Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 126–131
DOI: 10.29229/uzmj.2026-1-13

Distribution of prime divisors of non-homogeneous Beatty sequences

Low C.W., Sapar S.H., Deraman F., Johari M.A.M., Yunos F.

Abstract. Non-homogeneous Beatty sequence is a sequence of positive integer that taking the floor value of irrational numbers. This paper using the prime counting function, $\pi(x)$ to estimate the cardinality of total distinct prime divisors of a non-homogeneous Beatty sequence. When the parameters in the non-homogeneous Beatty sequence are sufficiently large, a better estimation can be obtained. From this study, we found that for a fixed irrational $\theta > 1$ and a real $\lambda > 0$, the the cardinality of total distinct prime divisors is less than or equals to the prime counting function of the last term. That is $|A| \leq \pi(\lfloor N\theta + \lambda \rfloor)$ where A is the set of total distinct prime divisors of a non-homogeneous Beatty sequence $(\lfloor n\theta + \lambda \rfloor)$ up to N^{th} term. Also, when parameters are sufficiently large, the following estimation is sharper, $|A| \leq \left(\sqrt{\theta^2 + \lambda}\right)(\log\ N)$.

Keywords: Beatty sequence, prime divisor, prime counting function
MSC (2020): 11B83, 11N05

1. Introduction

Numerous studies have been done on Beatty sequence. There are two types of Beatty sequences, namely homogeneous and non-homogeneous. The Beatty sequence that this paper considered is non-homogeneous case and it is defined as follows:

$$(\lfloor n\theta + \lambda \rfloor)_{n \leq N} = \lfloor 1\theta + \lambda \rfloor, \lfloor 2\theta + \lambda \rfloor, \lfloor 3\theta + \lambda \rfloor, ..., \lfloor N\theta + \lambda \rfloor \tag{1.1}$$

where $\theta > 1$ is an irrational number and $\lambda > 0$ is a real number.

In [1], the researchers used Maynard's methods to show the existence of gaps bounded by primes in a homogeneous Beatty sequence. They found that $\#\{x \leq n < 2x : \text{there exist } m \text{ distinct primes of the form } \lfloor \theta r \rfloor, r \in [n, n + \Delta_{\theta,m}]\} \gg \frac{x}{(logx)^B}$ where # is the number of elements in the set. [2] studied the non-homogeneous Beatty sequence. They used Diophantine properties of θ to established a better estimate bound for its character sums. Below are some important theorems:

Theorem 1.1. *Let θ be a fixed irrational number. For all real numbers λ, integers a, g, m with $gcd(ag, m) = 1$, and positive integers $N \leq t$, where t is the multiplicative order of g modulo m. The character sums of Beatty sequence $(\lfloor n\theta + \lambda \rfloor)_{n \leq N}$ modulo m, $S_m(\theta, \lambda, \chi; N)$ is given by:*

$$S_m(\theta, \lambda, \chi; N) \ll m^{\frac{1}{4}} N^{\frac{1}{2}} + ND_{\theta,\lambda}(N)$$

where $D_{\theta,\lambda}(N)$ is the discrepancy of the Beatty sequence $(\lfloor n\theta + \lambda \rfloor)_{n \leq N}$.

Theorem 1.2. *Let θ be a fixed irrational number. For any fixed $\delta > 0$, there exists a constant $\eta > 0$ such that for all real numbers λ, integers a, g and a prime p with $gcd(ag, p) = 1$, and positive integers $p^{\delta} < N \leq t$, where t is the multiplicative order of g modulo p, the following bound holds:*

$$S_p(\theta, \lambda, \chi; N) \ll Np^{-\eta} + ND_{\theta,\lambda}(N).$$

Then, [3] continue to estimate the sum of the cardinality of distinct prime divisors after the discovery of [2]. They found that the cardinality of distinct prime divisors $\omega(n)$ for a non-zero integer n, is approximate to $N \log \log N$. [4] studied the Beatty sequence in invariant games. They introduced an infinite binary sequence, called Sturmian word, in order to distinguish any two pairs of complementary Beatty sequences in non-homogeneous case. [5] studied the number on primes within the intersection of any non-homogeneous Beatty sequences and extended a few theorems under various compatibility conditions. [6] solved the trigonometric functions by using the complementary Beatty sequences. They use $\theta = \sqrt{3}$ for tangent function and $\theta = \sqrt{6}$ for sine function while solving the trigonometric inequalities. [7] worked on the disjoint complementary Beatty sequence which covering the rational non-homogeneous Beatty sequence.

[8] studied the new approach found in [2]. They obtained an estimation of the bound associated with composite moduli by using the method in [2] stated below.

Theorem 1.3. *Let θ be a fixed irrational number, λ be any real number and m is any composite number with primitive elements. For positive prime $P \leq m$, the set of prime $\mathcal{P}$ and $\#\mathcal{P}$ is the number of elements in set $\mathcal{P}$, the non-trivial multiplicative characters χ (mod m), the following bound holds:*

$$S_m(\theta, \lambda, \chi; P) \ll \phi(m)^{\frac{1}{4}} \#\mathcal{P}^{\frac{1}{2}} + \#\mathcal{P} D_{\theta,\lambda}(P).$$

[9] investigated the cardinality of character sums with Beatty sequences associated with composite modulo. The Beatty sequence considered is $\lfloor \alpha(n+k) + \beta \rfloor$. Character sums can be used to find the number of solutions of equations over a given finite field. This sums can be obtained by one character or more. Here, we provided the propositions stated in the book [10] on the character sums associated with prime modulo. Suppose that p is a odd prime and $\mathbb{F}_p^*$ is a multiplicative group. Then,

Proposition 1.4. *Let g be a primitive element of $\mathbb{F}_p$ with order $p-1$. For each fixed integer j where $0 \leq j \leq \phi(m) - 1$, the multiplicative character of $\mathbb{F}_p$, denoted by $\chi_j(g^k)$ is given by*

$$\chi_j(g^k) = e^{\frac{2\pi i j k}{p-1}} \text{ where } k = 0, 1, ..., p-1.$$

Proposition 1.5. *For additive character χ_a and χ_b where $a, b \in \mathbb{F}_p$. Then,*

$$\sum_{c \in \mathbb{F}_p} \chi_a(c)\overline{\chi_b(c)} = \begin{cases} p+1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

For multiplicative character, if $a, b \in \mathbb{F}_p^*$. Then,

$$\sum_{\chi} \chi_c(a)\overline{\chi_c(b)} = \begin{cases} p & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

where the sum is extended over all multiplicative character χ of $\mathbb{F}_p$.

In this paper, we are going to improve the result of [3] and introduce a better estimation when the parameters in Beatty sequence are sufficiently large. In our discussion, we let $\theta = \sqrt{k} > 1$ where k is a square-free integer and $\lambda > 0$ is a real number. Next, we expand the expressions in order to obtain the patterns. We will explain more in the next section.

2. Notation

For any real number x, $\lfloor x \rfloor$ denotes the greatest integer less or equals to x and $\{x\}$ denotes the fractional part of x, that is $x - \lfloor x \rfloor$.

In this paper, A denotes as the set of total distinct prime divisors of a non-homogeneous Beatty sequence $(\lfloor n\theta + \lambda \rfloor)$ up to N^{th} term. Our objective is to obtain the approximation for cardinality of A, $|A|$ where the result will be shown in the next section. Before we proceed, we provide an example so that the readers have a better understanding.

Example 2.1. Let $\theta = \sqrt{10}$, $\lambda = 5$ and $N = 100$. Then, (1.1) will be as follow

$$\begin{aligned}\Big(\lfloor n\sqrt{10} + 5 \rfloor\Big)_{n \leq 100} &= \lfloor 1\sqrt{10} + 5 \rfloor, \lfloor 2\sqrt{10} + 5 \rfloor, \lfloor 3\sqrt{10} + 5 \rfloor, ..., \lfloor 100\sqrt{10} + 5 \rfloor \\ &= 8, 11, 14, ..., 321.\end{aligned}$$

From above, we will find the prime divisors of each terms. Let A denotes as the set of prime divisors of whole sequence $8, 11, 14, ..., 321$. Then the elements of set A is as follow:

$$A = \{2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, 43, 47, 53, 59, 61, 67, 71, 73, 83, 89, 97, 103, 107, \\ 109, 113, 127, 131, 137, 151, 157, 163, 191, 197, 223, 229, 239, 251, 257, 283, 311\}.$$

Thus, the cardinality or the number of elements in set A, $|A| = 41$.

Next, the following is the definition of prime counting function and an example to illustrate it.

Definition 2.2. Prime counting function $\pi(x)$ is the number of primes less than or equal to the positive integer x.

Example 2.3. Let $x = 10$, then we list out the prime less than or equal to 10, which is $\{2, 3, 5, 7\}$. Thus $\pi(10) = 4$.

For a larger number x, we can use online prime counting function calculator to evaluate it.

3. Result and Discussion

The following is the main result of our research.

Theorem 3.1. *Let $\theta > 1$ be a fixed irrational and $\lambda > 0$ be a real number. The cardinality of distinct prime divisors, $|A|$ is given by*

$$|A| \leq \pi(\lfloor N\theta + \lambda \rfloor) \tag{3.1}$$

where $\pi(x)$ is the prime counting function for integer x and N is a natural number.

Proof. The Beatty sequence, $B_{\theta,\lambda}$ with $m_n = \lfloor n\theta + \lambda \rfloor$ for $n = 1, 2, 3, ..., N$ is as follows:

$$B_{\theta,\lambda} = m_1, m_2, m_3, ..., m_{N-2}, m_{N-1}, m_N.$$

Since $\theta > 1$, $B_{\theta,\lambda}$ is an increasing sequence, that is

$$m_1 < m_2 < m_3 < ... < m_N.$$

For each term of m_n, it can be written as a product of prime factors, if m_n is a composite. Then, arrange all primes by ordering number,

$$p_1 < p_2 < p_3 < ... < p_k < m_n.$$

Now, for the last term m_N, it can be either a prime or a composite. We consider two cases as follows:

Case 1: If m_N is a prime.
Let $m_N = p_{\max}$. For any term, m_n with $1 \leq n \leq N - 1$, we have

$$p_1 < p_2 < ... < p_j < m_n < p_{j+1} < ... < p_k < m_N = p_{\max}.$$

Obviously, $p_{\max}$ is the largest prime that $B_{\theta,\lambda}$ consists. Thus,

$$A = \{p_1, p_2, p_3, ..., p_{\max}\}.$$

Note that A may not consists of all the primes that less than $p_{\max}$. Let B be a set consists of all distinct primes that less than or equals to $p_{\max}$,

$$B = \{2, 3, 5, ..., p_{\max}\}.$$

We have $A \subseteq B$. Then,

$$|A| \leq |B|.$$

Since $|B| = \pi(p_{\max})$, therefore

$$|A| \leq \pi(m_N) = \pi(\lfloor N\theta + \lambda \rfloor).$$

We proved for Case 1.

Case 2: If m_N is a composite.
Let $m_N = p_{i_1}^{a_{i_1}} p_{i_2}^{a_{i_2}} p_{i_3}^{a_{i_3}} ... p_{i_k}^{a_{i_k}}$. For any term, m_n with $1 \leq n \leq N-1$, we have

$$p_1 < p_2 < ... < p_j < m_n < p_{j+1} < ... < p_k < m_N = p_{i_1}^{a_{i_1}} p_{i_2}^{a_{i_2}} p_{i_3}^{a_{i_3}} ... p_{i_k}^{a_{i_k}}$$

and

$$p_{i_1} < p_{i_2} < ... < p_{i_k} < m_N.$$

Then,

$$A = C \cup D$$

where

$$C = \{p_1, p_2, p_3, ..., p_k\} \text{ and } D = \{p_{i_1}, p_{i_2}, p_{i_3}, ..., p_{i_k}\}.$$

The largest prime in A is $\max\{p_k, p_{i_k}\}$, but m_N is the largest integer in $B_{\theta,\lambda}$. Thus,

$$\begin{aligned} \max\{p_k, p_{i_k}\} &< m_N \\ \pi(\max\{p_k, p_{i_k}\}) &\leq \pi(m_N) \\ |C \cup D| &\leq \pi(\lfloor N\theta + \lambda \rfloor). \end{aligned}$$

Therefore,

$$|A| \leq \pi(\lfloor N\theta + \lambda \rfloor).$$

We proved for Case 2. As a result, we showed that (3.1) is true,

$$|A| \leq \pi(\lfloor N\theta + \lambda \rfloor).$$

□

The problem of Theorem 3.1 is when θ, λ and N become bigger, the right hand side of (3.1) is much more bigger than the left hand side. Thus, if θ, λ and N are sufficiently large, the below theorem has a better estimation.

Theorem 3.2. *For sufficiently large fixed irrational θ, real λ and N with*

$$\theta \geq \sqrt{\frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2}}, \quad \lambda \geq \frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2} \quad \textit{and} \quad N \geq 10^{\sqrt{\pi(\lfloor N\theta + \lambda \rfloor)}}.$$

Then,

$$|A| \leq \left(\sqrt{\theta^2 + \lambda}\right)(\log N).$$

Proof. Firstly, we set

$$\theta \geq \sqrt{\frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2}}, \tag{3.2}$$

$$\lambda \geq \frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2} \tag{3.3}$$

and

$$N \geq 10^{\sqrt{\pi(\lfloor N\theta + \lambda \rfloor)}}. \tag{3.4}$$

By using (3.1) in Theorem 3.1, then (3.2) to (3.4) will become

$$\theta \geq \sqrt{\frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2}} \geq \sqrt{\frac{|A|}{2}}, \tag{3.5}$$

$$\lambda \geq \frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2} \geq \frac{|A|}{2} \tag{3.6}$$

and

$$N \geq 10^{\sqrt{\pi(\lfloor N\theta+\lambda \rfloor)}} \geq 10^{\sqrt{|A|}}. \tag{3.7}$$

From (3.5), we have

$$|A| \leq 2\theta^2. \tag{3.8}$$

From (3.6), we have

$$|A| \leq 2\lambda. \tag{3.9}$$

Then, we sum of (3.8) and (3.9)

$$|A| \leq \theta^2 + \lambda. \tag{3.10}$$

On the other hand, from (3.7) we have

$$|A| \leq (\log N)^2. \tag{3.11}$$

Now, we do the product of (3.10) and (3.11)

$$\begin{aligned}(|A|)^2 &\leq (\theta^2+\lambda)(\log N)^2 \\ &= \left(\left(\sqrt{\theta^2+\lambda}\right)(\log N)\right)^2.\end{aligned}$$

Since $|A|$ is a positive integer, we can remove square from both sides. Thus,

$$|A| \leq \left(\sqrt{\theta^2+\lambda}\right)(\log N).$$

□

4. Conclusion and Recommendation

We found $|A|$ by estimating the prime counting function which stated in Theorem 3.1. For a fixed irrational $\theta > 1$ and a real $\lambda > 0$. Then,

$$|A| \leq \pi(\lfloor N\theta + \lambda \rfloor)$$

where $\pi(x)$ is the prime counting function for integer x. On the other hand, when parameters are sufficiently large, the following estimation (Theorem 3.2) is sharper. For sufficiently large fixed irrational θ, real λ and N with

$$\theta \geq \sqrt{\frac{\pi(\lfloor N\theta+\lambda \rfloor)}{2}}, \; \lambda \geq \frac{\pi(\lfloor N\theta+\lambda \rfloor)}{2} \quad \text{and} \quad N \geq 10^{\sqrt{\pi(\lfloor N\theta+\lambda \rfloor)}}.$$

Then,

$$|A| \leq \left(\sqrt{\theta^2+\lambda}\right)(\log N).$$

In the future, the estimation of $|A|$ can be sharpened more. Next, the application of prime divisors in Beatty sequence can be an interesting topic to be studied.

5. Acknowledgments

We would like to thank Universiti Putra Malaysia, Serdang Selangor, Malaysia and Institute for Mathematical Research, (INSPEM), UPM Serdang Selangor, Malaysia for providing the necessary resources. The successful completion of this research would not be feasible without their contributions.

References

[1] Chua L., Park S., Smith G. D., Bounded gaps between primes in special sequences. Proc. Amer. Math. Soc. (2015) Vol. 143, P. 4597–4611.

[2] Bank W. D., Shparlinski I. E., Non-residues and primitive roots in Beatty sequences. Bull. Austral. Math. Soc. (2006) Vol. 73, P. 433–443.

[3] Bank W. D., Shparlinski I. E., Prime divisors in Beatty sequences. Journal of Number Theory, (2007) Vol. 123, P. 413–425.

[4] Cassaigne J., Duchêne E., Rigo M., Nonhomogeneous Beatty sequences leading to invariant games. SIAM Journal on Discrete Mathematics, (2016) Vol. 30, P. 1798–1829.

[5] Harman G., Primes in intersections of Beatty sequences. Journal of Integer Sequences, (2015) Vol. 18,

[6] Kimberling C., Beatty sequences and trigonometric functions. Integers, (2016) Vol. 16,

[7] Simpson J., Disjoint Beatty sequences. Integers: Electronic Journal of Combinatorial Number Theory, (2004) Vol. 4,

[8] Deraman F., Sapar S. H., Johari M. A. M., Mohd Atan K. A., Rasedee A. F. N., Extended bounds of Beatty sequence associated with primes. International Journal of Engineering and Advanced Technology (IJEAT), (2019) Vol. 8, P. 115–118.

[9] Allakov I., Deraman F., Sapar S. H., Ismail S., On cardinality of character sums with Beatty sequences associated with composite modules. Chebyshevskii Sbornik, (2023) Vol. 24, Iss. 5, P. 5–15.

[10] Lidl R., Niederreiter H., Introduction to finite fields and their applications. Cambridge University Press, Cambridge, (1986).

Low C. W.,
Institute for Mathematical Research, Universiti Putra Malaysia, Selangor Darul Ehsan, Malaysia.
e-mail: alexlowcheewai@gmail.com

Sapar S. H.,
Department of Mathematics and Statistics, Universiti Putra Malaysia, Selangor Darul Ehsan, Malaysia.
e-mail: sitihas@upm.edu.my

Deraman F.,
Faculty of Applied and Human Sciences, Universiti Malaysia Perlis, Perlis, Malaysia.
e-mail: fatanah@unimap.edu.my

Johari M. A. M.,
Department of Mathematics and Statistics, Universiti Putra Malaysia, Selangor Darul Ehsan, Malaysia.
e-mail: mamj@upm.edu.my

Yunos F.,
Department of Mathematics and Statistics, Universiti Putra Malaysia, Selangor Darul Ehsan, Malaysia.
e-mail: faridahy@upm.edu.my