\mathbf{k} \in \mathbb{T}^3$, corresponding to the Hamiltonian of a two-particle system on the three-dimensional lattice $\mathbb{Z}^3$, was considered in [6]. It was shown that the function $N(\mathbf{k}) = N(k_1, k_2, k_3)$, representing the number of eigenvalues lying below the essential spectrum of $H(\mathbf{k})$, is non-decreasing with respect to each component $k_i \in [0, \pi]$, for $i = 1, 2, 3$.

In the work [7], the spectral properties of the two-particle Schrödinger operator $H(\mathbf{k})$, acting in the ν-dimensional lattice space $\mathbb{Z}^\nu$, were studied. It was shown that the operator has only a finite number of negative eigenvalues under rather general assumptions on the interaction potential $\hat{v}$.

In [8], was considered the Hamiltonian of a two-particle bosonic system on the two-dimensional lattice $\mathbb{Z}^2$ under a specific type of interaction potential. The associated Schrödinger operator $H(\mathbf{k})$, with total quasi-momentum $\mathbf{k} \in \mathbb{T}^2$, was shown to possess an infinite number of eigenvalues when $\mathbf{k} = \boldsymbol{\pi} = (\pi, \pi)$. It has been established that the eigenvalue $z_0(\pi) = 4 - \bar{v}(0)$ is non-degenerate, $z_1(\pi) = 4 - \bar{v}(1)$ has multiplicity two, $z_2(\pi) = 4 - \bar{v}(2)$ appears with multiplicity four, while for all $n \geq 3$, the eigenvalues $z_n(\pi) = 4 - \bar{v}(n)$ each have multiplicity five. It was also shown that each multiple eigenvalue of the operator $H(\pi)$ becomes a simple eigenvalue under perturbation. In addition,

asymptotic expressions for the eigenvalues of $H(\pi - 2\beta, \pi)$ have been obtained with an accuracy up to terms of order β^2.

Let $\mathbb{Z}^2$ be the two-dimensional lattice, and let $(\mathbb{Z}^2)^2$ denote the Cartesian product of $\mathbb{Z}^2$. We denote by $\ell_2((\mathbb{Z}^2)^2)$ the Hilbert space of square-summable functions defined on $(\mathbb{Z}^2)^2$, and by $\ell_2^{as}((\mathbb{Z}^2)^2) \subset \ell_2((\mathbb{Z}^2)^2)$ denote the subspace consisting of antisymmetric functions.

The free Hamiltonian $\hat{H}_0$ for a pair of fermions with mass equal to one on the two-dimensional lattice $\mathbb{Z}^2$ acts as a bounded, self-adjoint operator in the space $\ell_2^{as}((\mathbb{Z}^2)^2)$, and is explicitly defined by

$$\hat{H}_0 = -\frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2.$$

Here $\Delta_1 = \Delta \otimes I$ and $\Delta_2 = I \otimes \Delta$, where I is the identity operator, the lattice Laplacian Δ is a difference operator describing the transfer of a particle from one site to a nearest-neighbor site:

$$(\Delta\hat{\psi})(\mathbf{x}) = \sum_{j=1}^{2}\left[\hat{\psi}(\mathbf{x}+\mathbf{e}_j) + \hat{\psi}(\mathbf{x}-\mathbf{e}_j) - 2\hat{\psi}(\mathbf{x})\right], \quad \mathbf{x} \in \mathbb{Z}^2, \quad \hat{\psi} \in \ell_2(\mathbb{Z}^2),$$

where the vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ form the standard basis in $\mathbb{Z}^2$.

The total Hamiltonian $\hat{H}$ of the two-fermion system acts in the $\ell_2^{as}((\mathbb{Z}^2)^2)$ and is defined as the difference between the free Hamiltonian $\hat{H}_0$ and the interaction operator $\hat{V}$, as described in [9]:

$$\hat{H} = \hat{H}_0 - \hat{V}.$$

The interaction potential $\hat{V}$ is a multiplication operator acting pointwise as

$$(\hat{V}\hat{\psi})(\mathbf{x},\mathbf{y}) = \hat{v}(\mathbf{x}-\mathbf{y})\hat{\psi}(\mathbf{x},\mathbf{y}), \quad \hat{\psi} \in \ell_2^{as}((\mathbb{Z}^2)^2),$$

where $\hat{v}$ is a potential depending on the relative position of the two particles.

Let us assume that the function $\hat{v}(\mathbf{n})$ is defined as:

$$\hat{v}(\mathbf{n}) = \hat{v}(n_1, n_2) = \begin{cases} 10^{-|\mathbf{n}|}, & \text{if} \quad |n_1| \leq 1 \\ 0, & \text{if} \quad |n_1| \geq 2, \end{cases} \tag{1.1}$$

where $|\mathbf{n}| = |n_1| + |n_2|$.

Let $\mathbb{T}^2$ be a two-dimensional torus and $L_2(\mathbb{T}^2 \times \mathbb{T}^2)$ be the Hilbert space of square-integrable functions defined on $\mathbb{T}^2 \times \mathbb{T}^2$, $L_2^{as}(\mathbb{T}^2 \times \mathbb{T}^2) \subset L_2(\mathbb{T}^2 \times \mathbb{T}^2)$ be the subspace of antisymmetric functions with respect to a permutation of variables. Let $F : \ell_2(\mathbb{Z}^2 \times \mathbb{Z}^2) \to L_2(\mathbb{T}^2 \times \mathbb{T}^2)$ be the standard Fourier transform. Let us denote by $\hat{F} : \ell_2^{as}(\mathbb{Z}^2 \times \mathbb{Z}^2) \to L_2^{as}(\mathbb{T}^2 \times \mathbb{T}^2)$ the restriction F in $\ell_2^{as}(\mathbb{Z}^2 \times \mathbb{Z}^2)$. The Hamiltonian $H = H_0 - V = \hat{F}\hat{H}\hat{F}^{-1}$ in the momentum representation commutes with the unitary operators $U_{\mathbf{s}}, \quad \mathbf{s} \in \mathbb{Z}^2$:

$$(U_{\mathbf{s}}f)(\mathbf{k}_1,\mathbf{k}_2) = e^{-i(\mathbf{s},\mathbf{k}_1+\mathbf{k}_2)}f(\mathbf{k}_1,\mathbf{k}_2), \quad f \in L_2^{as}((\mathbb{T}^2)^2).$$

This implies [1] that there are decompositions of the space $L_2^{as}(\mathbb{T}^2 \times \mathbb{T}^2)$ and the operators $U_{\mathbf{s}}$, and H into direct integrals:

$$L_2^{as}((\mathbb{T}^2)^2) = \int_{\mathbb{T}^2} \oplus L_2(F_{\mathbf{k}})\,d\mathbf{k}, \quad U_{\mathbf{s}} = \int_{\mathbb{T}^2} \oplus U_{\mathbf{s}}(\mathbf{k})\,d\mathbf{k}, \quad H = \int_{\mathbb{T}^2} \oplus \tilde{H}(\mathbf{k})\,d\mathbf{k},$$

where

$$F_{\mathbf{k}} = \{(\mathbf{k}_1,\mathbf{k}_2) \in (\mathbb{T}^2)^2 : \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}\}, \quad \mathbf{k} = (k_1, k_2) \in \mathbb{T}^2.$$

The fiber operator $\tilde{H}(\mathbf{k})$, associated with H, acts in the space $L_2(F_{\mathbf{k}})$ and is unitarily equivalent to the two-particle discrete Schrödinger operator $H(\mathbf{k}) := H_0(\mathbf{k}) - V$ that acts in the Hilbert space

$$L_2^o(\mathbb{T}^2) := \left\{f \in L_2(\mathbb{T}^2) : f(-\mathbf{q}) = -f(\mathbf{q})\right\}.$$

The unperturbed operator $H_0(\mathbf{k})$ is an operator of multiplication by the function

$$\varepsilon_{\mathbf{k}}(\mathbf{q}) = \varepsilon\left(\frac{\mathbf{k}}{2} + \mathbf{q}\right) + \varepsilon\left(\frac{\mathbf{k}}{2} - \mathbf{q}\right) = 4 - 2\cos\frac{k_1}{2}\cos q_1 - 2\cos\frac{k_2}{2}\cos q_2, \tag{1.2}$$

$$\varepsilon(\mathbf{q}) = \sum_{i=1}^{2}(1 - \cos q_i), \quad \mathbf{k} = (k_1, k_2) \in \mathbb{T}^2.$$

The integral operator V is defined by the kernel $\frac{1}{2\pi}v(\mathbf{q} - \mathbf{s})$, which admits the representation

$$v(\mathbf{q} - \mathbf{s}) = \frac{1}{2\pi}\sum_{\mathbf{n}\in\mathbb{Z}^2}\hat{v}(\mathbf{n})e^{i(\mathbf{q}-\mathbf{s},\mathbf{n})},$$

where the potential function $\hat{v}$ is defined by the equality (1.1).

Note that due to the specific form of $\hat{v}(\mathbf{n})$, the kernel of the integral operator V can be expressed in a separable form as $v(\mathbf{q}) = v_1(q_1)v_2(q_2)$, where the component functions are given by

$$v_1(q_1) = \frac{1}{\sqrt{2\pi}}\{1 + \frac{2}{10}\cos q_1\}, \quad v_2(q_2) = \frac{1}{\sqrt{2\pi}}\{1 + 2\sum_{m=1}^{\infty}10^{-m}\cos mq_2\}. \tag{1.3}$$

Let $L_2^{oe}(\mathbb{T}^2) = L_2^o(\mathbb{T}) \otimes L_2^e(\mathbb{T})$ and $L_2^{eo}(\mathbb{T}^2) = L_2^e(\mathbb{T}) \otimes L_2^o(\mathbb{T})$, where $L_2^o(\mathbb{T})$ and $L_2^e(\mathbb{T})$ denote the spaces of odd and even functions, respectively (see [10]). Then the space $L_2^o(\mathbb{T}^2)$ can be express as a direct sum $L_2^o(\mathbb{T}^2) = L_2^{oe}(\mathbb{T}^2) \oplus L_2^{eo}(\mathbb{T}^2)$.

Observe that the subspace $L_2^{eo}(\mathbb{T}^2)$ and $L_2^{oe}(\mathbb{T}^2)$ are invariant with respect to the operator $H(\mathbf{k})$ (see Lemma 3.1). By $H^{eo}(\mathbf{k})$ and $H^{oe}(\mathbf{k})$ we denote the restrictions of the operator $H(\mathbf{k})$ in the subspaces $L_2^{eo}(\mathbb{T}^2)$ and $L_2^{oe}(\mathbb{T}^2)$, respectively.

Theorem 1.1. *For any $k_1 \in (-\pi, \pi]$ the operator $H^{eo}(k_1, \pi)$ has an infinite number of eigenvalues lying to the left of the essential spectrum.*

Let $\mathcal{N}(k_1)$ be the number of the eigenvalues of the operator $H^{oe}(k_1, \pi)$ lying to the left of the essential spectrum. Then the following statement holds for the number $\mathcal{N}(k_1)$ as $k_1 \to \pi$:

Theorem 1.2. *For any $k_1 \in (-\pi, \pi)$, the operator $H^{oe}(k_1, \pi)$ has a finite number of eigenvalues lying to the left of the essential spectrum. The number of eigenvalues $\mathcal{N}(k_1)$ increases as $k_1 \to \pi$ and the following asymptotic formula holds*

$$\lim_{k_1\to\pi}\frac{\mathcal{N}(k_1)}{|\lg\cos\frac{k_1}{2}|} = 1. \tag{1.4}$$

2. Essential spectrum of the operator $H(\mathbf{k})$

As is known, the spectrum of the operator $H_0(\mathbf{k})$ consists of the range of values of the function $\varepsilon_{\mathbf{k}}$:

$$\sigma(H_0(\mathbf{k})) = [m(\mathbf{k}), M(\mathbf{k})],$$

where

$$m(\mathbf{k}) = \min_{\mathbf{q}\in\mathbb{T}^2}\varepsilon_{\mathbf{k}}(\mathbf{q}) = \varepsilon_{\mathbf{k}}(\mathbf{0}) = 2\varepsilon(\frac{\mathbf{k}}{2}), \quad M(\mathbf{k}) = \max_{\mathbf{q}\in\mathbb{T}^2}\varepsilon_{\mathbf{k}}(\mathbf{q}) = \varepsilon_{\mathbf{k}}(\boldsymbol{\pi}) = 8 - 2\varepsilon(\frac{\mathbf{k}}{2}).$$

The spectrum of the operator V consists of the set $\{0, \frac{1}{10^n}, n \in \mathbb{N}\}$ where $\frac{1}{10^n}$ is the eigenvalue of the operator V. Condition (1.1) implies that the operator V is Hilbert-Schmidt. By Weyl's theorem, since V is compact, the essential spectrum of $H(\mathbf{k})$ coincides with the spectrum of the operator $H_0(\mathbf{k})$, (see [1]), i.e.

$$\sigma_{ess}(H(\mathbf{k})) = [m(\mathbf{k}), M(\mathbf{k})].$$

We define $w(\mathbf{k})$ as the width of the essential spectrum of $H(\mathbf{k})$. Accordingly, we obtain:

$$w(\mathbf{k}) = M(\mathbf{k}) - m(\mathbf{k}) = 8 - 4\varepsilon(\frac{\mathbf{k}}{2}) = 4\cos\frac{k_1}{2} + 4\cos\frac{k_2}{2} \tag{2.1}$$

and

$$\min_{\mathbf{k}\in\mathbb{T}^2} w(\mathbf{k}) = w(\boldsymbol{\pi}) = 0, \qquad \max_{\mathbf{k}\in\mathbb{T}^2} w(\mathbf{k}) = w(\mathbf{0}) = 8,$$

where $\boldsymbol{\pi} = (\pi, \pi)$, $\mathbf{0} = (0, 0)$.

It follows from (2.1) that if $k_j \in [0, \pi]$, $j = 1, 2$, increases, then the width of the essential spectrum $w(\mathbf{k})$ decreases.

Let us determine the width of the essential spectrum $H(\mathbf{k})$ in the direction $\mathbf{e}_j, j = 1, 2$, as:

$$w_j(\mathbf{k}) = \max_{p_j\in[-\pi,\pi]} \varepsilon_{\mathbf{k}}(\mathbf{p}) - \min_{p_j\in[-\pi,\pi]} \varepsilon_{\mathbf{k}}(\mathbf{p}) = 4\cos\frac{k_j}{2}, \, j = 1, 2.$$

Then we have

$$w(\mathbf{k}) = w_1(\mathbf{k}) + w_2(\mathbf{k}).$$

If $\mathbf{k} = \boldsymbol{\pi}$, then the essential spectrum is concentrates at the point $\{4\}$, i.e. $w(\boldsymbol{\pi}) = 0$. The spectrum of the operator $H(\boldsymbol{\pi}) = 4I - V$ consists of eigenvalues of the form $4 - \frac{1}{10^n}$, $n \in \mathbb{N}$. The narrowing of the essential spectrum results in an increased number of eigenvalues for the Schrödinger operator $H(\mathbf{k})$.

From the self-adjointness of the operator $H(\mathbf{k}) = H_0(\mathbf{k}) - V$ and the positivity of V it follows that

$$\sigma(H(\mathbf{k})) \cap (M(\mathbf{k}), \infty) = \emptyset$$

i.e. $\sigma_{disc}(H(\mathbf{k})) \subset (-\infty, m(\mathbf{k}))$.

3. Invariant subspaces under the operator $H(\mathbf{k})$

We state the lemma on the invariant subspaces with respect to the operator $H(\mathbf{k})$.

Lemma 3.1. *The subspaces $L_2^{eo}(\mathbb{T}^2)$ and $L_2^{oe}(\mathbb{T}^2)$ invariant under the operator $H(\mathbf{k})$.*

Proof. Let us prove the invariance of the subspace $L_2^{eo}(\mathbb{T}^2)$ with respect to $H(\mathbf{k})$. From (1) it is clear that the function $\varepsilon_{\mathbf{k}}$ belongs to $L_2^{ee}(\mathbb{T}^2)$, where $L_2^{ee}(\mathbb{T}^2) = L_2^e(\mathbb{T}) \otimes L_2^e(\mathbb{T})$. From here we conclude that if $f \in L_2^{eo}(\mathbb{T}^2)$, then $\varepsilon_{\mathbf{k}}f \in L_2^{eo}(\mathbb{T}^2)$. This relation proves the invariance of the subspace $L_2^{eo}(\mathbb{T}^2)$ under the operator $H_0(\mathbf{k})$. According to condition (1.1), the kernel $v(p_1, p_2)$ of the operator V belongs to the subspace $L_2^{ee}(\mathbb{T}^2)$. It follows that $g = Vf \in L_2^{eo}(\mathbb{T}^2)$ for any $f \in L_2^{eo}(\mathbb{T}^2)$, i.e.:

$$(Vf)(p_1, p_2) = \frac{1}{2\pi} \int_{\mathbb{T}^2} v(p_1 - s_1, p_2 - s_2) f(s_1, s_2) ds_1 ds_2 \in L_2^{eo}(\mathbb{T}^2).$$

From the above relations, we obtain the invariance of the subspace $L_2^{eo}(\mathbb{T}^2)$ under the operator $H(\mathbf{k}) = H_0(\mathbf{k}) - V$.

According to (1.1) and (1.3), the operator $V^{eo} = V|_{L_2^{eo}(\mathbb{T}^2)}$ has following explicit form:

$$(V^{eo}f)(\mathbf{p}) = \frac{1}{2\pi^2} \int_{\mathbb{T}^2} \{\frac{1}{10}\sin p_2 \sin q_2 + 2\sum_{n=2}^{\infty} \frac{1}{10^n}[\sin np_2 \sin nq_2 +$$

$$+2\cos p_1 \sin(n-1)p_2 \cos q_1 \sin(n-1)q_2]\}f(\mathbf{q})d\mathbf{q}, \quad f \in L_2^{eo}(\mathbb{T}^2).$$

The invariance of the subspace $L_2^{oe}(\mathbb{T}^2)$ with respect to the operator $H(\mathbf{k})$ is also proved in this way, there operator $V^{oe} = V|_{L_2^{oe}(\mathbb{T}^2)}$ acts to $f \in L_2^{oe}(\mathbb{T}^2)$ as:

$$(V^{oe}f)(\mathbf{p}) = \frac{1}{2\pi^2} \int_{\mathbb{T}^2} \left\{\frac{1}{10}\sin p_1 \sin q_1 + 2\sum_{n=2}^{\infty} \frac{1}{10^n}[\sin p_1 \cos(n-1)p_2 \sin q_1 \cos(n-1)q_2]\right\}f(\mathbf{q})\,d\mathbf{q}.$$

$\square$

Note that the systems $\{\psi_n^o(q) = \frac{1}{\sqrt{\pi}}\sin nq\}_{n\in\mathbb{N}}$ and $\psi_0^e(q) = \frac{1}{\sqrt{2\pi}}$, $\{\psi_n^e(q) = \frac{1}{\sqrt{\pi}}\cos nq\}_{n\in\mathbb{N}}$ are an orthonormal bases in the subspaces $L_2^o(\mathbb{T})$ and $L_2^e(\mathbb{T})$, respectively. Let $L^o(n)$ and $L^e(n)$ be one-dimensional subspaces spanned by the vectors $\{\psi_n^o(q)\}, n \in \mathbb{N}$ and $\psi_0^e(q) = \frac{1}{\sqrt{2\pi}}$, $\{\psi_n^e(q) = \frac{1}{\sqrt{\pi}}\cos nq\}_{n\in\mathbb{N}}$. Then we have the following equalities

$$L_2^o(\mathbb{T}) = \sum_{n=1}^{\infty} \oplus L^o(n), \quad L_2^e(\mathbb{T}) = \sum_{n=0}^{\infty} \oplus L^e(n)$$

From here we obtain

$$L_2^e(\mathbb{T}) \otimes L_2^o(\mathbb{T}) = \sum_{n=1}^{\infty} \oplus\{L_2^e(\mathbb{T}) \otimes L^o(n)\} = \sum_{n=1}^{\infty} \oplus \mathfrak{B}_n^{eo},$$

$$L_2^o(\mathbb{T}) \otimes L_2^e(\mathbb{T}) = \sum_{n=0}^{\infty} \oplus\{L_2^o(\mathbb{T}) \otimes L^e(n)\} = \sum_{n=0}^{\infty} \oplus \mathfrak{B}_n^{oe},$$

where $\mathfrak{B}_n^{eo} := L_2^e(\mathbb{T}) \otimes L^o(n)$ and $\mathfrak{B}_n^{oe} := L_2^o(\mathbb{T}) \otimes L^e o(n)$.

Lemma 3.2. *For each $n \in \mathbb{N}$, the subspace $\mathfrak{B}_n^{eo}$ is invariant under the operator $H^{eo}(k_1, \pi)$.*

Proof. We choose an arbitrary element of $\mathfrak{B}_n^{eo}$ in the form $(f\psi_n^o)(p_1, p_2) := f(p_1)\psi_n^o(p_2)$, where $f \in L_2^e(\mathbb{T})$, then the action of $H^{eo}(k_1, \pi) = H_0(k_1, \pi) - V^{eo}$ on the space $\mathfrak{B}_n^{eo}$ is as follows:

$$(H_0(k_1,\pi)f\psi_n^o)(p_1,p_2) = \left[\varepsilon_{k_1}(p_1)f(p_1)\right]\psi_n^o(p_2) \in \mathfrak{B}_n^{eo}, \tag{3.1}$$

here $\varepsilon_{k_1}(p) = 4 - 2\cos\frac{k_1}{2}\cos p$,

$$(V^{eo}f\psi_n^o)(p_1,p_2) = \frac{1}{2\pi^2}\int_{\mathbb{T}^2}\left\{\frac{1}{10}\sin p_2\sin q_2 + \right.$$

$$\left. + \sum_{n=2}^{\infty}\frac{1}{10^n}[\sin np_2\sin nq_2 + 2\cos p_1\sin(n-1)p_2\cos q_1\sin(n-1)q_2]\right\}(f\psi_n^o)(q_1,q_2)dq_1dq_2 =$$

$$= \left[\frac{1}{2\pi}\int_{\mathbb{T}}(\frac{1}{10^n} + \frac{2}{10^{n+1}}\cos p_1\cos q_1)f(q_1)dq_1\right]\psi_n^o(p_2) \in \mathfrak{B}_n^{eo}. \tag{3.2}$$

In obtaining (3.2) we used the orthogonality of the system $\{\psi_n^o\}_{n\in\mathbb{N}}$. According to (3.1) and (3.2) we obtain the relation

$$(H^{eo}(k_1,\pi)f\psi_n^o)(p_1,p_2) = \left[\varepsilon_{k_1}(p_1)f(p_1) - \frac{1}{2\pi}\int_{\mathbb{T}}\left\{\frac{1}{10^n} + \frac{2}{10^{n+1}}\cos p_1\cos q_1\right\}f(q_1)dq_1\right]\psi_n^e(p_2) \in \mathfrak{B}_n^{eo}$$

which proves the lemma. $\qquad\qquad\square$

From expressions (3.1) and (3.2) it can be seen that the restriction $H_n^{eo}(k_1,\pi)$ of the operator $H^{eo}(k_1,\pi)$ in the subspace $\mathfrak{B}_n^{eo} := L_2^e(\mathbb{T}) \otimes L^o(n)$ is of the form:

$$H_n^{eo}(k_1,\pi) = [H_0(k_1) - V_n^e] \otimes I_n, \tag{3.3}$$

where I_n is the identity operator in $L^o(n)$, and $H_n^e(k_1) := H_0(k_1) - V_n^e$ acts in the space $L_2^e(\mathbb{T})$ as follows:

$$(H_n^e(k_1)f)(p) = \varepsilon_{k_1}(p)f(p) - \frac{1}{2\pi}\int_{\mathbb{T}}\left\{\frac{1}{10^n} + \frac{2}{10^{n+1}}\cos p\cos q\right\}f(q)dq. \tag{3.4}$$

Lemma 3.3. *For each $n \in \mathbb{Z}_+$, the subspace $\mathfrak{B}_n^{oe}$ is invariant under the operator $H^{oe}(k_1, \pi)$.*

Proof. We choose an arbitrary element of $\mathfrak{B}_n^{oe}$ in the form $(f\,\psi_n^e)(p_1,p_2) := f(p_1)\,\psi_n^e(p_2)$, where $f \in L_2^o(\mathbb{T})$, then the action of $H^{oe}(k_1,\pi) = H_0(k_1,\pi) - V^{oe}$ on $\mathfrak{B}_n^{oe}$ is as

$$(H_0(k_1,\pi)f\psi_n^e)(p_1,p_2) = \left[\varepsilon_{k_1}(p_1)f(p_1)\right]\psi_n^e(p_2) \in \mathfrak{B}_n^{oe}, \tag{3.5}$$

$$(V^{oe}f\psi_n^e)(p_1,p_2) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\left\{4\sum_{m=2}^{\infty}\frac{1}{10^m}\sin p_1\cos(m-1)p_2\sin q_1\cos(m-1)q_2\right\}(f\psi_n^e)(q_1,q_2)\,dq_1dq_2 =$$

$$= \left[\frac{1}{\pi}\int_{\mathbb{T}}\frac{1}{10^{n+1}}\sin p_1\sin q_1 f(q_1)dq_1\right]\psi_n^e(p_2) \in \mathfrak{B}_n^{oe}. \tag{3.6}$$

(3.6) is obtained based on the orthogonality of the system $\{\psi_n^e\}_{n\in\mathbb{Z}_+}$. $\qquad\square$

From (3.5) and (3.6), it can be seen that the restriction of the operator $H^{oe}(k_1,\pi)$ in the subspace $\mathfrak{B}_n^{oe} := L_2^o(\mathbb{T}) \otimes L^e(n)$ has the form

$$H_n^{oe}(k_1,\pi) = [H_0(k_1) - V_n^o] \otimes I_n, \tag{3.7}$$

where I_n is the identity operator in $L^e(n)$. Then the action of $H_n^o(k_1) := H_0(k_1) - V_n^o$ on $L_2^o(\mathbb{T})$:

$$(H_n^o(k_1)f)(p) = \varepsilon_{k_1}(p)f(p) - \frac{1}{\pi}\int_{\mathbb{T}}\frac{1}{10^{n+1}}\sin p\sin q f(q)dq. \tag{3.8}$$

Thus, according to (3.3) and (3.7), we have the following representation for the operators $H^{eo}(k_1,\pi)$ and $H^{oe}(k_1,\pi)$:

$$H^{eo}(k_1,\pi) = \sum_{n=1}^{\infty}\oplus H_n^{eo}(k_1,\pi), \quad H^{oe}(k_1,\pi) = \sum_{n=0}^{\infty}\oplus H_n^{oe}(k_1,\pi). \tag{3.9}$$

4. On the spectrum of the operator $H_n^e(k_1)$

This section is devoted to the analysis of the discrete spectrum of the operator $H^{eo}(k_1,\pi)$. According to (3.3) and (3.4) it is sufficient to study the eigenvalues of operator $H_n^e(k_1)$, $n \in \mathbb{N}$ defined by (3.9). It should be noted that the width $w(k_1) := w(k_1,\pi)$ of the essential spectrum of the operator $H_n^e(k_1)$ is independent of n and is given by

$$w(k_1) = w_1(k_1) = 4\cos\frac{k_1}{2}.$$

It is known that the study of the discrete spectrum of the operator $H_n^e(k_1)$ lying to the left of the essential spectrum reduces to the study of the eigenvalues of the self-adjoint, compact and positive operator $T_n^e(k_1,z) = r_0^{\frac{1}{2}}(k_1,z)V_n^e r_0^{\frac{1}{2}}(k_1,z)$, $z \in (-\infty, m(k_1))$, where $r_0(k_1,z)$ is the resolvent of the unperturbed operator $H_0(k_1)$ (see [7]). The operator $T_n^e(k_1,z)$ acts on the space $L_2^e(\mathbb{T})$ according to the following formula:

$$(T_n^e(k_1,z)g)(p) = \frac{1}{2\pi}\int_{\mathbb{T}}\frac{[10^{-n}+2\cdot 10^{-n-1}\cos p\cos q]g(q)dq}{\sqrt{\varepsilon_{k_1}(p)-z}\sqrt{\varepsilon_{k_1}(q)-z}}, \quad z \in (-\infty, m(k_1)).$$

By definition of the operator $T_n^e(k_1,z)$, it has rank two for each $n \in \mathbb{N}$.

Lemma 4.1. *The number $z \in (-\infty, m(k_1))$, is an eigenvalue of the operator $H_n^e(k_1)$ if and only if 1 is the eigenvalue of the operator $T_n^e(k_1,z)$.*

The proof proceeds analogously to that of Theorem 1 in [5].

Let $\mathfrak{m}[\mu, B]$ denote the number of eigenvalues of the self-adjoint operator B that lie above μ, where $\mu > \sup\sigma_{ess}(B)$ and B acts in a Hilbert space $\mathfrak{H}$.

Lemma 4.2. *If $z < m(k_1)$, then the number of eigenvalues of $H_n^e(k_1)$ below z is equal to the number of eigenvalues of $T_n^e(k_1, z)$ that are greater than 1:*

$$\mathfrak{m}\left[1,\, T_n^e(k_1, z)\right] = \mathfrak{m}\left[-z,\, -H_n^e(k_1)\right].$$

The proof proceeds analogously to that of Lemma 2 in [5].

Theorem 4.3. *For each $n \in \mathbb{N}$ there exists at least one eigenvalue of the operator $H_n^e(k_1)$ lying to the left of the essential spectrum.*

Proof. By Lemma 4.2, we show that for some $z_0 < m(k_1)$ the operator $T_n^e(k_1, z)$ has at least one eigenvalue greater than 1.

Let $g_0(q) = \dfrac{\sqrt{C_0(z)}}{\sqrt{2\pi}\sqrt{\varepsilon_{k_1}(q)-z}} \in L_2^e(\mathbb{T})$ and $\|g_0\| = 1$, where $C_0(z)$ is the normalizing multiplier. Observe that the result below is valid for all $z < m(k_1)$:

$$C_0(z) = \left(\frac{1}{2\pi}\int_{\mathbb{T}}\frac{dq}{\varepsilon_{k_1}(q)-z}\right)^{-1} = \sqrt{(4-z)^2 - 4\cos^2\frac{k_1}{2}} \tag{4.1}$$

and

$$\lim_{z\to m(k_1)} C_0(z) = 0. \tag{4.2}$$

Consequently, we arrive at the following expression for the inner product $(T_n^e(k_1, z)g_0, g_0)$:

$$(T_n^e(k_1, z)g_0, g_0) = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} T_n^e(p, q; k_1, z)g_0(q)dq\overline{g_0(p)}dp =$$

$$= \frac{10^{-n}}{2\pi}\left|\int_{\mathbb{T}}\frac{g_0(q)dq}{\sqrt{\varepsilon_{k_1}(q)-z}}\right|^2 + \frac{10^{-n-1}}{\pi}\left|\int_{\mathbb{T}}\frac{\cos q\, g_0(q)dq}{\sqrt{\varepsilon_{k_1}(q)-z}}\right|^2.$$

Both terms of this sum are nonnegative, so according to (4.1) we have

$$(T_n^e(k_1, z)g_0, g_0) \geq \frac{10^{-n}C_0(z)}{(2\pi)^2}\int_{\mathbb{T}}\frac{dp}{\varepsilon_{k_1}(p)-z}\int_{\mathbb{T}}\frac{dq}{\varepsilon_{k_1}(q)-z} = \frac{10^{-n}}{C_0(z)}.$$

From here and from (4.2) there exists $z = z_0(n) < m(k_1)$ such that for any $z \in (z_0(n), m(k_1))$ the relation $\frac{10^{-n}}{C_0(z)} > 1$ holds. Then, according to the Birman-Schwinger principle (see Lemma 4.2), it follows that $\mathfrak{m}[-m(k_1), -H_n^e(k_1)] > 1$. $\qquad\square$

Proof of Theorem 1.1. The proof of this theorem follows immediately from Theorem 4.3 and (3.9).

5. On the spectrum of the operator $H_n^o(k_1)$

In this section we give some statements about the spectrum of the operator $H_n^o(k_1), n \in \mathbb{Z}_+$ defined by the formula (3.8), and then give results for the operator $H^{oe}(k_1, \pi)$ using the representations (3.7) and (3.9).

Theorem 5.1. *Let $n \in \mathbb{Z}_+$ and $k_1 \in (-\pi, \pi)$. a) If $\frac{1}{10^{n+1}} > \cos\frac{k_1}{2}$, then the operator $H_n^o(k_1)$ has a simple eigenvalue lying below the essential spectrum;*
b) if $\frac{1}{10^{n+1}} \leq \cos\frac{k_1}{2}$, then the operator $H_n^o(k_1)$ has no eigenvalues lying outside of essential spectrum.

Proof. a) Suppose that the equation

$$(H_n^o(k_1)f)(p) = zf(p) \tag{5.1}$$

has a nonzero solution $f \in L_2^o(\mathbb{T})$. We express this equation in the form

$$\left(\varepsilon_{k_1}(p) - z\right) f(p) = \frac{1}{\pi} \int_{\mathbb{T}} \frac{1}{10^{n+1}} \sin p \sin q f(q) dq.$$

Denoting

$$a = \frac{1}{\pi} \int_{\mathbb{T}} \sin q f(q) dq \tag{5.2}$$

we arrive at the following expression for the eigenfunction:

$$f(p) = \frac{10^{-n-1} a \sin p}{4 - z - 2 \cos \frac{k_1}{2} \cos p}. \tag{5.3}$$

From here, by setting (5.3) to (5.2) and after some simplifications, we obtain the following Fredholm determinant $\Delta(k_1, z)$ for the operator $H_n^o(k_1)$:

$$\Delta(k_1, z) = 1 - \frac{10^{-n-1}}{\pi} \int_{\mathbb{T}} \frac{\sin^2 q \, dq}{4 - z - 2 \cos \frac{k_1}{2} \cos q} \tag{5.4}$$

This function is a continuous and monotonically decreasing function of z on the interval $(-\infty, m(k_1))$. Hence we have the following relationship:

$$\lim_{z \to -\infty} \Delta(k_1, z) = 1, \quad \lim_{z \to m(k_1)} \Delta(k_1, z) = \Delta(k_1, m(k_1)) < 0$$

and

$$\Delta(k_1, m(k_1)) = 1 - \frac{10^{-n-1}}{2\pi \cos \frac{k_1}{2}} \int_{\mathbb{T}} \frac{\sin^2 q \, dq}{1 - \cos q} = 1 - \frac{10^{-n-1}}{\cos \frac{k_1}{2}}.$$

Since $10^{-n-1} > \cos \frac{k_1}{2}$, then the function $\Delta(k_1, z)$ has zero on the interval $(-\infty, m(k_1))$, i.e. the operator $H_n^o(k_1)$ has unique simple eigenvalue lying to the left of essential spectrum;
b) if $10^{-n-1} < \cos \frac{k_1}{2}$, then the determinant $\Delta(k_1, z)$ does not have zeros in the interval $(-\infty, m(k_1))$. If $10^{-n-1} = \cos \frac{k_1}{2}$, then the solution of the equation $(H_n^o(k_1)f)(p) = m(k_1)f(p)$ is $f(p) = \frac{\sin p}{1 - \cos p} \notin L_2^o(\mathbb{T})$. It follows that the number $z = m(k_1)$ is not an eigenvalue for the operator $H_n^o(k_1)$.

$\square$

Lemma 5.2. *Suppose that the condition a) of the theorem 5.1 holds. Then the operator $H_n^o(k_1)$ has a unique simple eigenvalue lying in a some neighborhood of the eigenvalue $z_n(\pi) = 4 - \frac{1}{10^{n+1}}$ of the operator $H(\pi)$ to the left of the essential spectrum:*

$$z_n(k_1) = 4 - \frac{1}{10^{n+1}} - 10^{n+1} \cos^2 \frac{k_1}{2}.$$

The proof of this and the following lemma is omitted due to its triviality.

Lemma 5.3. *Let $k_1 \in (-\pi, \pi)$. If for some number $n \in \mathbb{N}$ the inequality $\frac{1}{10^{n+2}} < \cos \frac{k_1}{2} < \frac{1}{10^{n+1}}$ holds, then the operator $H^{oe}(k_1, \pi)$ has exactly n eigenvalues to the left of the essential spectrum, i.e.*

$$\mathcal{N}(k_1) = n \tag{5.5}$$

Proof of theorem 1.2. Suppose that equality (5.5) holds, i.e. implies the relation

$$\frac{1}{10^{n+2}} < \cos \frac{k_1}{2} < \frac{1}{10^{n+1}}.$$

From here we have

$$\lg 10^{-n-2} < \lg \cos \frac{k_1}{2} < \lg 10^{-n-1} \text{ or } n < -1 - \lg \cos \frac{k_1}{2} < n + 1.$$

Considering (5.5), we deduce

$$1 + \frac{1}{\mathcal{N}(k_1)} < -\frac{\lg \cos \frac{k_1}{2}}{\mathcal{N}(k_1)} < 1 + \frac{2}{\mathcal{N}(k_1)}.$$

Thus, from here we obtain

$$\lim_{k_1 \to \pi} \frac{\mathcal{N}(k_1)}{|\lg \cos \frac{k_1}{2}|} = 1.$$

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