Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 174–178
DOI: 10.29229/uzmj.2026-1-19

Normality of quasitrace and AW*-completion of the real C*-subalgebras

Rakhmonova N.

Abstract. The paper studies quasitraces on real C*-algebras, and AW*-completion of C*-subalgebras with respect to the d_τ-metric generated by quasitrace τ. It is proved that the d_τ-closure of unital real C*-subalgebra B of real C*-algebra R is the smallest real AW*-subalgebra of R containing B. To prove this, it was necessary to obtain a key result concerning the maximal Abelian self-adjoint subalgebra (masa), in connection with which Abelian algebras are studied separately. It is proved that for a compact Hausdorff space X the algebra $C_r(X)$ of all continuous real functions on X is a real abelian AW*-algebra if and only if X is Stonean. Moreover, it has been proven that a unital real C*-algebra is a real AW*-algebra if and only if every masa has Stonean spectrum, and this is equivalent to the fact that every masa is monotone complete.

Keywords: Real C*-algebras, AW*-algebras, quasitrace, monotone completeness, maximal abelian self-adjoint subalgebra (masa).
MSC (2020): 46L10, 46K10, 46L05

1. Introduction

The theory of AW*-algebras, originally introduced by Kaplansky as a generalization of von Neumann algebras without assuming a distinguished Hilbert space, continues to play an important role in the study of non-commutative measure theory, projection lattices, and completions of C*-algebras. While the complex case has been extensively developed over the past decades, the corresponding theory for real C*-algebras and real AW*-algebras remains less explored, despite growing interest in real structures in operator algebras, Jordan algebras, and applications in mathematical physics.

This paper is devoted to the study of quasitraces on real C*-algebras and to the description of AW*-completions of real C*-subalgebras with respect to the metric d_τ induced by a quasitrace τ. The central result establishes that, under suitable conditions, the d_τ-closure of a unital real C*-subalgebra B inside a real C*-algebra R coincides with the smallest real AW*-subalgebra of R containing B.

A key step in proving this statement requires a detailed understanding of maximal abelian self-adjoint subalgebras (masas) in the real setting. For this reason, the paper first investigates abelian real C*-algebras and their AW*-properties.

It is shown that, for a compact Hausdorff space X, the algebra $C_r(X)$ of all continuous real-valued functions on X is a real abelian AW*-algebra if and only if X is Stonean (extremely disconnected). Furthermore, a unital real C*-algebra is a real AW*-algebra if and only if every masa has Stonean spectrum; this condition is in turn equivalent to every masa being monotone complete.

These characterizations closely parallel well-known results in the complex setting [1], but their proofs in the real case require careful adaptation, often via the method of passing to the enveloping complex C*-algebra $R + iR$ and applying known complex results there.

The notion of quasitrace plays a crucial role throughout the work. We adopt the definition suitable for real C*-algebras and establish its close relationship with quasitraces on the complexification. Building on this, we study normality of quasitraces (in the sense of additivity over orthogonal projections) and prove that if the closed unit ball is complete with respect to the d_τ-metric, then the algebra is a real AW*-algebra and the quasitrace is normal.

Finally, combining these tools, we obtain the main theorem concerning the AW*-completion in the real finite setting (when both R and its complexification are AW*-algebras), which provides a natural real analogue of corresponding completion results in the complex theory [1, 2].

The proofs rely heavily on the enveloping complex algebra technique, together with results from [1, 2] and structural properties of real AW*-algebras developed in [3, 4, 5]. The proofs of basically all the results were obtained by the so-called method of enveloping (complex) algebra. In this case, the results from the papers [1] and [2] were used, in which the above results were obtained in the complex case.

1.1. **Preliminaries.** A Banach *-algebra A over a field $\mathbb{C}$ is called a *C*-algebra* if $\|x^*x\| = \|x\|^2$, for any $x \in A$. By a *real C*-algebra* we mean a real Banach *-algebra A such that the relation $\|x^*x\| = \|x\|^2$ holds and the element $1 + x^*x$ is invertible for any $x \in A$. Let A be a ring and S a non-empty subset of A. Assume that $R(S) = \{x \in A \mid sx = 0 \text{ for all } s \in S\}$ and call $R(S)$ the right annihilator of S. Similarly, $L(S) = \{x \in A \mid xs = 0 \text{ for all } s \in S\}$ denotes the left annihilator of S. A *Baer *-ring* is a ring A such that for every non-empty subset S of A, $R(S) = gA$ for a suitable projection g. The equality $L(S) = ((R(S^*)))^* = ((hA))^* = Ah$ (for some projection h) shows that this definition can also be given through the left annihilator. *AW*-algebra* is a C*-algebra, which is also a Baer *-ring.

Let A be a C*-algebra with identity and let A_h be its Hermitian part. A *quasitrace* τ on A is a function $\tau : A \to \mathbb{C}$ satisfying the conditions:

(*i*) $\overline{\tau}(x^*x) = \overline{\tau}(xx^*) \geq 0$, for all $x \in A$;

(*ii*) $\overline{\tau}(a + ib) = \overline{\tau}(a) + i\overline{\tau}(b)$, for all $a, b \in A_h$;

(*iii*) $\overline{\tau}$ is linear on any abelian C*-subalgebra B of A.

Definition 1.1. Let R be a unital real C*-algebra. A *quasitrace* τ on R is a function $\tau : R \to \mathbb{R}$ that satisfies:

(i') $\tau(x^*x) = \tau(xx^*) \geq 0$, for $x \in R$;

(ii') $\tau(a + b) = \tau(a)$, for $a \in R_h$, $b \in R_k$, where $R_k = \{b = -b^*, b \in R\}$;

(iii') τ is linear on any abelian C*-subalgebra B of R.

Theorem 1.2. *[5] If $\overline{\tau}$ is a quasitrace on the C*-algebra $A = R + iR$, then its restriction to the real C*-algebra R, defined as $\tau(a + b) = \overline{\tau}(a)$, $a \in R_h, b \in R_k$ is a quasitrace on R.*

Conversely, if τ is a quasitrace on R, then its extension $\overline{\tau}$ to $A = R + iR$, defined as $\overline{\tau}(x + iy) = \tau(x) + i\tau(y)$, is a quasitrace on A, where $x, y \in R$.

2. Abelian and monotone complete AW*-algebras.

Proposition 2.1. *Let R be a real AW*-algebra. Let $Q \subseteq R$ be a *-subalgebra, and let Q' be the relative commutant of Q, that is, $Q' = \{x \in R : xy = yx, \forall y \in Q\}$.*

(1) *If $Q = Q''$, then Q is a real AW*-subalgebra.*

(2) *The center $Z(R)$ of R is a real AW*-subalgebra.*

Proof. 1) Let $N = Q + iQ$. Then $N' = Q' + iQ'$ (see [4]), therefore $N'' = Q'' + iQ'' = Q + iQ = N$. By [1, Proposition 1.8] N is AW*-subalgebra of $R + iR$, hence by [3, Proposition 4.3.1] Q is a real AW*-subalgebra of R.

2) By [1, Proposition 1.8] $Z(R + iR)$ is AW*-subalgebra of $R + iR$, therefore $Z(R)$ is a real AW*-subalgebra of R. □

Definition 2.2. A compact Hausdorff space X is called *Stonean* (or extremely disconnected) if the closure of every open set is open again. X is called *Hyperstonean* if it is Stonean (i.e., extremely disconnected) and the support of every positive Radon measure on X is clopen.

Theorem 2.3. *Let X be Stonean, and let $C_r(X)$ be the algebra of all continuous real functions on X. Then $C_r(X)$ is a real AW*-algebra.*

Proof. Let's consider algebra $C(X)$ of all continuous complex function on X. It is easily to see that $C(X) = C_r(X) + iC_r(X)$. By [1, Theorem 1.10] $C(X)$ is AW*-algebra. Then by [3, Proposition 4.3.1] $C_r(X)$ is a real AW*- algebra. □

Theorem 2.4. *Let X be compact Hausdorff space. If $C_r(X)$ is a real abelian AW*-algebra such that $C(X) = C_r(X) + iC_r(X)$ AW*-algebra, then X is Stonean.*

Proof. Since $C(X)$ is AW*-algebra, by [1, Theorem 1.11] the space X is Stonean. □

Remark 2.5. Following the same scheme of proof of [1, Theorem 1.11], one can prove Theorem 2.4 without the assumption that $C(X) = C_r(X) + iC_r(X)$ is AW*-algebra.

Let us recall that a complex or real C*-algebra called *monotone complete* if every upward directed and norm-bounded set of self-adjoint elements has a least upper bound.

Proposition 2.6. *Let R be a real C*-algebra. If $A = R + iR$ is monotone complete, then R is monotone complete.*

Proof. Let $(a_n) \subset R$ be a bounded monotone increasing sequence of self-adjoint elements. Since the algebra A is monotone complete, then the sequence $(a_n) \subset A$ has a least upper bound in A, which we denote by a. Since $a_n \in R$ $(n \in \mathbb{N})$, then $a \in R$, hence R is monotone complete. □

Theorem 2.7. *If X is Stonean space, then $C_r(X)$ is monoton complete.*

Proof. By [1, Theorem 1.13] *-algebra $C(X)$ is monotone complete. Then by Proposition2.6, $C_r(X)$ is also monoton complete. □

In the future, the *maximal abelian self-adjoint subalgebra* is briefly written as *masa*. Let us present one auxiliary result.

Lemma 2.8. *Let R be a unital real C^*-algebra such that every maximal abelian self-adjoint subalgebra (masa) is monotone complete. Let P be a family of commuting projections and L be the set of all projections that are lower bounds for P. Then:*

(1) *L is upward directed.*

(2) *P has the greatest lower bound.*

Proof. The proof of the lemma is similar to the proof of [1, Lemma 1.14]. □

Theorem 2.9. *Let X be a compact Hausdorff space. Then $C_r(X)$ is isomorphic to a real von Neumann algebra if and only if X is a Hyperstonean space.*

Proof. Since $C(X) = C_r(X)+iC_r(X)$ is isomorphic to a (complex) von Neumann algebra, then by [6, Theorem 1.18] X is a Hyperstonean space, and conversely. □

Theorem 2.10. *Let R be a unital real C*-algebra. Then the following are equivalent:*

1) *R is a real AW^*-algebra;*

2) *every masa has Stonean spectrum;*

3) *every masa is monotone complete.*

Proof. 1) $\Rightarrow$ 2). Let $Q \subseteq R$ be a masa. Then $Q = Q' = Q''$, hence by Proposition2.1, Q is a real AW*-subalgebra. From the Theorem 2.4 , we know that the spectrum X is Stonean.

2) $\Rightarrow$ 3). This follows from Theorem 2.7.

Further, the equivalence of these conditions to condition 1) is shown similarly to the complex case. □

3. Normality of τ and AW*-completion of the *-subalgebras with respect to the d_τ-metric.

Lemma 3.1. *Let R be a real C^* algebra and τ a faithful quasitrace on R. Then the closed unit ball of A is closed in d_τ.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the closed unit ball of R, converging to x in d_τ. Consider the sequence $a_n = x_n^* x_n$ and the element $a = x^*x$. Since the product is continuous in d_τ on norm-bounded sets, it is obvious that the sequence a_n converges to a in d_τ, and we can also deduce that for every $p \in \mathbb{N}$, the sequence a_n^p converges to a^p in d_τ. By continuity we obtain $\tau(a_n^p) \to \tau(a^p)$ for every $p \in \mathbb{N}$. Let μ_n be the measure on the spectrum $\sigma(a_n)$ given by the linear functional $\tau|_{C^*(a_n,1)}$, and let μ be the measure on the spectrum $\sigma(a)$ given by the linear functional $\tau|_{C^*(a,1)}$. We can consider all the measures as measures in the interval $J = [0, \max\{1, \|a\|\}]$, because all a_n are in the closed unit ball of A. Since $\tau(a_n^p) \to \tau(a^p)$ for all $p \in \mathbb{N}$, we see that μ_n converges to μ in the w^*-topology on $C(J)^*$. Furthermore, μ_n has support in $[0, 1]$ for all $n \in \mathbb{N}$, hence, μ has also support in $[0, 1]$. From the fact that τ is faithful, we obtain $\mathrm{supp}(\mu) = \sigma(a)$, and then the C^*-equation gives $\|x\|^2 = \|a\| \leq 1$. □

Let us recall that a quasitrace τ is called *normal* if for every orthogonal family of projections $(p_i)_{i\in I}$ the following holds: $\tau\Big(\sup_{i\in I} p_i\Big) = \sum_{i\in I} \tau(p_i)$. Put $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$ and $d_\tau(x, y) = ||x - y||_2^{2/3}$, $x, y \in A$. Then d is a metric on A (see [1],[2]).

Proposition 3.2. *Let R be a real C^*-algebra and τ a faithful normalized trace on R. If the closed unit ball of R is complete in the $\|\cdot\|_{2,\tau}$-norm, then R is a real von Neumann algebra and τ is normal.*

Proof. Let $A = R + iR$. It is not difficult to show that the closed unit ball of A is also complete in the $\|\cdot\|_{2,\overline{\tau}}$-norm. Then by [1, Lemma 2.20] A is a (complex) von Neumann algebra and $\overline{\tau}$ is normal. Hence R is a real von Neumann algebra and τ is normal. □

Theorem 3.3. *Let R be a real C^*-algebra and τ a faithful quasitrace on R. If the closed unit ball of R is complete in d_τ, then R is an real AW*-algebra and τ is normal.*

Proof. From the Theorem 2.10, we know that it suffices to show that every masa has Stonean spectrum. So, let Q be a masa in R. By Lemma 3.1 the closed unit ball of B is closed in d_τ, therefore it is also complete in d_τ. Since τ is linear on B, $\|.\|_{2,\tau}$ is a norm, and the closed unit ball is also complete in this norm. By Proposition 3.2 B is a real von Neumann algebra, and $\tau|_B$ is normal. By Theorem 2.9 B has Hyperstonean spectrum, in particular, it is Stonean. Then by Theorem 2.10 R is a real AW*-algebra. The normality of τ on every masa as a trace ensures that τ is also normal as a quasitrace. □

Now we will prove one of the main results of the paper.

Theorem 3.4. *Let R be a real finite AW*-algebra such that $R + iR$ is AW*-algebra. If B is a unital real C^*-subalgebra of R, then the d_τ-closure of B is the smallest real AW*-subalgebra of R containing B.*

Proof. Let τ a faithful normal quasitrace on R and let $\overline{\tau}$ be its extension to $A = R + iR$, which is also a faithful normal. Let $B_c = B + iB$. By [1, Theorem 2.26] algebra $\overline{B_c}^{d_{\overline{\tau}}} = \overline{B}^{d_\tau} + i\overline{B}^{d_\tau}$ is the smallest (complex) AW*-subalgebra of A containing B_c. Then by [3, Proposition 4.3.1] $\overline{B}^{d_\tau}$ is a real AW*-subalgebra of R, and in view of the above $\overline{B}^{d_\tau}$ is the smallest real AW*-subalgebra of R containing B. □

References

[1] Fehlker F., Quasitrace and AW*-bundles. University of Muenster, Muenster, (2018).

[2] Haagerup U., Quasitraces on exact C*-algebras are traces. C.R. Math. Acad. Sci. Soc. R. Can. (2014) Vol. 36, Iss. 2, P. 67–92.

[3] Ayupov S., Rakhimov A., Real W*-algebras. Actions of groups and Index theory for real factors. VDM Publishing House Ltd., Beau-Bassin, (2010).

[4] Ayupov S., Rakhimov A., Usmanov S., Jordan, Real and Lie Structures in Operator Algebras. Kluwer Academic Publishers, Dordrecht, (1997).

[5] Kim D., Rakhimov A., Quasitraces on real C*- and AW*-algebras. Uzbek Mathematical Journal, (2025), Iss. 3, P. 121–125.

[6] Takesaki M., Theory of Operator Algebras. Springer-Verlag, New York, (1979).

Nilufarkhon Rakhmonova,
Department of Digital technologies and Mathematics,
Kokand university, Kokand, Uzbekistan
e-mail: rahmonovanilufar406@gmail.com