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On the existence and uniqueness of a strong solution to the antiperiodic problem for a 2-parabolic equation with a deviating argument

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Abstract. This paper investigates the antiperiodic boundary value problem for a 2-parabolic equation with a time-deviating argument. A corresponding spectral problem is constructed, the symmetry of the differential operator is proven, and the properties of eigenvalues and eigenfunctions are established. It is shown that the eigenvalues have multiplicity two, and the corresponding eigenfunctions form an orthonormal basis in a Hilbert space. A strong solution to the problem is obtained in the form of a series expansion using the orthonormal basis of eigenfunctions corresponding to the spectrum of the operator generated by the boundary value problem. Conditions for the existence and uniqueness of a strong solution are established, and an explicit form of the inverse operator is constructed. Furthermore, it is proven that the problem operator is essentially self-adjoint. The proven statements complement the theoretical foundation for problems with deviating arguments in classes with antiperiodic boundary conditions, which is significant in modeling processes with memory and delay.

Keywords: deviating argument, 2-parabolic equation, spectral problem, eigenfunctions, eigenvalues
MSC (2020): 35D35, 35K35, 35P10, 47B25

1. Introduction and problem statement

The study of boundary value problems for partial differential equations plays a main role in theoretical and applied mathematics, especially in modeling complex physical processes described by parabolic-type equations. In recent years, there has been growing interest in problems with time-deviated arguments, which reflect memory effects or delays in the evolution of processes. Such problems naturally arise in thermal physics, biology, economics, and other applied fields.

Of particular interest is the formulation of antiperiodic boundary value problems, where the function and its derivatives at opposite ends of the interval differ in sign. Such conditions often model processes with symmetrical oscillatory regimes, where periodic influences reverse direction in each cycle.

Theoretical methods for analyzing differential equations with deviating arguments largely rely on the classical approaches presented in the monograph [1]. Fundamental works [2, 3, 4] on the theory of parabolic equations have formed a methodological basis for investigating a wide range of evolutionary problems in mathematical physics. The issues of well-posedness and construction of solutions to problems with time-deviating arguments have been addressed in numerous studies. Works [5, 6, 7, 8] yielded important results on the spectral properties of boundary value problems with a deviating argument. These studies laid the foundation for the further development of the theory of problems with non-traditional boundary conditions and time-deviating arguments. Significant contributions to the advancement of this field were made by works [9, 10, 11, 12, 13, 14, 15], which examined the existence, uniqueness, and regular and strong solutions of various classes of differential equations with deviating arguments.

A significant contribution to the development of the spectral theory of differential operators in functional spaces was made in the monograph [16], which consistently presents the modern concept of spectral geometry. The authors consider a wide range of problems related to studying the spectrum of differential and pseudo-differential operators on Riemannian manifolds and Lie groups, allowing for a deep analysis of the behavior of solutions to boundary value problems from the perspectives of geometry and functional analysis. Special attention is paid to the conditions for basis property of systems of eigenfunctions and associated functions, which are widely used in solving non-trivial boundary value problems of mathematical physics, including problems with antiperiodic conditions.

A substantial contribution to the study of spectral formulations of boundary value problems with deviating argument was made in the work [17], which conducted a systematic analysis of all possible boundary conditions for a first-order differential equation with involution. It has been shown that the structure of the spectrum and the properties of solutions significantly depend on the type of boundary conditions and the nature of the involutive operator.

In the work [18], the conditions for the basis property of the system of eigenfunctions and associated functions of a differential operator with involution are examined. Establishing the basis property plays a key role in constructing generalized expansions of solutions, which is especially relevant for problems with antiperiodic conditions, where the symmetric structure of boundary conditions is related to the involutive action of the operator. The results of [18] provide important analytical tools for proving the uniqueness and stability of solutions, as well as for developing spectral-analytical methods for studying boundary value problems with involution.

This work investigates the issues of unique strong solvability of the boundary value problem for a 2-parabolic equation with a deviating argument and homogeneous antiperiodic boundary conditions in the Hilbert space of square-integrable functions.

Let $\Omega = \{(x,t) : 0 < x < l,\ \ 0 < t < T\}$ and consider the following problem in the domain Ω:

Problem AP**.** To find the solution of the equation

$$Lu \equiv u_t\left(x, T-t\right) - u_{xxxx}\left(x,t\right) = f\left(x,t\right), \tag{1.1}$$

satisfying the boundary conditions

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} + \left.\frac{\partial^k u}{\partial x^k}\right|_{x=l} = 0, \quad 0 \le t \le T, \quad k = \overline{0,3}, \tag{1.2}$$

$$u(x,0) = 0, \quad 0 \le x \le l, \tag{1.3}$$

where $f(x,t)$ is a given function.

According to the classification proposed in the work [19], the equations of the form

$$\frac{\partial u}{\partial t} = (-1)^{p-1}\frac{\partial^{2p} u}{\partial x^{2p}} + f\left(x,t\right),$$

are called $p-$parabolic equations. At $p = 2$, the classical 2-parabolic equation is obtained

$$u_t\left(x,t\right) + u_{xxxx}\left(x,t\right) = f\left(x,t\right).$$

The deviating argument $T - t$ in equation (1.1) represents an involution of the variable t, since the double application of this transformation returns the original value $(T - (T - t))$. This involution significantly affects the structure of the equation and leads to a change in the sign before the fourth-order derivative compared to the classical Mihailov formula. Thus, equation (1.1) represents a 2-parabolic equation with involute deviation, which can be considered as an inverse problem for the classical 2-parabolic equation in the Mikhailov sense.

Let us introduce the following notations: $V(\Omega) = \{u(x,t) : u \in C^{3,0}_{x,t}(\bar{\Omega}) \cap C^{4,1}_{x,t}(\Omega)$, satisfies conditions (1.2),(1.3)$\}$.

Definition 1.1. The function $u(x,t) \in V(\Omega)$ is called a regular solution of the problem AP for $f(x,t) \in C(\Omega)$, if it satisfies equation (1.1) and conditions (1.2), (1.3) in the domain Ω.

Definition 1.2. The function $u(x,t) \in L_2(\Omega)$ is called a strong solution of the problem AP for $f(x,t) \in L_2(\Omega)$, if there exists a sequence $\{u_n(x,t)\}_{n=1}^{\infty}$ of regular solutions such that $\{u_n(x,t)\}_{n=1}^{\infty}$ and $\{Lu_n(x,t)\}_{n=1}^{\infty}$ converge in $L_2(\Omega)$ to $u(x,t)$ and $f(x,t)$ respectively.

Definition 1.3. A boundary value problem AP is called strongly solvable if a strong solution of the problem exists for any right-hand side $f(x,t) \in L_2(\Omega)$ and unique.

2. On the spectrum of the anti-periodic problem AP.

2.1. On symmetry. On the set $V(\Omega)$, we define the operator L_0, which acts from $V(\Omega)$ to $C(\Omega)$ from $\forall u \in V(\Omega)$ according to rule (1.1). Due to the relationship $C_0^{\infty}(\Omega) \in V(\Omega) \in L_2(\Omega)$, the domain $D(L_0) = V(\Omega)$ of the operator L_0 is densely packed in $L_2(\Omega)$. Let L be the closure of the operator L_0, in Hilbert space $L_2(\Omega)$, which is the minimum closed extension of the operator L_0.

Definition 2.1. [20]. An operator A acting in a Hilbert space H is called symmetric if $\overline{D(A)} = H$ and if for any $u, v \in D(A)$, the identity $(Au, v)_H = (u, Av)_H$ holds, where $(u, v)_H$ the inner product in the space H. In our case, $H = L_2(\Omega)$, and the inner product is defined by the formula

$$(u, v)_0 = (u, v)_{L_2(\Omega)} = \iint\limits_{\Omega} u(x,t)v(x,t)dxdt.$$

Lemma 2.2. *The operator L corresponding to the boundary value problem AP is symmetric.*

Proof. Note that $\overline{D(L)} = L_2(\Omega)$ is constructed. To prove the symmetry of the operator L, it is necessary to prove that for any $u, v \in D(L)$, the equality $(Lu, v)_0 = (u, Lv)_0$ holds.

$$(Lu, v) = \int\limits_0^T \int\limits_0^l [u_t(x, T-t) - u_{xxxx}(x,t)]v(x,t)dxdt,$$

using the substitution of the $s = T - t$ variable and integration by parts by t, we obtain

$$\int\limits_0^T \int\limits_0^l u_t(x, T-t)v(x,t)dxdt = \int\limits_0^T \int\limits_0^l u(x, T-t)v_t(x,t)dxdt.$$

The boundary conditions become zero due to the initial condition $u(x, 0) = 0$. Applying quadruple integration by parts by x, we have:

$$\int\limits_0^T \int\limits_0^l u_{xxxx}(x,t)v(x,t)dxdt = \int\limits_0^T \int\limits_0^l u(x,t)v_{xxxx}(x,t)dxdt,$$

all boundary conditions become zero due to the antiperiodic conditions (1.2). We get $(Lu, v)_0 = (u, Lv)_0$, which proves the symmetry of the operator.

□

2.2 On the basis property of eigenvectors. Let us consider the spectral problem for the operator L corresponding to the boundary value problem (1.1) - (1.3):

$$u_t(x, T-t) - u_{xxxx}(x,t) = \lambda u(x,t), \tag{2.1}$$

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} + \left.\frac{\partial^k u}{\partial x^k}\right|_{x=l} = 0, \quad 0 \le t \le T, \quad k = \overline{0,3}, \tag{2.2}$$

$$u(x, 0) = 0. \tag{2.3}$$

To solve the problem, we use the method of separation of variables. Assuming

$$u(x,t) = w(x) \cdot v(t), \tag{2.4}$$

and substituting (2.4) into (2.1), we have

$$\frac{v'(T-t)}{v(t)} = \frac{w^{IV}(x) + \lambda w(x)}{w(x)} = \gamma,$$

where is $\gamma-$the spectral parameter. Thus, if the solutions of problem (2.1) - (2.3) have the form (2.4), then the functions $w(x)$ and $v(t)$ are respectively solutions of the following spectral problems

$$\begin{cases} v'(T-t) - \gamma v(t) = 0 \\ v(0) = 0 \end{cases}, \tag{2.5}$$

$$\begin{cases} w^{IV}(x) - \beta w(x) = 0 \\ \left.\frac{d^k w}{dx^k}\right|_{x=0} + \left.\frac{d^k w(x)}{dx^k}\right|_{x=l} = 0, k = \overline{0,3}, \end{cases} \tag{2.6}$$

where $\beta = \gamma - \lambda$.

Lemma 2.3. *The spectral problem (2.6) has an infinite set of eigenvalues*

$$\beta_k = \frac{\pi^4(2k+1)^4}{l^4}, \quad k = 0, 1, 2..., \tag{2.7}$$

and corresponding eigenfunctions

$$w_k^{(1)}(x) = \sqrt{\frac{2}{l}} \cos \frac{\pi\,(2k+1)}{l}x, \qquad w_k^{(2)}(x) = \sqrt{\frac{2}{l}} \sin \frac{\pi\,(2k+1)}{l}x, \quad k = 0, 1, 2..., \tag{2.8}$$

which form an orthonormal basis in $L_2\,(0, l)$.

Proof. Let us find the eigenvalues of the antiperiodic problem. The characteristic equation is written in the form, $m^4 = \beta$, let us consider the cases $\beta > 0$, $\beta = 0$, $\beta < 0$.

Let $\beta = 0$, then the characteristic equation $m^4 = 0$ has four roots $m_{1,2,3,4} = 0$. The general solution is written as $w\,(x) = C_1x^3 + C_2x^2 + C_3x + C_4$ using the antiperiodic conditions of problem (2.2) we have $C_1 = C_2 = C_3 = C_4 = 0$. From this $X\,(x) = 0$.

Let $\beta < 0$, be $\beta = -4\mu^4$, $(\mu > 0)$, then the characteristic equation $m^4 = -4\mu^4$ has complex conjugate roots $m_{1,2} = \mu\,(1 \pm i)$; $m_{3,4} = \mu\,(-1 \pm i)$; the solution is written as:

$$w\,(x) = C_1 ch\mu x\, \cos\mu x + C_2 ch\mu x\, \sin\mu x + C_3 sh\mu x\, \cos\mu x + C_4 sh\mu x\, \sin\mu x,$$

using the conditions of problem (2.6), we have $C_1 = C_2 = C_3 = C_4 = 0$. From this $w\,(x) = 0$.

Thus, the problem (2.6) at $\beta \leq 0$ has only a trivial solution.

Let $\beta = \mu^4$, where $\mu > 0$. Then the characteristic equation $m^4 = \mu^4$ has roots $m_{1,2} = \pm\mu$; $m_{3,4} = \pm\mu i$, and the general solution can be written as:

$$w\,(x) = C_1e^{\mu x} + C_2e^{-\mu x} + C_3\cos\mu x + C_4\sin\mu x,$$

where $C_i, i = \overline{1,4}$ are arbitrary real numbers. Further, considering the boundary conditions of problem (2.6), to find these constants, we obtain the system

$$\begin{cases} \left(1 - e^{\mu l}\right) C_1 + \left(1 + e^{-\mu l}\right) C_2 + (1 + \cos\mu l)\, C_3 + \sin\mu l C_4 = 0, \\ \left(1 + e^{\mu l}\right) C_1 - \left(1 + e^{-\mu l}\right) C_2 - \sin\mu l C_3 + (1 + \cos\mu l)\, C_4 = 0, \\ \left(1 + e^{\mu l}\right) C_1 + \left(1 - e^{-\mu l}\right) C_2 - (1 + \cos\mu l)\, C_3 - \sin\mu l C_4 = 0, \\ \left(1 - e^{\mu l}\right) C_1 - \left(1 - e^{-\mu l}\right) C_2 + \sin\mu l C_3 - (1 + \cos\mu l)\, C_4 = 0. \end{cases} \tag{2.9}$$

The resulting system has a non-trivial solution only for the values μ, at which its determinant becomes zero. Let us denote the determinant of this system by $\Delta(\mu)$. Then it is not difficult to see, that $\Delta(\mu) = -16(1 + \cos\mu l)$. From this we find eigenvalues, that have the form (2.7).

Now let us examine the multiplicity of eigenvalues. Since the rank of the matrix corresponding to system (2.9) is equal to 2 when $\mu_k l = \pi + 2\pi k$, it follows that the geometric multiplicity of the eigenvalues is equal to 2. Therefore, each eigenvalue corresponds to a pair of eigenfunctions. The algebraic multiplicity is the order of multiplicity of the root in μ_k the equation $\Delta\,(\mu) = 0$. Since

$$\Delta'\,(\mu) = 16l\sin\mu l, \quad \Delta'\,(\mu_k) = 16l\sin\left[(2k+1)\,\pi\right] = 0,$$

$$\Delta''\,(\mu) = 16l^2\cos\mu l, \;\; \Delta''\,(\mu_k) = 16l^2\cos\left[(2k+1)\,\pi\right] = -16l^2 \neq 0,$$

therefore, the algebraic multiplicity of eigenvalues is also equal to 2. Consequently, all eigenvalues of problem (2.6) are of multiplicity two, and the eigenfunctions are the functions (2.8). The orthonormality of the resulting system in $L_2(0, 1)$ is verified directly. Then, by the Riesz-Fischer theorem, the system of eigenfunctions (2.8) of problem (2.6) forms an orthonormal basis in $L_2\,(0, 1)$. □

Lemma 2.4. *[15]. The spectral problem (2.5) has an infinite set of eigenvalues*

$$\gamma_n = (-1)^k \left(\frac{1}{2} + 2k\right)\frac{\pi}{T}, \quad k = 0, 1, 2.... \tag{2.10}$$

and corresponding eigenfunctions

$$v_k(t) = \sqrt{\frac{2}{T}} \sin \frac{\pi(2k+1)}{2T} t, \quad k = 0, 1, 2, ..., \tag{2.11}$$

which form an orthonormal basis of the space $L_2(0;T)$.

The following statements hold. [[20], p.65]:

Lemma 2.5. *If the system of functions* $\{\varphi_m(x)\}, m = 1, 2, ...$*forms an orthonormal basis of the space* $L_2(0, l)$*, and the system of functions* $\{\psi_n(x)\}, n = 1, 2, ...$*forms an orthonormal basis of the space* $L_2(0, T)$*, then the system of functions* $\{\varphi_m(x)\psi_n(x)\}$*,* $m, n = 1, 2, ...$ *forms an orthonormal basis of the space* $L_2[(0, l) \times (0, T)]$*.*

From this lemma and from formulas (2.4), (2.7), (2.11), it follows that:

Theorem 2.6. *The spectral problem (2.1) - (2.3) has an infinite set of eigenvalues*

$$\lambda_{kn} = (-1)^n \frac{\pi}{T}\left(\frac{1}{2} + 2n\right) + \frac{\pi^4 (2k+1)^4}{l^4}; \quad k, n = 0, 1, 2, ..., \tag{2.12}$$

and corresponding eigenfunctions

$$u_{kn}^{(1)}(x,t) = \frac{2}{\sqrt{Tl}} \cos \frac{\pi(2n+1)}{l} x \cdot \sin \frac{\pi(2k+1)}{2T} t, \quad k, n = 0, 1, 2, ..., \tag{2.13}$$

$$u_{kn}^{(2)}(x,t) = \frac{2}{\sqrt{Tl}} \sin \frac{\pi(2n+1)}{l} x \cdot \sin \frac{\pi(2k+1)}{2T} t, \quad k, n = 0, 1, 2, ..., \tag{2.14}$$

which form an orthonormal basis of the space $L_2(\Omega)$*.*

3. On the existence and uniqueness of a strong solution to the antiperiodic problem

Consider the linear operator L corresponding to the boundary value problem (1.1)-(1.3). Suppose that for some $u \in D(L)$, $u \neq 0$ the equality $Lu = 0$ holds. Then, due to the symmetry of the operator L we have the equality

$$0 = \left(Lu, u_{kn}^{(i)}\right) = \left(u, Lu_{kn}^{(i)}\right) = \lambda_{kn}\left(u, u_{kn}^{(i)}\right), \quad i = 1, 2.$$

If $\lambda_{kn} \neq 0$, then due to the completeness of the system (2.13), (2.14), we obtain $u = 0$, which contradicts our assumption. Therefore, for some values of the indices, the equality $\lambda_{kn} = 0$ holds. Conversely, if there is a zero eigenvalue among the eigenvalues, then for some $u \neq 0$ the equality $Lu = 0$ holds.

For the existence of the inverse operator L^{-1} it is necessary and sufficient that the kernel of the operator L consists only of the zero element, i.e.

$$\ker L = \{u \in D(L),\ Lu = 0\} = \{0\},$$

and for this, it is necessary and sufficient that the condition is satisfied$\lambda_{kn} \neq 0$, $\forall\, k, n = 0, 1, 2, ...$

If $\lambda_{kn} \neq 0$, $\forall\ k, n \in N$ then according to the theory, there exists a unique inverse operator L^{-1}, i.e. the solution to the problem AP exists and unique. Obviously, for $n = 2m$ we have

$$\lambda_{k,2m} = \frac{\pi}{T}\left(\frac{1}{2} + 4m\right) + \frac{\pi^4 (2k+1)^4}{l^4} \neq 0, \quad k, m = 0, 1, 2, ...$$

Therefore, from (2.12), when $n = 2m + 1$ we obtain the necessary and sufficient condition for the invertibility of the operator

$$\lambda_{k,2n+1} = \frac{\pi^4 (2k+1)^4}{l^4} - \frac{\pi}{T}\left(\frac{5}{2} + 4m\right) \neq 0, \quad k, m = 0, 1, 2, ...,$$

which holds when the following conditions are met

$$\frac{l^4}{2\pi^3 T} \neq \frac{(2k+1)^4}{4m+1}. \tag{3.1}$$

Thus, the necessary and sufficient condition for the invertibility of the operator L will be condition (3.1), which excludes the coincidence of eigenvalues (2.12) with the zero eigenvalue.

Since the right side of the inequality (3.1) is a rational number for any $k, m \in N$, and π is an irrational number, then, for example, this inequality holds for any fixed $l, T \in Q$.

Let us now construct the inverse operator L^{-1}. Let $u \in D(L)$, $f \in R(L)$ and the equality $Lu = f$ is hold. Expanding the left and right sides of this equality into a Fourier series with respect to the system $\{u_{kn}^{(i)}\}$, $k, n = 0, 1, 2, ...$, we have

$$Lu = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2}(Lu, u_{kn}^{(i)})u_{kn}^{(i)}(x,t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2}\lambda_{kn}(u, u_{kn}^{(i)})u_{kn}^{(i)}(x,t),$$

$$f = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2}(f, u_{kn}^{(i)})u_{kn}^{(i)}(x,t).$$

Substituting all of this into the equation , and comparing the coefficients, we get

$$(u, u_{kn}^{(i)}) = \frac{(f, u_{kn}^{(i)})}{\lambda_{kn}}.$$

Then the sought solution $u(x,t)$ of the equation $Lu = f$ can be written as

$$u(x,t) = L^{-1}f = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2}\frac{f_{kn}^{(i)}}{\lambda_{kn}}u_{kn}^{(i)}(x,t), \tag{3.2}$$

provided that $\lambda_{kn} \neq 0$ for $\forall\, k, n$. This is valid, since

$$\lambda_{kn} = (-1)^n \frac{\pi}{T}\left(\frac{1}{2} + 2n\right) + \frac{\pi^4(2k+1)^4}{l^4} \neq 0, \quad \forall\ k,\ n\ \in\ N_0.$$

Thus, the inverse operator $L^{-1}:\ R(L) \to D(L)$ is formally defined by expression (3.2). The resulting solution (3.2) is a strong solution to the problem (1.1)-(1.3) [9]. Let us denote by the closure of the operator $\overline{L}$, originally defined L on the set of regular functions $D(L)$. Then, if $R\left(\overline{L}\right) = L_2(\Omega)$, any function f can be the right-hand side of the equation, and there exists a strong solution; this condition is equivalent to the condition $\lambda_{kn} \neq 0,\ \forall\ k,\ n\ \in N_0$,i.e., there must be no zeros among the eigenfunctions. Thus, the following theorem is proven.

Theorem 3.1. *For the uniqueness of the strong solution of the boundary value problem (1.1) - (1.3), it is necessary and sufficient that condition (3.1) be satisfied. When this condition is satisfied, the strong solution of the problem exists and has the form*

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2}\frac{\left(f,\ u_{kn}^{(i)}\right)}{\lambda_{kn}}u_{kn}^{(i)}(x,t)\,,$$

for all $f(x,t) \in L_2(\Omega)$, satisfying the condition

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2}\left|\frac{\left(f,\ u_{kn}^{(i)}\right)}{\lambda_{kn}}\right|^2 < \infty,$$

where $u_{kn}^{(i)}(x,t)$, $i = 1,2$ and λ_{kn} are defined by (2.12) - (2.14).

4. On self-adjointness in the essential sense of an operator L.

The following statements hold [9]:

Lemma 4.1. *Let A be a symmetric linear operator in a Hilbert space H. If the operator A has a complete system of eigenvectors, then its closure $\overline{A}$ is a self-adjoint operator in H.*

Theorem 4.2. *The operator $Lu \equiv u_t(x, T-t) - u_{xxxx}(x,t)$, acting in the Hilbert space $H = L_2(\Omega)$, where $\Omega = (0,l) \times (0,T)$, with the domain*

$$D(L) := \left\{ u \in C^{4,1}(\Omega) \cap C(\overline{\Omega}) \;\middle|\; \begin{array}{ll} u(x,0) = 0, & 0 \le x \le l \\ \left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} + \left.\frac{\partial^k u}{\partial x^k}\right|_{x=l} = 0, \quad k = \overline{0,3}, & 0 \le t \le T \end{array} \right\}$$

which is symmetric and allows for a self-adjoint closure in $L_2(\Omega)$. That is, its closure $\overline{L}$coincides with the adjoint operator L^*, $\overline{L} = L^*$, and therefore the operator is essentially self-adjoint.

From Theorems 3.1 and 4.2, it follows

Theorem 4.3. *If*

$$\frac{l^4}{2\pi^3 T} \ne \frac{(2k+1)^4}{4n+1}, \quad \forall\, k, n \in N_0,$$

then the inverse operator L^{-1} exists and self-adjoint.

Proof. According to Lemma 4.1, a symmetric operator with a complete system of eigenfunctions is essentially self-adjoint, i.e. $\overline{L} = L^*$ is its closure. From this, it follows that the $L^{-1}-$inverse operator is also self-adjoint, $(L^{-1})^* = (L^*)^{-1} = (\overline{L})^{-1} = L^{-1}$. □

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