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Classification of three-dimensional complex Leibniz dialgebras with one-dimensional annihilator

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Abstract. This work focuses on classifying three-dimensional complex Leibniz dialgebras. We present a complete list of these algebras when the annihilator of the associated Leibniz algebra is one-dimensional.

Keywords: Associative algebra, Lie algebra, Leibniz algebra, dialgebra, Leibniz dialgebra, isomorphism, classification
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1. Introduction

In 1990s, Loday introduced the notion of Leibniz algebra, that is a generalization of Lie algebra, where the skew-symmetricity of the bracket is dropped and the Jacobi identity is replaced by the Leibniz identity (the identity has been called Leibniz identity by Loday due to its similarity to Leibniz rule, this is the reason for the class to be called by the name of Leibniz). In right (left) Leibniz algebras, the defining identity requires that each right (left) multiplication operator behaves as a derivation, i.e.,

$$[[x,y],z] = [[x,z],y] + [x,[y,z] \quad \text{or} \quad [x,[y,z]] = [[x,y],z] + [y,[x,z]],$$

respectively. He also introduced several new classes of algebras, including Leibniz algebras, diassociative algebras, dendriform algebras, Zinbiel algebras [1, 2] and showed that the link between them, in particular, Lie and associative algebras can be extended to analogous link between Leibniz algebras and so-called associative dialgebras which are a generalization of associative algebras possessing two composition laws.

By the definition [3], dialgebras are vector spaces with two bilinear products. A diassociative algebra is a vector space with two bilinear associative operations $\vdash$, $\dashv$, satisfying certain conditions [1] required for a dialgebra A to be associative are chosen in such a way that the new operation

$$ab = a \vdash b - b \dashv a \quad \text{or} \quad ab = a \dashv b - b \vdash a$$

turns A into a left (right) Leibniz algebra. Moreover, associative algebras are a particular case of diassociative algebras when two operations coincide. Some examples and applications of dialgebras are given in [4, 5, 6, 1].

The problem of classifying algebras of a particular type lies at the core of algebraic research. It provides the foundation for a deeper understanding of the structural behavior of the algebras involved. The study of structural properties of Leibniz algebras has been initiated by Ayupov and Omirov [7, 8]. Casas gave the list of isomorphism classes of three-dimensional complex Leibniz algebras [9] (two-dimensional case was given by Loday himself). There are classification results of low-dimensional dialgebras [10, 11, 12, 13, 14, 15].

By the motivation of the relation of dialgebras and conformal algebras, in 2008, Kolesnikov [16] gave the technique of how to define the notion of Var-dialgebra for a given variety of algebra Var. In this paper, first, we give the concept of "0-dialgebra" for introducing dialgebras. Then, we deal with the classification problem of 3-dimensional complex Leibniz dialgebras, particularly, we focus on the case where the annihilator

$$\mathcal{L}^{\text{ann}} = \text{ideal}\langle [x,x] \mid x \in \mathcal{L} \rangle$$

is one-dimensional, that is, $\dim(\mathcal{L}^{\text{ann}}) = 1$.

1.1. **Preliminaries.**

Definition 1.1. A vector space $\mathcal{D}$ with two multiplication operators $\dashv$ and $\vdash$ is called a "0-*dialgebra*" if

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \ x \dashv (y \vdash z) = x \dashv (y \dashv z),$$

for all $x, y, z \in \mathcal{D}$.

Definition 1.2. A 0-dialgebra $(\mathcal{D}, \dashv, \vdash)$ is called a *diassociative algebra* if

$$(x \vdash y) \vdash z = x \vdash (y \vdash z), \ (x \dashv y) \dashv z = x \dashv (y \dashv z), \ (x \vdash y) \dashv z = x \vdash (y \dashv z),$$

for all $x, y, z \in \mathcal{D}$.

Definition 1.3. A 0-dialgebra $(\mathcal{D}, \dashv, \vdash)$ is called a *Leibniz dialgebra* if

$$\begin{array}{l} (x \vdash y) \dashv z = x \vdash (y \dashv z) + (x \dashv z) \vdash y, \\ (x \dashv y) \vdash z = x \vdash (y \vdash z) + (x \vdash z) \dashv y, \\ (x \dashv y) \dashv z = x \dashv (y \dashv z) + (x \dashv z) \dashv y, \end{array}$$

for all $x, y, z \in \mathcal{D}$.

Let us consider an example of Lie dialgebras. Suppose $\sum = \{(x_1, x_2, x_3 - x_2 (x_1, x_3), x_1x_2 + x_2x_1)\}$ then the corresponding dialgebra identities include

$$x_1 \dashv x_2 + x_2 \vdash x_1.$$

A Lie dialgebra $\mathcal{D}$ considered as an ordinary algebra with respect to $[a, b] = a \vdash b$, $a, b \in \mathcal{D}$, is just a left Leibniz algebra. Conversely, every left Leibniz algebra $\mathcal{L}$ is a Lie dialgebra with respect to $a \vdash b = [a, b]$, $a \dashv b = -[b, a]$. Therefore, a Lie dialgebra is just the same as a Leibniz algebra.

Theorem 1.4. *[9] Any 3-dimensional complex non-Lie Leibniz algebra $\mathcal{L}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

Algebra	*Table of multiplication*	*Automorphisms*
$\mathcal{L}_1(\alpha)$, $\alpha \in \mathbb{C}$	$[e_2, e_2] = \alpha e_1,\ [e_3, e_2] = e_1,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = (a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2)e_1,$ $\varphi(e_2) = a_{21}e_1 + a_{22}e_2 + a_{23}e_3,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$
$\mathcal{L}_2$	$[e_3, e_3] = e_1.$	$\varphi(e_1) = a_{33}^2e_1,$ $\varphi(e_2) = a_{21}e_1 + a_{22}e_2,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$
$\mathcal{L}_3$	$[e_2, e_2] = e_1,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = (a_{32}^2 + a_{33}^2)e_1,$ $\varphi(e_2) = a_{21}e_1 - a_{33}e_2 + a_{32}e_3,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$
$\mathcal{L}_4$	$[e_1, e_3] = e_1.$	$\varphi(e_1) = a_{11}e_1,$ $\varphi(e_2) = a_{22}e_2,$ $\varphi(e_3) = a_{32}e_2 + e_3.$
$\mathcal{L}_5(\alpha)$, $\alpha \in \mathbb{C}\backslash\{0\}$	$[e_1, e_3] = \alpha e_1,\ [e_2, e_3] = e_2,\ [e_3, e_2] = -e_2.$	$\varphi(e_1) = a_{11}e_1,$ $\varphi(e_2) = a_{22}e_2,$ $\varphi(e_3) = a_{32}e_2 + e_3.$
$\mathcal{L}_6$	$[e_2, e_3] = e_2,\ [e_3, e_2] = -e_2,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = e_1,$ $\varphi(e_2) = a_{22}e_2,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + e_3.$
$\mathcal{L}_7$	$[e_1, e_3] = 2e_1,\ [e_2, e_2] = e_1,\ [e_2, e_3] = e_2,$ $[e_3, e_2] = -e_2,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = a_{22}^2e_1,$ $\varphi(e_2) = -a_{22}a_{32}e_1 + a_{22}e_2,$ $\varphi(e_3) = \frac{1}{2}(a_{22}^2 - a_{32}^2 - 1)e_1 + a_{32}e_2 + e_3.$

$\mathcal{L}_8(\alpha)$, $\alpha \in \mathbb{C}\backslash\{0\}$	$[e_1, e_3] = \alpha e_1,\ [e_2, e_3] = e_2.$	$\varphi_1(e_1) = a_{12}e_2,$ $\varphi_1(e_2) = a_{21}e_1,$ $\varphi_1(e_3) = -e_3,\ where\ \alpha = -1.$ $\varphi_2(e_1) = a_{11}e_1,$ $\varphi_2(e_2) = a_{22}e_2,$ $\varphi_2(e_3) = a_{33}e_3,\ where\ \alpha = -1.$ $\varphi_3(e_1) = a_{11}e_1 + a_{12}e_2,$ $\varphi_3(e_2) = a_{21}e_1 + a_{22}e_2,$ $\varphi_3(e_3) = e_3,\ where\ \alpha = 1,$ $\varphi_4(e_2) = a_{22}e_2,$ $\varphi_4(e_3) = e_3,\ where\ \alpha \neq \pm 1.$
$\mathcal{L}_9$	$[e_1, e_3] = e_1 + e_2,\ [e_2, e_3] = e_2.$	$\varphi(e_1) = a_{11}e_1 + a_{12}e_2,$ $\varphi(e_2) = a_{11}e_2,$ $\varphi(e_3) = e_3.$
$\mathcal{L}_{10}$	$[e_1, e_3] = e_2,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = a_{33}^2e_1 + a_{31}a_{33}e_2,$ $\varphi(e_2) = a_{33}^3e_2,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$
$\mathcal{L}_{11}$	$[e_1, e_3] = e_2,\ [e_2, e_3] = e_2,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = e_1 + (a_{22} - 1)e_2,$ $\varphi(e_2) = a_{22}e_2,$ $\varphi(e_3) = (a_{22} - a_{32} - 1)e_1 + a_{32}e_2 + e_3.$

2. Main result

In this section, we give lists of three-dimensional nontrivial complex Leibniz dialgebras. The idea is as follows. We choose the first part $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ of the Leibniz dialgebra from Theorem 1.4, restricting our consideration to the case where the annihilator of the corresponding Leibniz algebra $\mathcal{L}$ is one-dimensional i.e. they are algebras $\mathcal{L}_1, \dots \mathcal{L}_7$. Combining an algebra from this list (taking into account the Leibniz dialgebra axioms) with the second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$, we obtain constraints for the structural constants. Then we distinguish non-isomorphic algebras. The following theorem is one of the main results of this paper.

Theorem 2.1. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_1$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathcal{DL}_1(\alpha, m, n, q) : \begin{cases} e_2 \dashv e_2 = \alpha e_1,\ e_3 \dashv e_2 = e_1,\ e_3 \dashv e_3 = e_1, \\ e_2 \vdash e_2 = m e_1,\ e_2 \vdash e_3 = n e_1,\ e_3 \vdash e_2 = q e_1, \end{cases}$$

where $m \in \mathbb{C}\backslash\{0\}, \alpha, n, q \in \mathbb{C}$;

$\mathcal{DL}_2(\alpha, n, q) :\ e_2 \dashv e_2 = \alpha e_1,\ e_3 \dashv e_2 = e_1,\ e_3 \dashv e_3 = e_1,\ e_2 \vdash e_3 = n e_1,\ e_3 \vdash e_2 = q e_1,$
$\alpha \in \mathbb{C}\backslash\{0\}, n, q \in \mathbb{C}$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_2 \dashv e_2 = \alpha e_1, e_3 \dashv e_2 = e_1, e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by the following multiplication table:

$$\begin{array}{lll} e_1 \vdash e_1 = \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3, & e_2 \vdash e_1 = \alpha_4 e_1 + \beta_4 e_2 + \gamma_4 e_3, & e_3 \vdash e_1 = \alpha_7 e_1 + \beta_7 e_2 + \gamma_7 e_3, \\ e_1 \vdash e_2 = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3, & e_2 \vdash e_2 = \alpha_5 e_1 + \beta_5 e_2 + \gamma_5 e_3, & e_3 \vdash e_2 = \alpha_8 e_1 + \beta_8 e_2 + \gamma_8 e_3, \\ e_1 \vdash e_3 = \alpha_3 e_1 + \beta_3 e_2 + \gamma_3 e_3, & e_2 \vdash e_3 = \alpha_6 e_1 + \beta_6 e_2 + \gamma_6 e_3, & e_3 \vdash e_3 = \alpha_9 e_1 + \beta_9 e_2 + \gamma_9 e_3. \end{array} \quad (2.1)$$

Imposing the Leibniz dialgebra axioms, we obtain

$$e_2 \vdash e_2 = \alpha_5 e_1,\ e_2 \vdash e_3 = \alpha_6 e_1,\ e_3 \vdash e_2 = \alpha_8 e_1,\ e_3 \vdash e_3 = \alpha_9 e_1.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = (a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2)\, e_1,\ \varphi(e_2) = a_{21}e_1 + a_{22}e_2 + a_{23}e_3,\ \varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3,$$

with $\begin{cases} a_{22}^2\alpha + a_{22}a_{23} + a_{23}^2 = \alpha\left(a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2\right),\ a_{22}a_{32}\alpha + a_{23}a_{32} + a_{23}a_{33} = 0, \\ a_{22}a_{32}\alpha + a_{22}a_{33} + a_{23}a_{33} = a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2,\ a_{22}a_{33} -_{23} a_{32} \neq 0. \end{cases}$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha_5', \alpha_6', \alpha_8', \alpha_9'\}$ and $\{\alpha_5, \alpha_6, \alpha_8, \alpha_9\}$:

$$\begin{array}{lll} \varphi(e_2) \vdash \varphi(e_2) = \alpha_5'\varphi(e_1), & \Rightarrow & \alpha_5' = \frac{a_{22}^2\alpha_5 + a_{22}a_{23}(\alpha_6+\alpha_8) + a_{23}^2\alpha_9}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, \\ \varphi(e_2) \vdash \varphi(e_3) = \alpha_6'\varphi(e_1), & \Rightarrow & \alpha_6' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_6 + a_{23}a_{32}\alpha_8 + a_{23}a_{33}\alpha_9}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, \\ \varphi(e_3) \vdash \varphi(e_2) = \alpha_8'\varphi(e_1), & \Rightarrow & \alpha_8' = \frac{a_{22}a_{32}\alpha_5 + a_{23}a_{32}\alpha_6 + a_{22}a_{33}\alpha_8 + a_{23}a_{33}\alpha_9}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, \\ \varphi(e_3) \vdash \varphi(e_3) = \alpha_9'\varphi(e_1), & \Rightarrow & \alpha_9' = \frac{a_{32}^2\alpha_5 + a_{32}a_{33}(\alpha_6+\alpha_8) + a_{33}^2\alpha_9}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}. \end{array}$$

Then we have the following cases.

1. Let $(\alpha_5, \alpha_9) \neq (0,0)$. Without loss of generality, we assume that $\alpha_5 \neq 0$. Next, choosing $a_{32} = \frac{-(\alpha_6+\alpha_8) \pm \sqrt{(\alpha_6+\alpha_8)^2 - 4\alpha_5\alpha_9}}{2\alpha_5} a_{33}$, we can put $\alpha_9' = 0$. Thus we obtained the following algebra:

$$e_2 \vdash e_2 = \alpha_5 e_1,\ e_2 \vdash e_3 = \alpha_6 e_1,\ e_3 \vdash e_2 = \alpha_8 e_1.$$

   Again by using a change of basis we obtain the following relations:

$$\begin{array}{ll} \alpha_5' = \frac{a_{22}^2\alpha_5 + a_{22}a_{23}(\alpha_6+\alpha_8)}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, & \alpha_6' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_6 + a_{23}a_{32}\alpha_8}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, \\ \alpha_8' = \frac{a_{22}a_{32}\alpha_5 + a_{23}a_{32}\alpha_6 + a_{22}a_{33}\alpha_8}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, & a_{32}^2\alpha_5 + a_{32}a_{33}\left(\alpha_6 + \alpha_8\right) = 0, \end{array}$$

   with $\begin{cases} a_{22}^2\alpha + a_{22}a_{23} + a_{23}^2 = \alpha\left(a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2\right),\ a_{22}a_{32}\alpha + a_{23}a_{32} + a_{23}a_{33} = 0, \\ a_{22}a_{32}\alpha + a_{22}a_{33} + a_{23}a_{33} = a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2,\ a_{22}a_{33} - a_{23}a_{32} \neq 0. \end{cases}$

   (1) If $a_{32} \neq 0$ and $\alpha \neq 0$. Then we have $a_{32} = -\frac{a_{33}(\alpha_6+\alpha_8)}{\alpha_5} \neq 0$, $a_{22} = \frac{a_{23}(\alpha_5-\alpha_6-\alpha_8)}{\alpha(\alpha_6+\alpha_8)}$, $a_{33} \neq 0$, $\alpha_5(\alpha_5 - \alpha_6 - \alpha_8) + \alpha\left(\alpha_6 + \alpha_8\right)^2 \neq 0$, $a_{23} = \frac{a_{33}\alpha(\alpha_6+\alpha_8)}{\alpha_5} \neq 0$, and

$$\alpha_5' = \alpha_5 - \alpha_6 - \alpha_8,\ \alpha_6' = -\alpha_8,\ \alpha_8' = -\alpha_6.$$

   In this case we derive $\mathcal{DL}_1^1\left(\alpha, m, n, q\right)$, where $\alpha, m \neq 0, n + q \neq 0$.

   (2) If $a_{32} \neq 0$ and $\alpha = 0$. Then we have $a_{32} = -\frac{a_{33}(\alpha_6+\alpha_8)}{\alpha_5} \neq 0$, $a_{22} = \frac{a_{33}(\alpha_5-\alpha_6-\alpha_8)}{\alpha_5} \neq 0$, $a_{33} \neq 0$, $a_{23} = 0$, and

$$\alpha_5' = \alpha_5 - \alpha_6 - \alpha_8,\ \alpha_6' = \frac{\alpha_5\alpha_6 - \alpha_6 - \alpha_8}{\alpha_5},\ \alpha_8' = \frac{\alpha_5\alpha_8 - \alpha_6 - \alpha_8}{\alpha_5}.$$

   In this case we derive $\mathcal{DL}_1^2\left(0, m, n, q\right)$, where $m \neq 0, n + q \neq 0$.

   (3) If $a_{32} = 0$. Then we have $a_{23} = 0$, $a_{22} \neq 0$, $a_{33} \neq 0$, $a_{22} = a_{33}$, and

$$\alpha_5' = \alpha_5,\ \alpha_6' = \alpha_6,\ \alpha_8' = \alpha_8,$$

   we derive $\mathcal{DL}_1^3\left(\alpha, m, n, q\right)$, where $m \neq 0$.

In this case we obtain $\mathcal{DL}_1\left(\alpha, m, n, q\right)$, where $m \in \mathbb{C}\backslash\{0\}, \alpha, n, q \in \mathbb{C}$;

2. Let $\alpha_5 = \alpha_9 = 0$ and $\alpha_6 + \alpha_8 \neq 0$. Then we get $a_{23} = a_{32} = 0$. Thus we obtain $\mathcal{DL}_1^4\left(\alpha, 0, n, q\right)$, where $n + q \neq 0$.

3. Let $\alpha_5 = \alpha_9 = 0$ and $\alpha_6 + \alpha_8 = 0$. Then we get

$$\alpha_5' = 0,\ \alpha_6' = \alpha_6,\ \alpha_8' = -\alpha_6,\ \alpha_9' = 0.$$

   Thus we obtain $\mathcal{DL}_1^5\left(\alpha, 0, n, -n\right)$.

In this case we obtain $\mathcal{DL}_2\left(\alpha, n, q\right)$, where $\alpha \in \mathbb{C}\backslash\{0\}, n, q \in \mathbb{C}$;

□

Theorem 2.2. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_2$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{DL}_2^1(m,n):$ $e_3 \dashv e_3 = e_1, e_2 \vdash e_2 = e_1, e_2 \vdash e_3 = me_1, e_3 \vdash e_2 = ne_1,$ *where* $m, n \in \mathbb{C}$;
$\mathcal{DL}_2^2(q):$ $e_3 \dashv e_3 = e_1, e_3 \vdash e_3 = qe_1,$ *where* $q \in \mathbb{C}$;
$\mathcal{DL}_2^3:$ $e_3 \dashv e_3 = e_1, e_3 \vdash e_2 = e_1$;
$\mathcal{DL}_2^4(n):$ $e_3 \dashv e_3 = e_1, e_2 \vdash e_3 = e_1, e_3 \vdash e_2 = ne_1,$ *where* $n \in \mathbb{C}$;
$\mathcal{DL}_2^5:$ $e_3 \dashv e_3 = e_1, e_3 \vdash e_3 = e_2$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \le i \le 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_2 \vdash e_2 = \alpha_5 e_1,\ e_2 \vdash e_3 = \alpha_6 e_1,\ e_3 \vdash e_2 = \alpha_8 e_1,\ e_3 \vdash e_3 = \alpha_9 e_1 + \beta_9 e_2,$$

with the following constraints

$$\alpha_5\beta_9 = 0,\ \alpha_6\beta_9 = 0,\ \alpha_8\beta_9 = 0.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = a_{33}^2 e_1,\ \varphi(e_2) = a_{21}e_1 + a_{22}e_2,\ \varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha_5', \alpha_6', \alpha_8', \alpha_9', \beta_9'\}$ and $\{\alpha_5, \alpha_6, \alpha_8, \alpha_9, \beta_9\}$:

$$\begin{array}{ll}
\varphi(e_2) \vdash \varphi(e_2) = \alpha_5'\varphi(e_1), & \Rightarrow\ \alpha_5' = \frac{a_{22}^2}{a_{33}^2}\alpha_5,\\
\varphi(e_2) \vdash \varphi(e_3) = \alpha_6'\varphi(e_1), & \Rightarrow\ \alpha_6' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_6}{a_{33}^2},\\
\varphi(e_3) \vdash \varphi(e_2) = \alpha_8'\varphi(e_1), & \Rightarrow\ \alpha_8' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_8}{a_{33}^2},\\
\varphi(e_3) \vdash \varphi(e_3) = \alpha_9'\varphi(e_1) + \beta_9'\varphi(e_2), & \Rightarrow\ \begin{cases} \alpha_9' a_{33}^2 + \frac{a_{21}a_{33}^2}{a_{22}}\beta_9 = a_{32}^2\alpha_5 + a_{32}a_{33}(\alpha_6 + \alpha_8) + a_{33}^2\alpha_9,\\ \beta_9' = \frac{a_{33}^2}{a_{22}}\beta_9. \end{cases}
\end{array}$$

Then we have the following cases.

1. Let $\beta_9 = 0$. Then we have $\beta_9' = 0$ and

$$\begin{array}{ll}
\alpha_5' = \frac{a_{22}^2}{a_{33}^2}\alpha_5, & \alpha_6' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_6}{a_{33}^2},\\
\alpha_8' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_8}{a_{33}^2}, & \alpha_9' = \frac{a_{32}^2\alpha_5 + a_{32}a_{33}(\alpha_6+\alpha_8) + a_{33}^2\alpha_9}{a_{33}^2}.
\end{array}$$

   (1) If $\alpha_5 \neq 0$. Next, choosing $a_{33} = a_{22}\sqrt{\alpha_5}$ and $a_{32} = \frac{-(\alpha_6+\alpha_8) \pm \sqrt{(\alpha_6+\alpha_8)^2 - 4\alpha_5\alpha_9}}{2\alpha_5}a_{33}$, we can put $\alpha_5' = 1$ and $\alpha_9' = 0$. Thus we obtained $\mathcal{DL}_2^1(m,n)$, where $m, n \in \mathbb{C}$.

   (2) If $\alpha_5 = 0$. Then we obtain the following relations:

$$\begin{array}{ll}
\alpha_5' = 0, & \alpha_6' = \frac{a_{22}}{a_{33}}\alpha_6,\\
\alpha_8' = \frac{a_{22}}{a_{33}}\alpha_8, & \alpha_9' = \frac{a_{32}(\alpha_6+\alpha_8) + a_{33}\alpha_9}{a_{33}}.
\end{array}$$

   (2.1) If $\alpha_6 = 0$ and $\alpha_8 = 0$. Thus we obtain $\mathcal{DL}_2^2(q)$, where $q \in \mathbb{C}$.
   (2.2) If $\alpha_6 = 0$ and $\alpha_8 \neq 0$. Then choosing $a_{33} = a_{22}\alpha_8$, we get $\mathcal{DL}_2^3$.
   (2.3) If $\alpha_6 \neq 0$. Then choosing $a_{33} = a_{22}\alpha_8$, we get $\mathcal{DL}_2^4(n)$, where $n \in \mathbb{C}$.

2. Let $\beta_9 \neq 0$. Then we get $a_5 = a_6 = a_8 = 0$ and $\alpha_5' = \alpha_6' = \alpha_8' = 0$. Next, choosing $a_{22} = a_{33}^2\beta_9$, we can put $\beta_9' = 1$. In this case we obtain the algebra $\mathcal{DL}_2^5$.

□

Theorem 2.3. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_3$ is isomorphic to one of the following pairwise non-isomorphic algebras:*
$\mathcal{DL}_3(m,n,q):$ $e_2 \dashv e_2 = e_1,\ e_3 \dashv e_3 = e_1,\ e_2 \vdash e_3 = me_1,\ e_3 \vdash e_2 = ne_1,\ e_3 \vdash e_3 = qe_1,$
where $m,n,q \in \mathbb{C}$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_2 \dashv e_2 = e_1, e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \le i \le 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_2 \vdash e_2 = \alpha_5 e_1,\ e_2 \vdash e_3 = \alpha_6 e_1,\ e_3 \vdash e_2 = \alpha_8 e_1,\ e_3 \vdash e_3 = \alpha_9 e_1.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = (a_{32}^2 + a_{33}^2)\, e_1,\ \varphi(e_2) = a_{21}e_1 - a_{33}e_2 + a_{32}e_3,\ \varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3,$$

with $a_{22}a_{33} - a_{23}a_{32} \neq 0$.

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha_5', \alpha_6', \alpha_8', \alpha_9'\}$ and $\{\alpha_5, \alpha_6, \alpha_8, \alpha_9\}$:

$$\begin{aligned}
\varphi(e_2) \vdash \varphi(e_2) = \alpha_5'\varphi(e_1), &\quad \Rightarrow \quad \alpha_5' = \tfrac{a_{33}^2\alpha_5 - a_{32}a_{33}(\alpha_6+\alpha_8) + a_{32}^2\alpha_9}{a_{32}^2 + a_{33}^2},\\
\varphi(e_2) \vdash \varphi(e_3) = \alpha_6'\varphi(e_1), &\quad \Rightarrow \quad \alpha_6' = \tfrac{(\alpha_9-\alpha_5)a_{32}a_{33} - a_{33}^2\alpha_6 + a_{32}^2\alpha_8}{a_{32}^2 + a_{33}^2},\\
\varphi(e_3) \vdash \varphi(e_2) = \alpha_8'\varphi(e_1), &\quad \Rightarrow \quad \alpha_8' = \tfrac{(\alpha_9-\alpha_5)a_{32}a_{33} + a_{32}^2\alpha_6 - a_{33}^2\alpha_8}{a_{32}^2 + a_{33}^2},\\
\varphi(e_3) \vdash \varphi(e_3) = \alpha_9'\varphi(e_1), &\quad \Rightarrow \quad \alpha_9' = \tfrac{a_{32}^2\alpha_5 + a_{32}a_{33}(\alpha_6+\alpha_8) + a_{33}^2\alpha_9}{a_{32}^2 + a_{33}^2}.
\end{aligned}$$

Then we have the following cases.

1. Let $(\alpha_5 - \alpha_9)^2 + (\alpha_6 + \alpha_8)^2 = 0$. Then we have $\alpha_9 = \alpha_5 - i(\alpha_6 + \alpha_8)$ and

$$\begin{aligned}
\alpha_5' &= \tfrac{(a_{33}-ia_{32})\alpha_5 - a_{32}(\alpha_6+\alpha_8)}{a_{33}-ia_{32}}, & \alpha_6' &= \tfrac{a_{32}\alpha_8 + ia_{33}\alpha_6}{a_{32}-ia_{33}},\\
\alpha_8' &= \tfrac{a_{32}\alpha_6 + ia_{33}\alpha_8}{a_{32}-ia_{33}}, & \alpha_9' &= \tfrac{(a_{32}+ia_{33})\alpha_5 + a_{33}(\alpha_6+\alpha_8)}{a_{32}+ia_{33}}.
\end{aligned}$$

(1) Let $\alpha_5 + \alpha_9 = 0$ and $\alpha_6 - \alpha_8 = 0$. Then we have $\alpha_5 = -i\alpha_6$ and

$$\alpha_5' = \tfrac{3a_{32}+ia_{33}}{ia_{32}-a_{33}}\alpha_6,\quad \alpha_6' = \tfrac{a_{32}+ia_{33}}{a_{32}-ia_{33}}\alpha_6,\quad \alpha_8' = \tfrac{a_{32}+ia_{33}}{a_{32}-ia_{33}}\alpha_6,\quad \alpha_9' = \tfrac{3a_{33}-ia_{32}}{a_{32}+ia_{33}}\alpha_6.$$

Furthermore, by choosing $a_{33} = 3ia_{32}$, we obtain the following relations:

$$\alpha_5' = 0,\quad \alpha_6' = -\tfrac{1}{2}\alpha_6,\quad \alpha_8' = -\tfrac{1}{2}\alpha_6,\quad \alpha_9' = -4i\alpha_6.$$

In this case we obtain $\mathcal{DL}_3^1(m)$, where $m \in \mathbb{C}$.

(2) Let $\alpha_5 + \alpha_9 = 0$ and $\alpha_6 - \alpha_8 \neq 0$. Then we have $\alpha_5 = -\frac{i}{2}(\alpha_6 + \alpha_8)$ and

$$\begin{aligned}
\alpha_5' &= \tfrac{3a_{32}+ia_{33}}{2(ia_{32}-a_{33})}(\alpha_6+\alpha_8), & \alpha_6' &= \tfrac{a_{32}\alpha_8 + ia_{33}\alpha_6}{a_{32}-ia_{33}},\\
\alpha_8' &= \tfrac{a_{32}\alpha_6 + ia_{33}\alpha_8}{a_{32}-ia_{33}}, & \alpha_9' &= \tfrac{3a_{33}-ia_{32}}{2(a_{32}+ia_{33})}(\alpha_6+\alpha_8).
\end{aligned}$$

Then, by choosing $a_{33} = 3ia_{32}$, we obtain the following relations:

$$\alpha_5' = 0,\quad \alpha_6' = \tfrac{1}{4}(-3\alpha_6 + \alpha_8),\quad \alpha_8' = \tfrac{1}{4}(-3\alpha_8 + \alpha_6),\quad \alpha_9' = -2i(\alpha_6 + \alpha_8).$$

In this case we obatin $\mathcal{DL}_3^2(m,n)$, where $m,n \in \mathbb{C}$ and $m - n \neq 0$.

(3) Let $\alpha_5+\alpha_9\neq 0$ and $\alpha_6-\alpha_8\neq 0$. Then we have $2\alpha_5-i\left(\alpha_6+\alpha_8\right)\neq 0$. Therefore, by choosing $a_{32}=-\frac{i\alpha_5}{\alpha_5-i(\alpha_6+\alpha_8)}a_{33}$, we obtain the following relations:

$$\alpha_5'=0,\quad \alpha_6'=-i\alpha_5-\alpha_6,\quad \alpha_8'=-i\alpha_5-\alpha_8,\quad \alpha_9'=2\alpha_5-i\left(\alpha_6+\alpha_8\right).$$

In this case we obatin $\mathcal{DL}_3^3\left(m,n,q\right)$, where $m,n,q\in\mathbb{C}$, $n+q\neq 0$ and $2m-i\left(n+q\right)\neq 0$.

In this case we obatin $\mathcal{DL}_3^3\left(m,n,q\right)$, where $m,n,q\in\mathbb{C}$, $n+q\neq 0$, $2m-i\left(n+q\right)\neq 0$.

2. Let $(\alpha_5-\alpha_9)^2+(\alpha_6+\alpha_8)^2\neq 0$ and 1) let $(\alpha_5,\alpha_9)\neq(0,0)$. Without loss of generality, we assume that $\alpha_9\neq 0$. Next, choosing $a_{32}=\frac{(\alpha_6+\alpha_8)\pm\sqrt{(\alpha_6+\alpha_8)^2-4\alpha_5\alpha_9}}{2\alpha_9}a_{33}$, we can put $\alpha_5'=0$. Thus we obtained the following relations:

$$\begin{array}{ll} a_{32}^2\alpha_9-a_{32}a_{33}\left(\alpha_6+\alpha_8\right)=0, & \alpha_6'=\frac{a_{32}a_{33}\alpha_9-a_{33}^2\alpha_6+a_{32}^2\alpha_8}{a_{32}^2+a_{33}^2},\\ \alpha_8'=\frac{a_{32}a_{33}\alpha_9-a_{33}^2\alpha_8+a_{32}^2\alpha_6}{a_{32}^2+a_{33}^2}, & \alpha_9'=\frac{a_{32}a_{33}(\alpha_6+\alpha_8)+a_{33}^2\alpha_9}{a_{32}^2+a_{33}^2}.\end{array}$$

(1) If $a_{32}\neq 0$. Then we have $a_{32}=\frac{a_{33}(\alpha_6+\alpha_8)}{\alpha_9}$, and

$$\alpha_6'=\alpha_8,\ \alpha_8'=\alpha_6,\ \alpha_9'=\alpha_9.$$

In this case we obtain $\mathcal{DL}_3^4\left(m,n,q\right)$, where $q\neq 0$.

(2) If $a_{32}=0$. Then we have the following relations:

$$\alpha_6'=-\alpha_6,\ \alpha_8'=-\alpha_8,\ \alpha_9'=\alpha_9.$$

In this case we obtain $\mathcal{DL}_3^4\left(-m,-n,q\right)$, where $q\neq 0$.

2) Let $\alpha_5=\alpha_9=0$. Then we get $a_{32}=0$, and

$$\alpha_5'=0,\ \alpha_6'=-\alpha_6,\ \alpha_8'=-\alpha_8,\ \alpha_9'=0.$$

In this case we obtain $\mathcal{DL}_3^4\left(-m,-n,0\right)$.

In general, we obatin $\mathcal{DL}_3\left(m,n,q\right)$, where $m,n,q\in\mathbb{C}$.

□

Theorem 2.4. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_4$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{DL}_4^1:\ e_1\dashv e_3=e_1,\ e_3\vdash e_3=e_1;$
$\mathcal{DL}_4^2:\ e_1\dashv e_3=e_1,\ e_3\vdash e_1=-e_1;$
$\mathcal{DL}_4^3:\ e_1\dashv e_3=e_1,\ e_3\vdash e_1=-e_1,\ e_3\vdash e_3=e_1;$
$\mathcal{DL}_4^4:\ e_1\dashv e_3=e_1,\ e_1\vdash e_3=e_1;$
$\mathcal{DL}_4^5:\ e_1\dashv e_3=e_1,\ e_1\vdash e_3=e_1,\ e_3\vdash e_3=e_1;$

Proof. Consider the algebra $\mathcal{A}_1=(\mathcal{DL},\dashv)$ with multiplication table:

$$e_1\dashv e_3=e_1.$$

The second part $\mathcal{A}_2=(\mathcal{DL},\vdash)$ is defined by unknowns $\alpha_i,\beta_i,\gamma_i$ where $1\leq i\leq 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_1\vdash e_3=\alpha_3e_1,\ e_3\vdash e_1=\alpha_7e_1,\ e_3\vdash e_3=\beta_9e_1,$$

with the following constraints

$$\alpha_3\alpha_7=0,\ \alpha_3\left(\alpha_3-1\right)=0,\ \alpha_7\left(\alpha_7+1\right)=0.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1)=a_{11}e_1,\ \varphi(e_2)=a_{22}e_2,\ \varphi(e_3)=a_{32}e_2+e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_3, \alpha'_7, \beta'_9\}$ and $\{\alpha_3, \alpha_7, \beta_9\}$:

$$\begin{array}{ll} \varphi(e_1) \vdash \varphi(e_3) = \alpha'_3\varphi(e_1), & \Rightarrow \quad \alpha'_3 = \alpha_3, \\ \varphi(e_3) \vdash \varphi(e_1) = \alpha'_7\varphi(e_1), & \Rightarrow \quad \alpha'_7 = \alpha_7, \\ \varphi(e_3) \vdash \varphi(e_3) = \beta'_9\varphi(e_1), & \Rightarrow \quad \beta'_9 = \frac{1}{a_{22}}\beta_9. \end{array}$$

Hence, we see that if $\beta_9 = 0$, then $\beta'_9 = 0$. If $\beta_9 \neq 0$, then by choosing $a_{22} = \beta_9$, we obtain $\beta'_9 = 1$. Therefore, without loss of generality, β'_9 can be reduced to either 0 or 1. Thus we obtain the algebras $\mathcal{DL}_4^1$, $\mathcal{DL}_4^2$, $\mathcal{DL}_4^3$, $\mathcal{DL}_4^4$ and $\mathcal{DL}_4^5$.

□

Theorem 2.5. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_5$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{DL}_5^1$: $e_1 \dashv e_3 = \alpha e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2,$ *where* $\alpha \in \mathbb{C}\backslash\{0\}$;

$\mathcal{DL}_5^2$: $e_1 \dashv e_3 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_2 \vdash e_3 = e_1 + e_2,\ e_3 \vdash e_2 = -e_1 - e_2$;

$\mathcal{DL}_5^3$: $e_1 \dashv e_3 = \alpha e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_1 = -\alpha e_1,\ e_3 \vdash e_2 = -e_2,$ *where* $\alpha \in \mathbb{C}\backslash\{0\}$;

$\mathcal{DL}_5^4$: $e_1 \dashv e_3 = \alpha e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_1 \vdash e_3 = \alpha e_1,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2,$ *where* $\alpha \in \mathbb{C}\backslash\{0\}$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_1 \dashv e_3 = \alpha e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2, \text{ where } \alpha \in \mathbb{C}\backslash\{0\}.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \leq i \leq 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_1 \vdash e_3 = \alpha_3 e_1,\ e_2 \vdash e_3 = \alpha_6 e_1 + e_2,\ e_3 \vdash e_1 = \alpha_7 e_1,\ e_3 \vdash e_2 = -\alpha_6 e_1 - e_2.$$

with the following constraints

$$\alpha_3(\alpha_3 - \alpha) = 0,\ \alpha_3\alpha_6 = 0,\ \alpha_6(\alpha - 1) = 0,\ \alpha_3\alpha_7 = 0,\ \alpha_3\alpha_6 = 0,\ \alpha_6\alpha_7 = 0,\ \alpha_7(\alpha_7 + \alpha) = 0.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = a_{11}e_1,\ \varphi(e_2) = a_{22}e_2,\ \varphi(e_3) = a_{32}e_2 + e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_3, \alpha'_6, \alpha'_7\}$ and $\{\alpha_3, \alpha_6, \alpha_7\}$:

$$\begin{array}{ll} \varphi(e_1) \vdash \varphi(e_3) = \alpha'_3\varphi(e_1) & \Rightarrow \quad \alpha'_3 = \alpha_3, \\ \varphi(e_2) \vdash \varphi(e_3) = \alpha'_6\varphi(e_1) + \varphi(e_2) & \Rightarrow \quad \alpha'_6 = \frac{a_{22}}{a_{11}}\alpha_6, \\ \varphi(e_3) \vdash \varphi(e_1) = \alpha'_7\varphi(e_1) & \Rightarrow \quad \alpha'_7 = \alpha_7, \\ \varphi(e_3) \vdash \varphi(e_2) = -\alpha'_6\varphi(e_1) - \varphi(e_2) & \Rightarrow \quad \alpha'_6 = \frac{a_{22}}{a_{11}}\alpha_6. \end{array}$$

Hence, we see that if $\alpha_6 = 0$, then $\alpha'_6 = 0$. If $\alpha_6 \neq 0$, then by choosing $a_{11} = a_{22}\alpha_6$, we obtain $\alpha'_6 = 1$. Therefore, if $\alpha'_6 = 1$, then $\alpha = 1$. Thus, without loss of generality, α'_6 can be reduced to either 0 or 1. Then we obtain the algebras $\mathcal{DL}_5^1$, $\mathcal{DL}_5^2$, $\mathcal{DL}_5^3$ and $\mathcal{DL}_5^4$.

□

Theorem 2.6. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_6$ is isomorphic to the following algebra:*

$\mathcal{DL}_6^1$: $e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2,\ e_3 \vdash e_3 = qe_1,$ *where* $q \in \mathbb{C}$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \le i \le 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2,\ e_3 \vdash e_3 = \alpha_9 e_1.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = e_1,\ \varphi(e_2) = a_{22}e_2,\ \varphi(e_3) = a_{31}e_1 + a_{32}e_2 + e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters α'_9 and α_9 :

$$\varphi(e_3) \vdash \varphi(e_3) = \alpha'_9 \varphi(e_1) \quad \Rightarrow \quad \alpha'_9 = \alpha_9.$$

Hence, in this case we have $\mathcal{DL}_6^1$.

□

Theorem 2.7. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_7$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathcal{DL}_7^1 : \begin{cases} e_1 \dashv e_3 = 2e_1,\ e_2 \dashv e_2 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1, \\ e_1 \vdash e_3 = 2e_1,\ e_2 \vdash e_2 = e_1,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2; \end{cases}$$

$$\mathcal{DL}_7^2 : \begin{cases} e_1 \dashv e_3 = 2e_1,\ e_2 \dashv e_2 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1, \\ e_2 \vdash e_2 = -e_1,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_1 = -2e_1,\ e_3 \vdash e_2 = -e_2,\ e_3 \vdash e_3 = -e_1; \end{cases}$$

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_1 \dashv e_3 = 2e_1,\ e_2 \dashv e_2 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \le i \le 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$\begin{cases} e_1 \vdash e_3 = \alpha_3 e_1,\ e_2 \vdash e_2 = (\alpha_3 - 1)e_1,\ e_2 \vdash e_3 = e_2, \\ e_3 \vdash e_1 = (\alpha_3 - 2)e_1,\ e_3 \vdash e_2 = -e_2,\ e_3 \vdash e_3 = \alpha_9 e_1. \end{cases}$$

with the following constraints

$$\alpha_3(\alpha_3 - 2) = 0,\ (\alpha_3 - 2)(\alpha_9 + 1) = 0.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = a_{22}^2 e_1,\ \varphi(e_2) = -a_{22}a_{32}e_1 + a_{22}e_2,\ \varphi(e_3) = \tfrac{1}{2}\left(a_{22}^2 - a_{32}^2 - 1\right)e_1 + a_{32}e_2 + e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_3, \alpha'_9\}$ and $\{\alpha_3, \alpha_9\}$:

$$\varphi(e_1) \vdash \varphi(e_3) = \alpha'_3 \varphi(e_1) \quad \Rightarrow \quad \alpha'_3 = \alpha_3.$$

$$\varphi(e_2) \vdash \varphi(e_2) = (\alpha'_3 - 1)\varphi(e_1) \quad \Rightarrow \quad \alpha'_3 = \alpha_3.$$

$$\varphi(e_3) \vdash \varphi(e_1) = (\alpha'_3 - 2)\varphi(e_1) \quad \Rightarrow \quad \alpha'_3 = \alpha_3.$$

$$\varphi(e_3) \vdash \varphi(e_3) = \alpha'_9 \varphi(e_1) \quad \Rightarrow \quad \alpha'_9 = \tfrac{a_{32}^2\alpha_3 - a_{22}^2 + 1 + \alpha_9}{a_{22}^2}.$$

Then we have the following cases.

Case 1. Let $\alpha_3 \neq 0$. Hence, by setting $a_{32} = \sqrt{\frac{a_{22}^2 - \alpha_9 - 1}{2}}$, we obtain $\alpha'_9 = 0$. In this case we have $\mathcal{DL}_7^1$.

Case 2. If $\alpha_3 = 0$, then $\alpha_9 = -1$. In this case we obtain $\mathcal{DL}_7^2$.

□

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