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On the impact of the exponential functor on some types of continuous mappings

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Abstract. This paper investigates the influence of the symmetric product functor SP^n and the exponential functor exp on certain types of continuous mappings, focusing specifically on almost-open and pseudo-open mappings. The primary objective is to analyze how functors SP^n and exp interact with these mappings and alter their topological properties. Through the study, several key lemmas have been proven, providing insights into the behavior of the SP^n-induced mappings. Notably, it is demonstrated that for open sets $U_1, U_2, \ldots, U_n \subset X$, the set $[U_1, U_2, \ldots, U_n]$ retains its openness in $SP^n X$. These findings contribute to a deeper understanding of the topological implications of applying the SP^n functor on continuous mappings, offering new perspectives on its effect on almost-open and pseudo-open transformations.

Keywords: exponential functor, functor of permutation degree, almost-open map, pseudo-open map, sequence-covering map
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1. Introduction

The study of continuous mappings and their interactions with topological structures is a fundamental aspect of topology. Continuous mappings encode essential relationships between spaces, and understanding how their properties change under various transformations is a central problem. Functors such as the symmetric product functor SP^n and the exponential functor $\exp_n$ play a significant role in this context. These functors construct new spaces from given ones and induce corresponding mappings, providing a framework to analyze the preservation or alteration of topological properties.

In classical topology, concepts like openness, continuity, and compactness serve as cornerstones for understanding the behavior of mappings and spaces. Among these, almost-open and pseudo-open mappings are essential generalizations of open mappings that retain specific weaker properties. For instance, Michael's work on pseudo-open mappings [1] provides a foundational basis for analyzing mappings that are not strictly open but still preserve significant topological information.

The symmetric product functor SP^n, introduced and studied in depth by Dold [2], constructs the space of unordered n-tuples of points from a topological space X, equipped with a natural topology derived from X. On the other hand, the exponential functor $\exp_n$, closely associated with the Vietoris topology [3, 4], represents the space of non-empty closed subsets of X with at most n points. These constructions preserve certain topological structures of X while introducing new geometric and combinatorial features.

The purpose of this paper is to investigate how the functors SP^n and $\exp_n$ influence almost-open and pseudo-open mappings [5]. Key results presented here extend previous work by Arhangel'skii [6], Fedorchuk and Filippov [7], and Nagata [8] on general mappings, as well as Lin's studies on point-countable covers [9]. For example, we show that:

- For the symmetric product functor SP^n, the mapping $SP^n f\colon SP^n X \to SP^n Y$ is almost-open (Theorem 3.4) and pseudo-open (Theorem 3.5) whenever $f\colon X \to Y$ is almost-open or pseudo-open, respectively.
- For the exponential functor $\exp_n\colon Comp \to Comp$, the mapping retains these properties (Theorems 3.6 and 3.7).

By establishing these results, we contribute to a broader understanding of functorial transformations in topology and their implications for continuous mappings. This research builds upon earlier works in the field [10, 11, 12, 13, 14, 15, 16, 17] and opens avenues for further exploration of functorial interactions in generalized topological settings.

2. Preliminaries

Let X be a topological T_1-space. The set of all non-empty closed subsets of a topological space X is denoted by $\exp X$. The family of all sets of the form

$$O\langle U_1, ..., U_n\rangle = \left\{ F: \ F \in \exp X, \ F \subset \bigcup_{i=1}^{n} U_i, \ F \bigcap U_i \neq \emptyset, \ i = 1, 2, ..., n \right\}.$$

where $U_1, ..., U_n$ are open subsets of X, generates a base of the topology on the set $\exp X$. This topology is called the Vietoris topology. The set $\exp X$ with the Vietoris topology is called exponential space or the hyperspace of a space X [18]. Note that the space $\exp X$ is compact for any compact space X.

Define the following subspaces of $\exp X$:

$$\exp_n X = \{F \in \exp X : |F| \leq n\},$$

and

$$\exp_\omega X = \bigcup \{\exp_n X : n = 1, 2, ...\}.$$

Let $f : X \to Y$ be a continuous mapping between topological spaces. For a nonempty closed subset $C \subset X$, define

$$(\exp f)(C) = f(C). \tag{1}$$

Then $\exp f : \exp X \to \exp Y$ is well-defined and continuous.

Equality (1) defines a functor $\exp_n : Comp \to Comp$, which assigns to each topological space X the hyperspace $\exp X$, and to each continuous map $f : X \to Y$ the continuous map $\exp f : \exp X \to \exp Y$.

Let X be a compact Hausdorff space. Consider the mapping

$$\pi_n : X^n \to \exp_n X$$

that assigns to each point $x = (x_1, x_2, \ldots, x_n) \in X^n$ the set of its coordinates $\{x_1, x_2, \ldots, x_n\}$.

Then π_n is a continuous mapping of the compact space X^n onto the compact space $\exp_n X$. Thus, the hypersymmetric n power of the compact space X is the quotient space of its n power with respect to the partition generated by the following equivalence relation: points $x, y \in X^n$ are equivalent if they have the same set of coordinates.

On the n^{th} power X^n of the compact X, the permutation group S^n acts as the group of coordinate permutations. The set of orbits of this action with the quotient topology is denoted by $SP^n X$. Consider the quotient mapping

$$\pi_n^s : X^n \to SP^n X$$

that associates to each point $x = (x_1, x_2, \ldots, x_n) \in X^n$ the orbit of this point. Thus, the points of the space $SP^n X$ are finite subsets (equivalence classes) of the product X^n.

In this setting, two points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are considered equivalent if there exists a permutation $\sigma \in S^n$ such that $y_i = x_{\sigma(i)}$ for all $i = 1, 2, \ldots, n$ [13].

The space $SP^n X$ is called the n symmetric power of the space X. Equivalence relations by which the spaces $SP^n X$ and $\exp_n X$ are obtained from X^n are called symmetric and hypersymmetric equivalence relations, respectively. Any two points that are symmetrically equivalent in X^n will also be hypersymmetrically equivalent. However, in general, the converse does not hold. For example, for distinct elements $x, y \in X$ such that $x \neq y$, the points (x, x, y) and $(x, y, y) \in X^3$ are hypersymmetrically equivalent but not symmetrically equivalent [7].

Let $f : X \to Y$ be a continuous mapping between compact Hausdorf spaces X and Y. For the equivalence class $[(x_1, x_2, \ldots, x_n)] \in SP^n X$, put

$$(SP^n f)([(x_1, x_2, \ldots, x_n)]) = [(f(x_1), f(x_2), \ldots, f(x_n))].$$

This defines a mapping

$$SP^n f : SP^n X \to SP^n Y.$$

It is easy to verify that the operation SP^n constructed in this way is a covariant functor in the category $Comp$ of compact spaces and their continuous mappings [7].

Definition 2.1. A mapping $f\colon X \to Y$ is called almost-open if, for each $y \in Y$, there exists an element $x \in f^{-1}(y)$ such that for every open neighborhood U of x, the image $f(U)$ is a neighborhood of y (i.e. $y \in \mathrm{Int}(f(U))$) [19, 9].

Example 2.2. The notions of almost-open and open maps differ essentially. Let X be the disjoint union of the unit circle S^1 and the interval $(0,1)$; for brevity write $X = S^1 \cup (0,1)$ with the disjoint-union topology. Fix a point $r \in S^1$ and define a map

$$f : X \to S^1, \qquad f(x) = \begin{cases} x, & x \in S^1, \\ r, & x \in (0,1). \end{cases}$$

The map f is continuous and almost-open. Indeed, let $y \in S^1$. Choose the point $x = y \in X$ (here x is regarded as an element of the domain X, while y is the corresponding element of the range S^1). For every open neighborhood $U \subset X$ of x, we have $U \subset S^1$, so $f(U) = U$, which is an open neighborhood of y in S^1. Thus the condition of almost-openness is satisfied.

However, f is not an open map: the set $(0,1) \subset X$ is open, while $f((0,1)) = \{r\}$, which is not open in S^1. Moreover, there exist open neighborhoods in X whose images are neighborhoods in S^1 but not open. For instance, take $y_0 \in S^1$ with $y_0 \neq r$, and let $x_0 = y_0 \in X$. Let $V \subset S^1$ be an open arc containing y_0 but not r, and define $U = V \cup (0,1) \subset X$. Then U is an open neighborhood of x_0 in X, while

$$f(U) = V \cup \{r\}.$$

This set is a neighborhood of y_0 in S^1 (since it contains the open arc V), but it is not open because of the isolated point r. Hence an almost-open map may send an open neighborhood of a preimage point to a neighborhood of the image point without that image being open.

Definition 2.3. A mapping f is called pseudo-open if for each $y \in Y$ and each open neighborhood U of $f^{-1}(y)$ in X , $f(U)$ is a neighborhood of y in Y [6, 19].

Definition 2.4. A mapping f is called a 1-sequence-covering mapping if for each $y \in Y$ there exists $x \in f^{-1}(y)$, such that whenever $\{y_n\}$ is a sequence converging to y in Y, there exists a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$ [6, 19].

Definition 2.5. A mapping f is called a sequence-covering mapping if whenever $\{y_n\}$ is a convergent sequence in Y, there exists a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$ [6, 19].

Definition 2.6. A covariant functor $F\colon Comp \to Comp$ acting in the category of compact Hausdorff spaces and their continuous mappings, is called normal [18], if it
1) preserves the weight;
2) preserves singletons and empty set;
3) monomorphic (preserves embeddings);
4) epimorphic (preserves surjections);
5) preserves intersections of closed subsets;
6) preserves inverse images;
7) is continuous with respect to inverse limits.

Note that the functors exp and SP^n are normal.

3. Main results

For subsets $M_1, M_2, \dots, M_n$ of a space X define:

$$[(M_1, M_2, \dots, M_n)] = \big\{[(x_1, x_2, \dots, x_n)] \in SP^n X : \exists \sigma \in S^n, x_{\sigma(i)} \in M_i, i = 1, \dots, n\big\} \subset SP^n X.$$

Lemma 3.1. *For open sets $U_1, U_2, \dots, U_n \subset X$ the set $[(U_1, U_2, \dots, U_n)]$ is also open in $SP^n X$.*

Proof. By the construction of the set $SP^n X$, for the quotient map $\pi_n^s\colon X^n \to SP^n X$ we have the equality

$$(\pi_n^s)^{-1}([(U_1, U_2, \dots, U_n)]) = \bigcup_{g \in S^n} U_{g(1)} \times U_{g(2)} \times \dots \times U_{g(n)}.$$

Therefore, the set $(\pi_n^s)^{-1}([(U_1, U_2, \ldots, U_n)])$ is open in X^n as the union of $n!$ number of open sets in the form $U_{g(1)} \times U_{g(2)} \times \cdots \times U_{g(n)}$. Consequently, the set $[(U_1, U_2, \ldots, U_n)]$ is open in $SP^n X$. Lemma 3.1 is proved. □

Lemma 3.2. *For every sequence of subsets $A_1, A_2, \ldots A_n \subset X$ we have*

$$[(IntA_1, IntA_2, \ldots IntA_n)] \subset Int\,[(A_1, A_2, \ldots A_n)]$$

Proof. Get an arbitrary element $[(x_1, x_2, \ldots, x_n)] \in [(IntA_1, IntA_2, \ldots IntA_n)]$. Then there exists $g \in S^n$ such that $x_i \in IntA_{g(i)}$ for every $i = 1, \ldots, n$. This means that there are open sets $U_1, U_2, \ldots, U_n$ such that $x_i \in U_i \subset IntA_{g(i)}$ for every $i = 1, \ldots, n$. Therefore, we obtain $[(x_1, x_2, \ldots, x_n)] \in [(U_1, U_2, \ldots, U_n)]$. Note that by Lemma 3.1 the set $[(U_1, U_2, \ldots, U_n)]$ is an open neighborhood of the point $[(x_1, x_2, \ldots, x_n)]$ in the space $SP^n X$. Now for an arbitrary element $[(y_1, y_2, \ldots, y_n)] \in [(U_1, U_2, \ldots, U_n)]$ there exists $g' \in S_n$ with $y_i \in U_{g'(i)}$ for every $i = 1, \ldots, n$. Since $U_i \subset IntA_{g(i)}$, we have $y_i \in A_{g(g'(i))} = A_{(gg')(i)}$. Consequently, $[(y_1, y_2, \ldots, y_n)] \in [(A_1, A_2, \ldots, A_n)]$. By the arbitrariness of the choice of the element $[(y_1, y_2, \ldots, y_n)]$, we obtain the relation $[(U_1, U_2, \ldots, U_n)] \subset [(A_1, A_2, \ldots, A_n)]$. This implies $[(IntA_1, IntA_2, \ldots IntA_n)] \subset Int\,[(A_1, A_2, \ldots A_n)]$. Lemma 3.2 is proved.

□

Proposition 3.3. *The collection of all sets in the form $[(U_1, U_2, \ldots, U_n)]$, where $U_1, U_2, \ldots, U_n$ are open subsets in X, generates a base in $SP^n X$.*

Proof. It is clear that $SP^n X = [X]$. For two elements $[(U_1, U_2, \ldots, U_n)], [(V_1, V_2, \ldots, V_n)] \in SP^n X$ with

$$[(U_1, U_2, \ldots, U_n)] \cap [(V_1, V_2, \ldots, V_n)] \neq \emptyset,$$

consider any element

$$[(x_1, x_2, \ldots, x_n)] \in [(U_1, U_2, \ldots, U_n)] \cap [(V_1, V_2, \ldots, V_n)].$$

Then there exist $g_1, g_2 \in S^n$ such that $x_i \in U_{g_1(i)} \cap V_{g_2(i)}$ for each $i = 1, \ldots, n$. Put $W_i = U_i \cap V_{(g_1^{-1}g_2)(i)}$ (Note that for each $i = 1, \ldots, n$ we have $U_i \cap V_{(g_1^{-1}g_2)(i)} \neq \emptyset$, since $x_i \in U_{g_1(i)} \cap V_{g_2(i)}$). Let us take an arbitrary element $[(y_1, y_2, \ldots, y_n)] \in [(W_1, W_2, \ldots, W_n)]$. Then there exists $g \in S^n$ such that $y_i \in W_{g(i)}$ for each $i = 1, \ldots, n$. Taking into account that $y_i \in U_{g(i)} \cap V_{(g_1^{-1}g_2)(i)}$ we obtain

$$[(y_1, y_2, \ldots, y_n)] \in [(U_1, U_2, \ldots, U_n)] \cap [(V_1, V_2, \ldots, V_n)].$$

As a result, we have

$$[(W_1, W_2, \ldots, W_n)] \subset [(U_1, U_2, \ldots, U_n)] \cap [(V_1, V_2, \ldots, V_n)].$$

Proposition 3.3 is proved □

Theorem 3.4. *Let $f\colon X \to Y$ be an almost-open mapping. Then the induced map*

$$SP^n f\colon SP^n X \to SP^n Y$$

is also almost-open for every $n \in \mathbb{N}$.

Proof. Get an arbitrary element $[(y_1, y_2, \ldots, y_n)] \in SP^n Y$. Since $f\colon X \to Y$ is an almost-open mapping, for each y_i there is x_i such that $f(U)$ is an neighborhood of y_i for every open neighborhood U of x_i (i.e. $y_i \in Int(f(U))$). In this case, we have

$$[(x_1, x_2, \ldots, x_n)] \in (SP^n f)^{-1}([(y_1, y_2, \ldots, y_n)]). \tag{3.1}$$

By Proposition 3.3, without loss of generality, we can choose an arbitrary neighborhood of the point $[(x_1, x_2, \ldots, x_n)] \in SP^n X$ in the form $[(U_1, U_2, \ldots, U_n)]$, where $U_1, U_2, \ldots, U_n$ are open neighborhoods of points $x_1, x_2, \ldots, x_n$, respectively. Clearly, we have

$$[(y_1, y_2, \ldots, y_n)] \in (SP^n f)([(U_1, U_2, \ldots, U_n)]).$$

Further, using Lemma 3.2 we obtain the following relations:

$$[(\operatorname{Int} f(U_1), \operatorname{Int} f(U_2), \ldots, \operatorname{Int} f(U_n))] \subset \operatorname{Int}[(f(U_1), f(U_2), \ldots, f(U_n))] \subset$$

$$\subset [(f(U_1), f(U_2), \ldots, f(U_n))] = (SP^n(f))([(U_1, U_2, \ldots, U_n)]).$$

On the other hand, for each $i = 1, 2, \ldots, n$ we have $y_i \in Int(f(U_i))$. Consequently,

$$[(y_1, y_2, \ldots, y_n)] \in [(Int(f(U_1)), Int(f(U_2)), \ldots, Int(f(U_n)))].$$

Therefore, we obtain $[(y_1, y_2, \ldots, y_n)] \in Int(SP^n f([(U_1, U_2, \ldots, U_n)]))$. The last relation means that the mapping $SP^n f \colon SP^n X \to SP^n Y$ is almost-open. Theorem 3.4 is proved. □

Theorem 3.5. *Let $f \colon X \to Y$ be a pseudo-open mapping. Then the induced map*

$$SP^n f \colon SP^n X \to SP^n Y$$

is also pseudo-open for every $n \in \mathbb{N}$.

Proof. Let $[(y_1, \ldots, y_n)] \in SP^n Y$ be arbitrary and put

$$A := (SP^n f)^{-1}\big([(y_1, \ldots, y_n)]\big) \subset SP^n X.$$

By Proposition 3.3, we can choose open sets

$$U_1, \ldots, U_n \subset X$$

such that each U_i is an open neighborhood of the fiber $f^{-1}(y_i)$ in X, and

$$[(U_1, \ldots, U_n)]$$

is a neighborhood of some point of A in $SP^n X$. Since f is pseudo-open, for each $i = 1, \ldots, n$ we have

$$y_i \in \operatorname{Int}(f(U_i)).$$

Now consider the image of $[(U_1, \ldots, U_n)]$ under $SP^n f$. As in the proof of Theorem 3.4,

$$(SP^n f)([(U_1, \ldots, U_n)]) = [(f(U_1), \ldots, f(U_n))].$$

By Lemma 3.2,

$$[(\operatorname{Int} f(U_1), \ldots, \operatorname{Int} f(U_n))] \subseteq \operatorname{Int}[(f(U_1), \ldots, f(U_n))].$$

Since $y_i \in \operatorname{Int}(f(U_i))$ for each i, it follows that

$$[(y_1, \ldots, y_n)] \in [(\operatorname{Int} f(U_1), \ldots, \operatorname{Int} f(U_n))].$$

Therefore,

$$[(y_1, \ldots, y_n)] \in \operatorname{Int}\big((SP^n f)([(U_1, \ldots, U_n)])\big).$$

Thus every neighborhood in $SP^n X$ of a point of $(SP^n f)^{-1}([(y_1, \ldots, y_n)])$ is mapped by $SP^n f$ to a neighborhood of $[(y_1, \ldots, y_n)]$. Hence $SP^n f$ is pseudo-open. □

Theorem 3.6. *Let $f \colon X \to Y$ be an almost-open and surjective mapping. Then the induced map*

$$\exp_n f \colon \exp_n X \to \exp_n Y$$

is also almost-open for every $n \in \mathbb{N}$.

Proof. Assume that $f\colon X \to Y$ is almost-open and surjective. Let $F = \{y_1, y_2, \dots, y_n\} \in \exp_n Y$ be arbitrary.

Since $\exp_n$ is a normal functor in the category of compact spaces, the induced map $\exp_n f$ is also surjective. We will show that $\exp_n f$ is almost-open.

By the definition of almost-openness, for each $y_i \in F$, there exists a point $x_i \in f^{-1}(y_i)$ such that for every open neighborhood U_i of x_i, the image $f(U_i)$ is a neighborhood of y_i. Let us define the finite set

$$C = \{x_1, x_2, \dots, x_n\} \in \exp_n X.$$

Clearly, $(\exp_n f)(C) = f(C) = \{f(x_1), \dots, f(x_n)\} = F$, so $C \in (\exp_n f)^{-1}(F)$.

Now let $\langle U_1, U_2, \dots, U_n \rangle$ be a basic open neighborhood of C in $\exp_n X$, where each U_i is an open neighborhood of x_i in X.

We want to show that

$$F \in \operatorname{Int}\left((\exp_n f)\left(\langle U_1, U_2, \dots, U_n \rangle\right)\right).$$

First, observe that by the properties of the functor $\exp_n$,

$$(\exp_n f)\left(\langle U_1, U_2, \dots, U_n \rangle\right) = \langle f(U_1), f(U_2), \dots, f(U_n) \rangle.$$

According to Lemma 2.3 in [16], we have:

$$\langle \operatorname{Int}(f(U_1)), \operatorname{Int}(f(U_2)), \dots, \operatorname{Int}(f(U_n)) \rangle \subset \langle f(U_1), f(U_2), \dots, f(U_n) \rangle,$$

and this inclusion holds within $\exp_n Y$, where the left-hand side is an open set.

Therefore,

$$\langle \operatorname{Int}(f(U_1)), \dots, \operatorname{Int}(f(U_n)) \rangle \subset \operatorname{Int}\left(\langle f(U_1), \dots, f(U_n) \rangle\right).$$

Since $y_i \in \operatorname{Int}(f(U_i))$ for each $i = 1, \dots, n$, we conclude that

$$F = \{y_1, \dots, y_n\} \in \langle \operatorname{Int}(f(U_1)), \dots, \operatorname{Int}(f(U_n)) \rangle \subset \operatorname{Int}\left((\exp_n f)(\langle U_1, \dots, U_n \rangle)\right).$$

Hence, $\exp_n f$ is almost-open. □

Theorem 3.7. *Let $f\colon X \to Y$ be a pseudo-open and surjective mapping. Then the induced mapping*

$$\exp_n f\colon \exp_n X \to \exp_n Y$$

is pseudo-open for every $n \in \mathbb{N}$.

Proof. Let $f\colon X \to Y$ be pseudo-open and surjective. Take an arbitrary element $F = \{y_1, y_2, \dots, y_n\} \in \exp_n Y$.

Since $\exp_n$ is a normal functor in the category of compact spaces, the induced map $\exp_n f$ is surjective as well. We will show that $\exp_n f$ is pseudo-open.

By the definition of pseudo-openness, for each $y_i \in F$, there exists a point $x_i^j \in f^{-1}(y_i)$, indexed by $j \in A$, such that for every open neighborhood U_i of x_i^j, the image $f(U_i)$ is a neighborhood of y_i. Here the index $j \in A$ is used to distinguish different possible choices of preimages of y_i, since in general the fiber $f^{-1}(y_i)$ may contain more than one element.

Fix such a collection of points $\{x_1^j, x_2^j, \dots, x_n^j\} \subset X$ and define

$$C^j = \{x_1^j, x_2^j, \dots, x_n^j\} \in \exp_n X.$$

Clearly, $\exp_n f(C^j) = f(C^j) = \{f(x_1^j), \dots, f(x_n^j)\} = F$, so $C^j \in (\exp_n f)^{-1}(F)$.

Let $\langle U_1, U_2, \dots, U_n \rangle$ be a basic open neighborhood of C^j in $\exp_n X$, where each U_i is an open neighborhood of x_i^j. We want to prove that

$$F \in \operatorname{Int}\left((\exp_n f)(\langle U_1, U_2, \dots, U_n \rangle)\right).$$

Using the properties of the $\exp_n$ functor, we have

$$(\exp_n f)(\langle U_1, U_2, \dots, U_n \rangle) = \langle f(U_1), f(U_2), \dots, f(U_n) \rangle. \tag{3.2}$$

By Lemma 2.3 in [16], the following inclusion holds:

$$\langle \mathrm{Int}(f(U_1)), \mathrm{Int}(f(U_2)), \dots, \mathrm{Int}(f(U_n)) \rangle \subset \langle f(U_1), f(U_2), \dots, f(U_n) \rangle,$$

and this set is open in $\exp_n Y$. Moreover, since $y_i \in \mathrm{Int}(f(U_i))$ for each $i = 1, \dots, n$, it follows that

$$F = \{y_1, \dots, y_n\} \in \langle \mathrm{Int}(f(U_1)), \dots, \mathrm{Int}(f(U_n)) \rangle.$$

Therefore, by (3.2), we conclude that

$$F \in \mathrm{Int}\left(\langle f(U_1), f(U_2), \dots, f(U_n) \rangle\right) = \mathrm{Int}\left((\exp_n f)(\langle U_1, \dots, U_n \rangle)\right).$$

Hence, $\exp_n f$ is pseudo-open. □

Theorem 3.8. *Let $f\colon X \to Y$ be an almost-open surjective mapping between compact spaces X and Y. Then the induced mapping $\exp f\colon \exp X \to \exp Y$ is also almost-open.*

Proof. Take an arbitrary point $E \in \exp Y$. Since f is surjective, the preimage $F = f^{-1}(E) \in \exp X$ we have $F \in (\exp f)^{-1}(E)$.

Consider an arbitrary open neighborhood $O = \langle U_1, \dots, U_n \rangle$ of F in $\exp X$, where each U_i is an open subset of X. Then:

$$F \cap U_i \neq \emptyset \quad \text{for each } i = 1, 2, \dots, n,$$

and

$$F \subset \bigcup_{i=1}^{n} U_i.$$

This implies:

$$E \cap f(U_i) \neq \emptyset \quad \text{for each } i, \quad \text{and} \quad E \subset \bigcup_{i=1}^{n} f(U_i).$$

Thus,

$$E \in \langle f(U_1), \dots, f(U_n) \rangle.$$

We now show that $E \in \langle \mathrm{Int}\, f(U_1), \dots, \mathrm{Int}\, f(U_n) \rangle \subset \langle f(U_1), \dots, f(U_n) \rangle$. Let us define $F_i = F \cap U_i$. Then:

$$\bigcup_{i=1}^{n} F_i = F \cap \left(\bigcup_{i=1}^{n} U_i \right) = F.$$

Set $E_i = f(F_i)$. Then:

$$\bigcup_{i=1}^{n} E_i = f\left(\bigcup_{i=1}^{n} F_i \right) = f(F) = E.$$

Now take an arbitrary $y \in E$. Then $y \in E_i$ for some i, so there exists $x_y \in F_i \subset U_i$ with $f(x_y) = y$. Since f is almost-open we clearly have $y \in \mathrm{Int}\, f(U_i)$.

Thus:

$$E \subset \bigcup_{i=1}^{n} \mathrm{Int}\, f(U_i). \tag{3.3}$$

Moreover, for each $j = 1, 2, \dots, n$, there exists $x \in F_j = F \cap U_j$ such that $f(x) \in \mathrm{Int}\, f(U_j)$. Hence:

$$E \cap \mathrm{Int}\, f(U_j) \neq \emptyset. \tag{3.4}$$

From (3.3), (3.4) and Lemma 2.3.1 in [16], we conclude that

$$E \in \langle \mathrm{Int}\, f(U_1), \dots, \mathrm{Int}\, f(U_n) \rangle \subset \mathrm{Int}\langle f(U_1), \dots, f(U_n) \rangle.$$

Therefore, $E \in \mathrm{Int}\langle f(U_1), \dots, f(U_n) \rangle$, and so $\exp f$ is almost-open. □

Theorem 3.9. *Let $f\colon X \to Y$ be a pseudo-open and surjective mapping between compact spaces. Then the induced mapping $\exp f\colon \exp X \to \exp Y$ is also pseudo-open.*

Proof. Take an arbitrary $E \in \exp Y$. Clearly, $F = f^{-1}(E) \in \exp X$. For any F, consider an arbitrary neighborhood $O\langle U_1, ..., U_n\rangle$. We have $F \cap U_i \neq \emptyset$ for each $i = 1, 2, ..., n$ and $F \subset \bigcup_{i=1}^{n} U_i$. This implies $E \cap f(U_i) \neq \emptyset$ for each i , and $E \subset \bigcup_{i=1}^{n} f(U_i)$. Hence, $E \in O\langle f(U_1), ..., f(U_n)\rangle$.

Now we need to show pseudo-openness. For any $y \in E$, since f is pseudo-open, there exists $x_y \in f^{-1}(y)$ such that $y \in \mathrm{Int}(f(U_i))$ because $x_y \in F \cap U_i$. Therefore,

$$E \subset \bigcup_{i=1}^{n} \mathrm{Int}(f(U_i)).$$

Moreover, for each $j = 1, 2, ..., n$, we have $E \cap \mathrm{Int}(f(U_j)) \neq \emptyset$. Consequently, E lies inside a pseudo-open neighborhood.

Thus, $\exp f : \exp X \to \exp Y$ is pseudo-open. □

Theorem 3.10. *Let $f\colon X \to Y$ be an 1-sequence-covering mapping . Then the induced map $\exp_n f$: $\exp_n X \to \exp_n Y$ is also 1-sequence-covering.*

Proof. Let $F \in \exp_n Y$ and let (F_k) be a sequence in $\exp_n Y$ converging to F. We aim to find a sequence (C_k) in $\exp_n X$ converging to some $C \in \exp_n X$ such that

$$(\exp_n f)(C_k) = F_k \quad \text{for all } k, \quad \text{and} \quad (\exp_n f)(C) = F.$$

Write

$$F_k = \{y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)}\}, \quad F = \{y_1, y_2, \ldots, y_n\}.$$

Since convergence in $\exp_n Y$ implies that for each $i = 1, \ldots, n$, the sequences $(y_i^{(k)})$ converge to y_i in Y (possibly after reindexing or relabeling elements if necessary), we can apply the 1-sequence-covering property of f individually to each sequence $(y_i^{(k)})$.

Thus, for each i, there exists a sequence $(x_i^{(k)})$ in X and a point $x_i \in f^{-1}(y_i)$ such that:

$$x_i^{(k)} \to x_i \quad \text{in } X, \quad \text{and} \quad f(x_i^{(k)}) = y_i^{(k)} \quad \text{for all } k.$$

Now define

$$C_k = \{x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\}, \quad C = \{x_1, x_2, \ldots, x_n\}.$$

Then $C_k \to C$ in $\exp_n X$ because each coordinate sequence converges, $(\exp_n f)(C_k) = F_k$ for all k, and $(\exp_n f)(C) = F$.

Therefore, $\exp_n f$ is 1-sequence-covering. □

Theorem 3.11. *Let $f\colon X \to Y$ be a sequence-covering mapping . Then the induced map $\exp_n f\colon \exp_n X \to \exp_n Y$ is also sequence-covering.*

Proof. Let (F_k) be a sequence in $\exp_n Y$ converging to F, where

$$F_k = \{y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)}\}, \quad F = \{y_1, y_2, \ldots, y_n\}.$$

By the definition of convergence in $\exp_n Y$, for each $i = 1, \ldots, n$, the sequences $(y_i^{(k)})$ converge to y_i in Y.

Since f is sequence-covering, for each i there exists a sequence $(x_i^{(k)})$ in X and a point $x_i \in X$ such that:

$$x_i^{(k)} \to x_i \quad \text{in } X, \quad f(x_i^{(k)}) = y_i^{(k)} \quad \text{for all } k, \quad \text{and} \quad f(x_i) = y_i.$$

Define

$$C_k = \{x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\}, \quad C = \{x_1, x_2, \ldots, x_n\}.$$

Then $C_k \to C$ in $\exp_n X$ since each $x_i^{(k)} \to x_i$, $(\exp_n f)(C_k) = F_k$ for all k, and $(\exp_n f)(C) = F$.

Thus, $\exp_n f$ is a sequence-covering mapping. □

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References

[1] Michael E., Quotient and pseudo-open images of separable metric spaces. Journal of Topology, (1968) Vol. 11, Iss. 2, P. 305–320.

[2] Dold A., The fixed point index of fibre-preserving maps. Inventiones Mathematicae, (1958) Vol. 6, Iss. 4, P. 344–356.

[3] Kelley J., General Topology. Van Nostrand Reinhold Company, (1955).

[4] Vietoris L., Bereiche zweiter Ordnung. Mathematische Annalen, (1922) Vol. 85, P. 116–171.

[5] Toshbuvayev B., Acting of covariant functors on mappings. International Journal of Applied Mathematics, (2024) Vol. 37, Iss. 1, P. 47–51.

[6] Arhangelaskii A., Mappings and spaces. Russian Math. Surveys, (1966) Vol. 21, P. 115–162.

[7] Fedorchuk V., Covariant functors in a category of compacta. absolute retracts and μ-manifolds. Uspekhi Mat. Nauk, (1981) Vol. 36, Iss. 3, P. 177–195.

[8] Nagata J., On pseudo-open mappings. Mathematica Scandinavica, (1974) Vol. 34, Iss. 1, P. 55–68.

[9] Lin S., Point-Countable Covers and Sequence-Covering Mappings. Chinese Science Press, Beijing, (2002).

[10] Beshimov R., Georgiou D., Mamadaliev N., Sereti F., On local density and local weak density of the hyperspace of sets with finitely many components. Filomat, (2023) Vol. 37, Iss. 14, P. 4659–4669.

[11] Beshimov R., Georgiou D., Sereti F., Zhuraev R., Metric, stratifiable and uniform spaces of G-permutation degree. Mathematica Slovaca, (2024) Vol. 72, Iss. 1, P. 235–248.

[12] Beshimov R., Georgiou D., Juraev R., Manasipova R., Sereti F., Some topological properties of e-space and description of τ-closed sets. Filomat, (2025) Vol. 39, Iss. 8, P. 2625–2637.

[13] Fedorchuk V., Filippov V., Topology of hyperspaces and its applications. Mathematica, Cybernetica, (1989) Vol. 4, P. 48.

[14] Georgiou D., Mamadaliev N., Zhuraev R., A note on functional tightness and minitightness of space of the G-permutation degree. Comment. Math. Univ. Carolin. (2023) Vol. 64, Iss. 1, P. 97–108.

[15] Hatcher A., Algebraic Topology. Cambridge University Press, (2002).

[16] Michael E., Topologies on spaces of subsets. Transactions of the American Mathematical Society, (1951) Vol. 71, Iss. 2, P. 152–182.

[17] Whitehead J., Combinatorial homotopy. I. Bulletin of the American Mathematical Society, (1949) Vol. 55, Iss. 3, P. 213–226.

[18] Fedorchuk V., Filippov V., General Topology. The Basic Foundation. Fizmatlit, Moscow, (2006).

[19] Ge X., Notes on almost open mappings. Matematicki Vesnik, (2008) Vol. 60, P. 181–186.

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