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Quasitraces on exact real C*-algebras are traces

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Abstract. In this paper, n-quasitraces on real C*-algebras are studied. It is proved that if R is a real C*-algebra, then the natural extension of an n-quasitrace of R to $R+iR$ is also an n-quasitrace, and conversely, the restriction of an n-quasitrace from $R+iR$ to R is also an n-quasitrace. In 1982, Blackadar and Handelman proved that every quasitrace on an AW*-algebra is a 2-quasitrace. In this paper, a real analogue of that result is obtained. However, in the general case (i.e., for C*-algebras), this result does not hold. Using Kirchberg's example – where a unital C*-algebra and its quasitrace are constructed such that the quasitrace is not a 2-quasitrace (and therefore is not a trace) – a similar example is constructed in the real case. The paper also studies the properties of 2-quasitrace on real C*-algebras. As is known, Kaplansky asked whether every (2-) quasitrace on a C*-algebra linear, i.e., a trace. This question remains open to this day. Haagerup has a positive answer to this question in the case where the C*-algebra is unital and exact. In this paper, a real analogue of Haagerup's result is proved.

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1. Introduction

It is known that AW*-algebras are a generalization of W*-algebras (von Neumann algebras), and naturally, the question arises about the generalization of results obtained for W*-algebras to AW*-algebras, which is quite relevant. It is also pertinent to investigate under which conditions (or which) AW*-algebras are W*-algebras. As is well know, in the study and classification W*-algebras, the concept of a trace–alongside the role of projections–plays a significant part. For example, as shown by Takesaki, a W*-algebra is finite if and only if there exists a separating family of finite normal traces on it. This illustrates why C*-algebras are relatively less studied: some of them do not even have non-trivial projections, let alone traces.

On the other hand, AW*-algebras are relatively better studied because these algebras possess a sufficient number of projections. However, there are also problems concerning traces for these algebras. In some works (for example, those by Wright), the existence of a trace on an AW*-algebra is assumed for convenience.

In 1982, Blackadar and Handelman introduced an analogue of the trace called a quasitrace. Although this concept does not fully replace the trace, it has allowed some researchers to obtain results that are analogous to those available for traces. In partcular, U. Haagerup studied certain properties of quasitraces and proved that, in an exact C*-algebra, every quasitrace is in fact a trace.

In this paper, we study quasitraces on real C*-algebras and obtain a real analogue of Haagerup's result.

2. Preliminaries

A complex Banach *-algebra A is called *C*-algebra* if $\|x^*x\| = \|x\|^2$ for all $x \in A$. By a *real C*-algebra* we mean a real Banach *-algebra R such that the relation $\|a^*a\| = \|a\|^2$ holds and the element $1 + a^*a$ is invertible for any $a \in R$ (see [1], [2]). A bijective linear mapping $\alpha : A \to A$ is called a **-antiautomorphism*, if $\alpha(x^*) = \alpha(x)^*$ and $\alpha(xy) = \alpha(y)\alpha(x)$, for all $x, y \in A$. A mapping α is called *involutive* if $\alpha^2 = id$. It is directly shown that a real C*-algebra R generates a natural involutive *-antiautomorphism of $A = R + iR$, namely $\alpha(x + iy) = x^* + iy^*$, where $x, y \in R$. It is clear that $R = \{z \in A : \alpha(z) = z^*\}$. Conversely, given a C*-algebra A and any involutive *-antiautomorphism α on A, the set $\{z \in A : \alpha(z) = z^*\}$ is real C*-algebra.

Let A be an *-algebra and let S be a nonempty subset of A. The sets $R(S) = \{x \in A \ : \ sx = 0$ for all $s \in S\}$ and $L(S) = \{x \in A \ : \ xs = 0$ for all $s \in S\}$ are called the right-annihilator and the left-annihilator of S, respectively. An *-algebra A is called a Baer *-algebra if for any non-empty $S \subset A$ we have $R(S) = gA$ for an appropriate projection g. Since $L(S) = (R(S^*))^* = (hA)^* = Ah$ the definition is symmetric and can be given in terms of the left-annihilator and a suitable projection h. Here $S^* = \{s^* | \ s \ \in \ S\}$. A real (or complex) C*-algebra R which is a Baer *-algebra is called a real (or complex) AW*-algebra. A linear functional τ on A is called *positive* if $\tau(x^*x) \geq 0$ for all $x \in A$. A positive linear functional with $\|\tau\| = 1$ is called a *state*. A state is called *trace* if it satisfies the condition $\tau(xy) = \tau(yx)$, for all $x, y \in A_+$.

Let everywhere R be a unital real C*-algebra and $A = R + iR$ be an enveloping C*-algebra of R.

Definition 2.1. [3] A quasitrace τ on A is a function $\tau : A \to \mathbb{C}$ that satisfies the following conditions

(*i*) $\tau(x^*x) = \tau(xx^*) \geq 0$, $x \in A$;

(*ii*) $\tau(a + ib) = \tau(a) + i\tau(b)$, for $a, b \in A_h$;

(*iii*) τ is linear on an abelian C*-subalgebra B of A.

Furthermore, τ is called an $n-quasitrace$ for $n \in \mathbb{N}, n \geq 2$ if there exists a 1-quasitrace $\tau_n : M_n(A) \to \mathbb{C}$ such that $\tau(x) = \tau_n(x \otimes e_{11})$.

Definition 2.2. [4] A quasitrace τ on R is a function $\tau : R \to \mathbb{R}$ that satisfies the following conditions

(i') $\tau(x^*x) = \tau(xx^*) \geq 0$, $x \in R$;

(ii') $\tau(a + b) = \tau(a)$, for $a \in R_h, b \in R_k$;

(iii') τ is linear on an abelian C*-subalgebra B of R.

Furthermore, τ is called an $n-quasitrace$ for $n \in \mathbb{N}, n \geq 2$ if there exists a 1-quasitrace $\tau_n : M_n(R) \to \mathbb{R}$ such that $\tau(x) = \tau_n(x \otimes e_{11})$.

We can see that definitions of quasitrace in real and complex cases are slightly different. In the next two theorems we naturally consider the *restriction* of a quasitrace from A to R, and conversely, the *extension* of a quasitrace from R to A

Theorem 2.3. *[4] If $\overline{\tau}$ is a quasitrace on the C*-algebra $A = R + iR$, then its restriction to the real C*-algebra R, defined as $\tau(a + b) = \overline{\tau}(a)$, $a \in R_h, b \in R_k$ is a quasitrace on R.*

Conversely, If τ is a quasitrace on R, then its extension $\overline{\tau}$ to $A = R + iR$, defined as $\overline{\tau}(x + iy) = \tau(x) + i\tau(y)$, is a quasitrace on A, where $x, y \in R$.

3. 2-quasitraces on a real C*-algebras.

Using Theorem 2.3 the following results are directly proved.

Proposition 3.1. *Let $\overline{\tau}$ be an n-quasitrace on A. Then $\tau(a + b) = \overline{\tau}(a)$, $a \in R_h$, $b \in R_k$ is an n-quasitrace on R.*

Proof. Let $\overline{\tau}$ be an n-quasitrace on A. Then according to the definition of n-quasitrace, there exists a 1-quasitrace $\overline{\tau}_n : M_n(A) \to \mathbb{C}$, such that

$$\overline{\tau}(x) = \overline{\tau}_n(x \otimes e_{11}), \qquad x \in A.$$

If we define the restriction of $\overline{\tau}_n$ to $M_n(\mathbb{R})$ as follows

$$\tau_n : M_n(R) \to \mathbb{R}, \ \tau_n((a + b) \otimes e_{11}) = \overline{\tau}_n(a \otimes e_{11}).$$

Hence $\tau : R \to \mathbb{R}$ and $\tau(a + b) = \tau_n((a + b) \otimes e_{11}) = \overline{\tau}_n(a \otimes e_{11}) = \overline{\tau}(a)$. Therefore, τ is an n-quasitrace on R. □

Proposition 3.2. *Let τ be an n-quasitrace on R. Then $\overline{\tau}(x + iy) = \tau(x) + i\tau(y)$, $x, y \in R$ is an n-quasitrace on A.*

Proof. Let τ be an n-quasitrace on R. According to the definition of n-quasitrace, there exists a 1-quasitrace $\tau_n : M_n(R) \to \mathbb{R}$, such that

$$\tau(c) = \tau_n(c \otimes e_{11}), \qquad c \in R.$$

So, we can extend τ to quasitrace $\overline{\tau}$ on $M_n(A)$. So $\overline{\tau}_n : M_n(A) \to \mathbb{C}$, $\overline{\tau}_n((x+iy)\otimes e_{11}) = \tau_n(x \otimes e_{11}) + i\tau_n(y \otimes e_{11})$. This $\overline{\tau}_n$ is a 1-quasitrace on $M_n(A)$. Then for arbitrary $z \in A$, we have

$$\begin{aligned} \overline{\tau}(z) &= \overline{\tau}(x+iy) = \tau(x) + i\tau(y) = \tau_n(x \otimes e_{11}) + i\tau_n(y \otimes e_{11}) \\ &= \overline{\tau}_n((x+iy)\otimes e_{11}) = \overline{\tau}(z \otimes e_{11}). \end{aligned}$$

Therefore, $\overline{\tau}$ is an n-quasitrace on A. □

In [3, Corollary II.1.10] Blackadar and Handelman proved that *every quasitrace on an AW*-algebra is a 2-quasitrace*. Below, we will prove a real analogue of this result.

Theorem 3.3. *Every quasitrace on a real AW*-algebra is a 2-quasitrace.*

Proof. Let τ be a quasitrace on a real AW*-algebra. Then, by Theorem 2.3, its extension $\overline{\tau}$ to $A = R + iR$, such that $\overline{\tau}(x+iy) = \tau(x) + i\tau(y),\ x, y \in R$ is a quasitrace on A. Then by [3, Corollary II.1.10] $\overline{\tau}$ is a 2-quasitrace on A. According to Proposition 3.1, τ is a 2-quasitrace on R, defined as $\tau(a+b) = \overline{\tau}(a),\ a \in R_h,\ b \in R_k$. □

In the general case, i.e. for C*-algebras, this result is not true. Kirchberg gave an example of a quasitrace on a unital C*-algebra which is not a 2-quasitrace (see [5, 23§, 1274p.]). In particular, this quasitrace is not a trace. Using Kirchberg's example we can construct its real analogue as follows.

Example 3.4. Let A be a unital C*-algebra, constructed in [5, 23§, 1274p.] and let $\overline{\tau}$ is a quasitrace on A, which is not a 2-quasitrace (therefore it is not a trace – nonlinear). Let $\alpha : A \to A$ be an arbitrary involutive *-antiautomorphism. Then consider a real C*-algebra $R = \{x \in A : \alpha(x) = x^*\}$. Then we have $A = R + iR$. By Theorem 2.3 $\tau(a+ib) = \overline{\tau}(a)$ $(a \in R_s, b \in R_k)$ is a quasitrace on R, which according to Propositions 3.1 and 3.2 is not a 2-quasitrace, therefore, it is also not a trace.

In [3, Proposition II.4.1] Blackadar and Handelman also proved, that *every 2-quasitrace is an n-quasitrace*, for every $n \in N$. We will prove the real analogue of this result in a more general form. More specifically, we will prove the real analogue of F.Fehlker's result [6, Corollary 2.11].

Theorem 3.5. *Let τ be a 2-quasitrace on a real C*-algebra R. Then*

(i) *τ is an n-quasitrace for every $n \in \mathbb{N}$;*

(ii) *τ is order-preserving on R_{sa};*

(iii) *τ is continuous;*

(iv) *τ is bounded.*

Proof. (i) By Proposition 3.2, $\overline{\tau}(x+iy) = \tau(x) + i\tau(y)$ $(x, y \in R)$ is a 2-quasitrace on A. Then by [6, Corollary 2.11], $\overline{\tau}$ is an n-quasitrace for every $n \in \mathbb{N}$. Therefore, according to Proposition 3.1, a quasitrace τ is an n-quasitrace on R.
(ii) Since $\overline{\tau}$ is a 2-quasitrace, by [6, Corollary 2.11] $\overline{\tau}$ is order-preserving on A_{sa}, therefore τ is order-preserving on R_{sa}.
(iii) By [6, Corollary 2.11] $\overline{\tau}$ is continuous, therefore, τ is also continuous.
(iv) By [6, Corollary 2.11] $\overline{\tau}$ is bounded, hence τ is also bounded. □

In [7, Corollary 6.4] Alex Gow demonstrated that if a unital C*-algebra A admits a 2-quasitrace $\tau : A \to \mathbb{C}$, then A also admits a tracial functional. In particular, this implies that τ is linear. We have proved this result in the real case as a theorem.

Corollary 3.6. *If a unital real C*-algebra R admits a 2-quasitrace $\tau : R \to \mathbb{R}$, then it admits a tracial functional. In particular, τ is linear.*

Proof. By Proposition 3.2, the extension $\overline{\tau}$ to A is a 2-quasitrace on A. By [7, Corollary 6.4], A admits a tracial functional and $\overline{\tau}$ is linear. Then τ is also linear. □

4. Quasitraces on exact real C*-algebras.

Recall that a sequence $0 \to M \to N \to F \to 0$ is called *exact*, if there exists a monomorphism f from the algebra M to N and an epimorphism g from the algebra N to F such that $Im(f) = Ker(g)$. A real or complex C*-algebra A is called *exact* if for all pairs (B, J) of a C*-algebra B and a closed two-sided ideal J in B, the sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes B/J \to 0$$

is exact.

Whether every (2-) quasitrace on a C*-algebra is linear, i.e. a trace, is a well-known open question (as asked by Kaplansky). Haagerup has a positive answer to this question in the case that C*-algebra is unital and exact [8, Theorem 5.11]. In [9, Lemma 6.2] was proved that a real C*-algebra R is exact if and only if $A = R + iR$ is exact. Using this below we will prove a real analogue of Haagerup's result.

Theorem 4.1. *Quasitraces on exact unital real C*-algebras are traces.*

Proof. Let R be an exact unital real C*-algebra and let τ be a quasitrace on R. Then by [9, Lemma 6.2], A is also exact. By Theorem 2.3, the extension $\overline{\tau}$ to $A = R + iR$ of τ is a quasitrace on A. By [8, Theorem 5.11] $\overline{\tau}$ is a trace on A. Then τ is also trace on R. □

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