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Inverse source problem for a Hopf-type system

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Abstract. A one-dimensional inverse problem for a quasilinear hyperbolic system with an unknown excitation source is considered. The Cauchy problem for a nonlinear Hopf-type system is studied. A Fourier transform is used to reduce the inverse problem to a direct problem, and an existence and uniqueness theorem for the solution is proved. The approach used can form the basis for constructing an efficient numerical algorithm for the inverse problem.

Keywords: Two-velocity hydrodynamics, viscous fluid, relative velocity, direct problem, inverse problem, Darcy coefficient
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1. Introduction

The theory of two-phase filtration finds important application in solving problems of petroleum engineering, soil science, biomechanics and others practical areas. Recently, increasing attention has been paid to modeling multiphase flows in connection with the disposal of radioactive waste. Simulation and numerical analysis of two-phase filtration in elastically deformable porous media are an important element in the development of cost-effective and safe treatment devices, allowing for a reduction in the number of laboratory and field experiments, identification of the main mechanisms, optimization of existing strategies, and assessment of potential risks. In recent years, interest in multiphase filtration processes in fractured porous reservoirs with low permeability has increased significantly. One important reason for this is that fractured hydrocarbon deposits contain more than 20 percent of the world's oil reserves [1].

In this paper, we study the inverse problem for a system of Hopf-type equations with an unknown source under the condition of overdetermination of solutions given on a fixed line. The original problem is reduced to the study of the Cauchy problem for a system of ordinary nonlinear integro-differential equations containing a convolution for which a unique solvability has been proven. The unique solvability of the inverse problem is proved, and a representation of its solution is obtained by solving the above-mentioned Cauchy problem [2]. Similar problems for linear and semilinear equations are considered in [3, 4, 5]. Inverse problems with final overdetermination are studied for parabolic equations and equations of viscous incompressible fluids in [6, 7, 8]. For the study of direct problems for Burgers-type equations and systems, see, for example, [9, 10, 11]. The issues of correctness of the linear inverse problem for a three-dimensional, second-order, mixed-type equation of the second kind in an unbounded parallelepiped are considered in [12].

In recent decades, a significant number of publications have been devoted to the study of inverse problems for partial differential equations due to their applied significance, but for the proposed quasilinear equations, this aspect remains poorly understood.

In [13] an inverse problem is considered, which consists of determining a solution-dependent coefficient of a quasilinear system of equations based on additional information about one of the components of the solution of the system, defined at a fixed point in space and being a function of time. The uniqueness of the solution to the inverse problem has been proven. In [14] a theorem on the existence of an inverse problem of determining an unknown coefficient of a quasilinear hyperbolic equation that depends on its solution is proved. Similar problems for a system of Hopf-type equations in the class of analytic functions are considered in [15, 16, 17, 18].

In this paper, we investigate a new inverse source problem for a Hopf-type system, assuming that its solutions are once continuously differentiable with respect to spatial variables and that a Fourier transform exists. The original problem reduces to studying the Cauchy problem for a system of ordinary nonlinear integro-differential equations containing a convolution for which a unique solvability has been proven. An example of a class of such functions is the space of rapidly decreasing functions [19].

2. Hopf type system of equations

The Cauchy problem in a strip $\Pi_{[0,T]} = \{(t,x) : 0 \le t \le T, x \in R\}$ for a system of Hopf-type equations is considered [20, 21, 22]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -b(u-v) + f(x)g_1(t), \tag{2.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \varepsilon b(u-v) + f(x)g_2(t), \tag{2.2}$$

$$u|_{t=0} = u^0(x), \quad v|_{t=0} = v^0(x). \tag{2.3}$$

where the function $f(x)$ is given, $\varepsilon = \frac{\rho_1}{\rho_2}$ is a dimensionless positive constant, b is a positive constant. The unknown functions $g_k = g_k(t)$ $(k = 1, 2)$, $t \in [0, T]$, and solutions u, v of the system of equations (2.1), (2.2) must be determined. The system (2.1), (2.2) differs from the system of two-velocity hydrodynamics in the dissipative case due to the coefficient of friction, the absence of pressure and the condition of incompressibility. For this reason, problems arise associated with the Hopf-type system, which gives the simplest quasi-linear system of equations [23].

Let us define the direct and inverse problems for a Hopf-type system.

Definition 2.1. The problem of determining functions u, v from system (2.1), (2.2) for given parameters ε, b and functions $f(x)$, $u^0(x)$, $v^0(x)$, $g_k = g_k(t)$ $(k = 1, 2)$ will be called the direct problem for a Hopf-type system.

3. Inverse source problem for a Hopf-type system

Let us assume we have additional override conditions

$$u|_{x=0} = \varphi(t), \quad v|_{x=0} = \psi(t), \quad t \in [0, T], \tag{3.1}$$

and the functions $\varphi(t)$, $\psi(t)$ satisfy the matching conditions

$$\varphi(0) = u^0(0), \quad \psi(0) = v^0(0). \tag{3.2}$$

Definition 3.1. The problem of determining functions u, v, $g_k = g_k(t)$ $(k = 1, 2)$ from system (2.1), (2.2) for given parameters ε, b and functions $f(x)$, $u^0(x)$, $v^0(x)$, $\varphi(t)$, $\psi(t)$ will be called the inverse problem for a Hopf-type system.

The functions $u^0(x)$, $v^0(x)$, $f(x)$ and $\varphi(t)$, $\psi(t)$ are assumed to be real. Next, we study the real solution to the classical inverse problem.

Suppose that there exist the Fourier transforms $U(t,y)$, $V(t,y)$ (with respect to x) of the solution $u(t,x)$, $v(t,x)$ for (2.1)–(2.3)

$$(U(t,y), V(t,y)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(u(t,x), v(t,x))e^{ixy}dx = F(u,v)(t,y), \tag{3.3}$$

$$(u(t,x), v(t,x)) = \int_{-\infty}^{\infty}(U(t,y), V(t,y))e^{-ixy}dx = F^{-1}(U,V)(t,x).$$

Applying the Fourier transform in variables x to (2.1), (2.2) we have

$$\frac{\partial U(t,y)}{\partial t} + \frac{1}{2}F\left(\frac{\partial u^2}{\partial x}\right)(t,y) = -b(U(t,y) - V(t,y)) + \tilde{F}(y)g_1(t), \tag{3.4}$$

$$\frac{\partial V(t,y)}{\partial t} + \frac{1}{2}F\left(\frac{\partial v^2}{\partial x}\right)(t,y) = \varepsilon b(U(t,y) - V(t,y)) + \tilde{F}(y)g_2(t), \tag{3.5}$$

where $\tilde{F}(y) = F(f)(y)$. Let in (2.1) and (2.2) $x = 0$. Using (2.3) and (3.3), taking into account the properties of the Fourier transform, we obtain

$$\varphi_t(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy = -b(\varphi(t) - \psi(t)) + f(0)g_1(t), \tag{3.6}$$

$$\psi_t(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy = \varepsilon b(\varphi(t) - \psi(t)) + f(0)g_2(t), \tag{3.7}$$

what does it mean

$$g_1(t) = \frac{1}{f(0)}\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy\right\}, \tag{3.8}$$

$$g_2(t) = \frac{1}{f(0)}\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy\right\}. \tag{3.9}$$

In formulas (3.8) and (3.9)

$$\tilde{\varphi}(t) = \varphi_t(t) + b(\varphi(t) - \psi(t)), \quad \tilde{\psi}(t) = \psi_t(t) - \varepsilon b(\varphi(t) - \psi(t)).$$

Further, without loss of generality, we can assume that $f(0) = 1$.

Since we are looking for a real solution $u(t,x)$, $v(t,x)$, $g_1(t)$, $g_2(t)$, it is worth considering the real parts of the functions $g_1(t)$, $g_2(t)$ in (3.8) and (3.9) (see remark 3.1 [24])

$$Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy\right\}, \quad Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy\right\}.$$

Suppose that the functions $U^0(y) = F(u^0)(y)$, $V^0(y) = F(v^0)(y)$ are continuously differentiable on $(-\infty,\infty)$, $\tilde{F}(y)$ and $\tilde{F}_y(y)$ continuous on $(-\infty,\infty)$, the functions $\varphi(t)$, $\psi(t)$ are continuously differentiable in $[0,T]$ and

$$\left(1+|y|^{k+\lambda}\right)\left|U^0(y)\right| + \left(1+|y|^{k+\lambda}\right)\left|\tilde{F}(y)\right| + \left|\frac{\partial}{\partial y}U^0(y)\right| + \left|\frac{\partial}{\partial y}\tilde{F}(y)\right| \le d_1(k), y \in (-\infty,\infty), \tag{3.10}$$

$$\left(1+|y|^{k+\lambda}\right)\left|V^0(y)\right| + \left(1+|y|^{k+\lambda}\right)\left|\tilde{F}(y)\right| + \left|\frac{\partial}{\partial y}V^0(y)\right| + \left|\frac{\partial}{\partial y}\tilde{F}(y)\right| \le d_2(k), y \in (-\infty,\infty), \tag{3.11}$$

where $\lambda = const > 0$, $d_1(k)$, $d_2(k)$ are positive constants and $k > 0$ is an integer.

Since

$$F(u^2)(t,y) = \int_{-\infty}^{\infty} U(t,z)U(t,y-z)dz$$

we represent

$$F\left(\frac{\partial u^2}{\partial x}\right)(t,y) = iyF(u^2)(t,y) = iy\int_{-\infty}^{\infty} U(t,z)U(t,y-z)dz$$

and substitute the real parts $g_1(t)$, $g_2(t)$ from (3.8), (3.9) into (3.4), (3.5) to obtain the integro-differential equation

$$\frac{\partial U(t,y)}{\partial t} + iy\int_{-\infty}^{\infty} U(t,z)U(t,y-z)dz = -b(U(t,y) - V(t,y)) + Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy\right\}\tilde{F}(y), \tag{3.12}$$

$$\frac{\partial V(t,y)}{\partial t} + iy\int_{-\infty}^{\infty} V(t,z)V(t,y-z)dz = \varepsilon b(U(t,y) - V(t,y)) + Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy\right\}\tilde{F}(y), \tag{3.13}$$

with parameter and initial Cauchy data

$$U(0,y) = U^0(y), \quad V(0,y) = V^0(y). \tag{3.14}$$

Note that system (3.12) and (3.13) are not the result of applying the Fourier transform to system (2.1) and (2.2), since instead of , in (3.8), (3.9) we take only their real parts.

We will prove the existence and uniqueness of solution (3.12)–(3.14) using the method of cutting functions [24]. The essence of the method of cutting functions is as follows. We introduce a sequence of cutting functions in the class such that

$$S_N(y) = \begin{cases} 1, & |y| \le N-2, \\ 0, & |y| > N, \end{cases} \tag{3.15}$$

or we approximate (3.12)–(3.14) by the problem

$$U_t^N(t,y) + iy\int_{-\infty}^{\infty} S_N(z)U^N(t,z)S_N(y-z)U^N(t,y-z)dz =$$

$$= -b(U^N(t,y) - V^N(t,y)) + Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yS_N(y)U^N(t,y)dy\right\}\tilde{F}(y), \tag{3.16}$$

$$V_t^N(t,y) + iy\int_{-\infty}^{\infty} S_N(z)V^N(t,z)S_N(y-z)V^N(t,y-z)dz =$$

$$= \varepsilon b(U^N(t,y) - V^N(t,y)) + Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yS_N(y)V^N(t,y)dy\right\}\tilde{F}(y), \tag{3.17}$$

$$U^N(0,y) = S_N(y)U^0(y), \quad V^N(0,y) = S_N(y)V^0(y), \quad N \ge 3. \tag{3.18}$$

By virtue of (3.15), we can replace the integrals in (3.16), (3.17) with integrals over a segment $[-N, N]$ and obtain

$$U_t^N(t,y) + iy\int_{-N}^{N} S_N(z)U^N(t,z)S_N(y-z)U^N(t,y-z)dz =$$

$$= -b(U^N(t,y) - V^N(t,y)) + Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-N}^{N} yS_N(y)U^N(t,y)dy\right\}\tilde{F}(y), \tag{3.19}$$

$$V_t^N(t,y) + iy\int_{-N}^{N} S_N(z)V^N(t,z)S_N(y-z)V^N(t,y-z)dz =$$

$$= \varepsilon b(U^N(t,y) - V^N(t,y)) + Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-N}^{N} yS_N(y)V^N(t,y)dy\right\}\tilde{F}(y). \tag{3.20}$$

Solving the Cauchy problem for system (3.18)-(3.20) we obtain a system of nonlinear integral Volterra equations of the second kind

$$U^N(t,y) = \frac{\varepsilon + e^{-b(1+\varepsilon)t}}{1+\varepsilon}S_N(y)U^0(y) + \frac{1 - e^{-b(1+\varepsilon)t}}{1+\varepsilon}S_N(y)V^0(y) +$$

$$+\frac{\varepsilon + e^{-b(1+\varepsilon)t}}{(1+\varepsilon)^2}\int_0^t \left[\left(\varepsilon + e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\varphi}(\tau) + i\varphi(\tau)\int_{-N}^{N} yS_N(y)U^N(\tau,y)dy\right\}\tilde{F}(y) -\right.\right.$$

$$\left. - iy\int_{-N}^{N} S_N(z)U^N(\tau,z)S_N(y-z)U^N(\tau,y-z)dz\right] +$$

$$+\left(1 - e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\psi}(\tau) + i\psi(\tau)\int_{-N}^{N} yS_N(y)V^N(\tau,y)dy\right\}\tilde{F}(y) -\right.$$

$$\left.\left. - iy\int_{-N}^{N} S_N(z)V^N(\tau,z)S_N(y-z)V^N(\tau,y-z)dz\right]\right]d\tau +$$

$$+\frac{1 - e^{-b(1+\varepsilon)t}}{(1+\varepsilon)^2}\int_0^t \left[\varepsilon\left(1 - e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\varphi}(\tau) + i\varphi(\tau)\int_{-N}^{N} yS_N(y)U^N(\tau,y)dy\right\}\tilde{F}(y) -\right.\right.$$

$$-iy\int_{-N}^{N}S_N(z)U^N(\tau,z)S_N(y-z)U^N(\tau,y-z)dz\Big]+$$

$$+\left(1+\varepsilon e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\psi}(\tau)+i\psi(\tau)\int_{-N}^{N}yS_N(y)V^N(\tau,y)dy\right\}\tilde{F}(y)-\right.$$

$$\left.-iy\int_{-N}^{N}S_N(z)V^N(\tau,z)S_N(y-z)V^N(\tau,y-z)dz\Big]\right]d\tau, \tag{3.21}$$

$$V^N(t,y)=\frac{\varepsilon\left(1-e^{-b(1+\varepsilon)t}\right)}{1+\varepsilon}S_N(y)U^0(y)+\frac{1+\varepsilon e^{-b(1+\varepsilon)t}}{1+\varepsilon}S_N(y)V^0(y)+$$

$$+\frac{\varepsilon\left(1-e^{-b(1+\varepsilon)t}\right)}{(1+\varepsilon)^2}\int_0^t\left[\left(\varepsilon+e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\varphi}(\tau)+i\varphi(\tau)\int_{-N}^{N}yS_N(y)U^N(\tau,y)dy\right\}\tilde{F}(y)-\right.\right.$$

$$-iy\int_{-N}^{N}S_N(z)U^N(\tau,z)S_N(y-z)U^N(\tau,y-z)dz\Big]+$$

$$+\left(1-e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\psi}(\tau)+i\psi(\tau)\int_{-N}^{N}yS_N(y)V^N(\tau,y)dy\right\}\tilde{F}(y)-\right.$$

$$\left.-iy\int_{-N}^{N}S_N(z)V^N(\tau,z)S_N(y-z)V^N(\tau,y-z)dz\Big]\right]d\tau+$$

$$+\frac{1+\varepsilon e^{-b(1+\varepsilon)t}}{(1+\varepsilon)^2}\int_0^t\left[\varepsilon\left(1-e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\varphi}(\tau)+i\varphi(\tau)\int_{-N}^{N}yS_N(y)U^N(\tau,y)dy\right\}\tilde{F}(y)-\right.\right.$$

$$-iy\int_{-N}^{N}S_N(z)U^N(\tau,z)S_N(y-z)U^N(\tau,y-z)dz\Big]+$$

$$+\left(1+\varepsilon e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\psi}(\tau)+i\psi(\tau)\int_{-N}^{N}yS_N(y)V^N(\tau,y)dy\right\}\tilde{F}(y)-\right.$$

$$\left.-iy\int_{-N}^{N}S_N(z)V^N(\tau,z)S_N(y-z)V^N(\tau,y-z)dz\Big]\right]d\tau. \tag{3.22}$$

Using the method of contraction mappings, it can be shown that for fixed $N\geq 3$, there exist classical solutions $U^N(t,y)$, $V^N(t,y)\in C^{1,0}_{t,y}\left(\Pi_{[0,t_N]}\right)$ of problem (3.18)–(3.20) in $\Pi_{[0,t_N]}$. Here the constant t_N is positive and, generally speaking, depends on N.

Following [5], taking into account Lemma 3.1 [24], a priori estimates of solutions $U^N(t,y)$, $V^N(t,y)$ are established:

$$|y|^{3+\lambda}\left|U^N(t,y)\right|\leq c_1,\quad |y|^{3+\lambda}\left|V^N(t,y)\right|\leq c_2,\quad (t,y)\in\Pi_{[0,t_*]}. \tag{3.23}$$

Here and below, the constants c_1, c_2 do not depend on N, while t_* depends on the constants $d_1(4)$, $d_2(4)$, $\|\varphi\|_{C^1[0,T]}$, $\|\psi\|_{C^1[0,T]}$ and does not depend on N, for all $N\geq 3$. From equations (3.19), (3.20) we obtain

$$\left|U^N_t(t,y)\right|\leq c_3,\quad \left|V^N_t(t,y)\right|\leq c_4,\quad (t,y)\in\Pi_{[0,t_*]}. \tag{3.24}$$

Differentiating both parts of system (3.22), (3.23) with respect to y, we can show that the estimates are valid

$$\left|U^N_y(t,y)\right|\leq c_5,\quad \left|V^N_y(t,y)\right|\leq c_4,\quad (t,y)\in\Pi_{[0,t_*]}. \tag{3.25}$$

Using (3.23)–(3.25) and Arzela's compactness theorem in C, we can choose subsequences $\{U^{N_k}\}$, $\{V^{N_k}\}$ such that

$$U^{N_k}\to U,\quad V^{N_k}\to V,\quad N_k\to\infty, \tag{3.26}$$

uniformly on each compact K in $\Pi_{[0,t_*]}$.

The uniqueness of the solution is proved in the usual way. Thus, we arrive at the following theorem.

Theorem 3.1. *Let conditions* (3.10), (3.11) $f(0) = 1$, *be satisfied and* $\varphi, \psi \in C^1[0,T]$. *Then there exists a unique solution* $U(t,y)$, $V(t,y)$ *to system* (3.12)–(3.14) *in the strip* $\Pi_{[0,t_*]}$. *The value* $0 < t_* \leq T$ *depends only on the constants* $d_1(4)$, $d_2(4)$ *and* $\|\varphi\|_{C^1[0,T]}$, $\|\psi\|_{C^1[0,T]}$.

Let us prove that the solution $u(t,x)$, $v(t,x)$, $g_1(t)$, $g_2(t)$ to the original problem (2.1)-(3.1) is

$$(u(t,x), v(t,x)) = \int_{-\infty}^{\infty} (U(t,y), V(t,y)) e^{-ixy} dx, \tag{3.27}$$

$$g_1(t) = Re\left\{\tilde{\varphi}(t) + i\varphi(t) \int_{-\infty}^{\infty} yU(t,y)dy\right\}, \tag{3.28}$$

$$g_2(t) = Re\left\{\tilde{\psi}(t) + i\psi(t) \int_{-\infty}^{\infty} yV(t,y)dy\right\}. \tag{3.29}$$

It is easy to see that $g_1(t)$ and $g_2(t)$ are real functions. We will show that $u(t,x)$, $v(t,x)$ are also real functions and satisfy (2.1)–(2.3), (3.1) (where $g_1(t)$ and $g_2(t)$ are defined in (3.28) and (3.29), respectively). We apply the inverse Fourier transform to (3.12)–(3.14) by y and see that $u(t,x)$, $v(t,x)$ are a solution to the problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -b(u-v) + f(x)g_1(t), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.30}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \varepsilon b(u-v) + f(x)g_2(t), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.31}$$

$$u|_{t=0} = u^0(x), \quad v|_{t=0} = v^0(x). \tag{3.32}$$

or

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} - u_2\frac{\partial u_2}{\partial x} = -b(u_1 - v_1) + f(x)g_1(t), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.33}$$

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x} - v_2\frac{\partial v_2}{\partial x} = \varepsilon b(u_1 - v_1) + f(x)g_2(t), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.34}$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} = -b(u_2 - v_2), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.35}$$

$$\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_1}{\partial x} = \varepsilon b(u_2 - v_2), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.36}$$

$$u_1|_{t=0} = u^0(x), \quad v_1|_{t=0} = v^0(x), \quad u_2|_{t=0} = 0, \quad v_2|_{t=0} = 0. \tag{3.37}$$

where u_1, v_1 and u_2, v_2 are the real and imaginary parts of the function u, v ($u = u_1 + iu_2$, $v = v_1 + iv_2$), and $g_1(t)$, $g_2(t)$ are the functions in (3.28), (3.29). Since u_1, v_1 and u_2, v_2 is a classical bounded solution of (3.33)–(3.37) (see (26)), we can consider system (3.35), (3.36) as a linear system with respect to u_2, v_2 and apply the method of characteristics (see, for example, [**25**, 10]) to obtain $u_2 = 0$, $v_2 = 0$. Consequently, $u = u_1$, $v = v_1$ is a real solution of (3.30)–(3.32) or (which is the same) (2.1)–(2.3), and $g_1(t)$, $g_2(t)$ are given by the second equations of (3.28), (3.29).

Let us show that the redefinition conditions (3.1) are satisfied: $u(t,0) = \varphi(t)$, $v(t,0) = \psi(t)$. Let $x = 0$ in (3.30), (3.31), then

$$u_t(t,0) + iu(t,0)\int_{-\infty}^{\infty} yU(t,y)dy = -b(u(t,0) - v(t,0)) + \varphi_t(t) + b(\varphi(t) - \psi(t)) + \varphi(t)i\int_{-\infty}^{\infty} yU(t,y)dy,$$

$$v_t(t,0) + iv(t,0)\int_{-\infty}^{\infty} yV(t,y)dy = \varepsilon b(u(t,0) - v(t,0)) + \psi_t(t) - \varepsilon b(\varphi(t) - \psi(t)) + \psi(t)i\int_{-\infty}^{\infty} yV(t,y)dy,$$

or

$$\Phi_t(t) + \mathrm{K}(t)\Phi(t) = -b(\Phi(t) - \Psi(t)), \tag{3.38}$$

$$\Psi_t(t) + \Lambda(t)\Psi(t) = \varepsilon b(\Phi(t) - \Psi(t)), \tag{3.39}$$

where

$$\Phi(t) = u(t,0) - \varphi(t), \quad \Psi(t) = v(t,0) - \psi(t),$$

$$\mathrm{K}(t) = i\int_{-\infty}^{\infty} yU(t,y)dy, \quad \Lambda(t) = i\int_{-\infty}^{\infty} yV(t,y)dy.$$

Note that the functions $\mathrm{K}(t)$, $\Lambda(t)$ are real functions [24].

By virtue of (3.1) and (3.32)

$$\Phi(t)|_{t=0} = 0, \quad \Psi(t)|_{t=0} = 0. \tag{3.40}$$

The only solutions to the Cauchy problem (3.38)-(3.40) are $\Phi(t) \equiv 0$, $\Psi(t) \equiv 0$ [25] and, therefore, $u(t,0) \equiv \varphi(t)$, $v(t,0) \equiv \psi(t)$. Therefore, the functions $u(t,x)$, $v(t,x)$, $g_1(t)$, $g_2(t)$ are a solution to (2.1)-(2.3), (3.1).

Thus, we have proved the following theorem.

Theorem 3.2. *Let* (3.10) *and* (3.11) *be satisfied and* $\varphi, \psi \in C^1[0,T]$, $f(0) = 1$. *Then problem* (2.1)-(2.3), (3.1) *has a solution* $u(t,x)$, $v(t,x) \in C^1(\Pi_{[0,t_*]})$, $g_1(t)$, $g_2(t) \in C[0,t_*]$, *which is determined by formulas* (3.27)-(3.29). *The value of* t_*, $0 < t_* \leq T$, *depends only on the constants* $d_1(4)$, $d_2(4)$ *and* $\|\varphi\|_{C^1[0,T]}$, $\|\psi\|_{C^1[0,T]}$.

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