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Periodic quasi Gibbs measures for the p-adic Potts model with an external field

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Abstract. In the present paper, we study G_2-periodic p-adic quasi Gibbs measures for the p-adic Potts model with an external field on a Cayley tree of order two. We find G_2-periodic (non-translation-invariant) p-adic quasi Gibbs measures. Moreover, for the corresponding model, we show that if $|q(q-1)|_p = 1$, $\sqrt{1-q} \in \mathbb{Q}_p$ then a phase transition occurs; If $|q|_p < 1$, a quasi phase transition occurs.

Keywords: p-adic numbers, Potts model, external field, p-adic quasi Gibbs measure, phase transition, quasi phase transition
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1. Introduction

A central problem in statistical mechanics involves understanding how infinite systems behave based on their energy (Hamiltonian). This includes identifying phase transitions, where the system can exist in multiple distinct stable states. While determining all possible stable states for a given system is often extremely complex, researchers often focus their efforts on studying these states within specific simplified structures known as Cayley trees. The problem of phase transitions for some models on a Cayley tree was studied (for example, see [1, 2, 3, 4, 5, 6]).

Theories of p-adic and non-Archimedean stochastic processes have been established in previous research [7, 8]. Building upon these theories, researchers have constructed a wide range of stochastic processes using finite-dimensional probability distributions [9, 10, 11, 7, 12]. Furthermore, p-adic statistical mechanics has been developed within the framework of p-adic probability and stochastic processes [11, 13, 14, 15, 16, 17, 18]. This research has specifically focused on investigating the p-adic Ising and Potts models on the Cayley tree.

In the present paper, we study p-adic quasi Gibbs measures (including p-adic Gibbs measures) for the Potts model with an external field on the Cayley tree of order two. Note that p-adic quasi Gibbs measures were first introduced by F. Mukhamedov [5]. p-adic quasi Gibbs measures for the Potts model (without external field) studied in [5, 19, 20]. In the real case, Gibbs measures for the Potts model with external field were studied in [21, 22]. In [20] it was found that a phase transition occurs for any p for the three state Potts model. Moreover, in [5] it was proved that if $|q|_p = 1$, a quasi phase transition occurs for the $(q+1)$- state Potts model. By comparing these works, we prove that, if $|q(q-1)|_p = 1$, $\sqrt{1-q} \in \mathbb{Q}_p$ a phase transition occurs, if $|q|_p < 1$, a quasi phase transition occurs for the p-adic Potts model with an external field.

2. Preliminaries

2.1. **p-adic numbers and p-adic measure.** Let $\mathbb{Q}$ be a field of rational numbers. For a fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$ where, $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, \, x \neq 0, \\ 0, \quad x = 0. \end{cases}$$

This norm is non-Archimedean, i.e. it satisfies the strong triangle inequality: for all $x, y \in \mathbb{Q}$ $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. From this property, one gets the following facts:
1) if $|x|_p \neq |y|_p$, then $|x \pm y|_p = \max\{|x|_p, |y|_p\}$;
2) if $|x|_p = |y|_p$, then $|x - y|_p \leq |x|_p$.

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$. Any p-adic number $y \neq 0$ can be uniquely represented in the canonical form

$$y = p^{\alpha(y)}(y_0 + y_1 p + y_2 p^2 + ...),$$

where $\alpha(y) \in \mathbb{Z}$ and the integers y_j satisfy: $y_0 > 0$, $0 \leq y_j \leq p-1$. In this case $|y|_p = p^{-\alpha(y)}$. An integer $a \in \mathbb{Z}$ is called *quadratic residue modulo* p if the congruent equation $x^2 \equiv a (\mathrm{mod}\, p)$ has a solution $x \in \mathbb{Z}$.

Lemma 2.1. *[23] The equation $y^2 = a$, $0 \neq a = p^{\alpha(a)}(a_0 + a_1 p + a_2 p^2 + ...)$, $0 \leq a_j \leq p-1$, $a_0 > 0$ has a solution in $y \in \mathbb{Q}_p$ if and only if the following conditions hold:*
i) $\alpha(a)$ is even;
ii) $y^2 \equiv a_0 (\mathrm{mod}\, p)$ is solvable for $p \neq 2$; the equality $a_1 = a_2 = 0$ hold if $p = 2$.

In [24] authors have introduced new symbols "O" and "o" which allowed to simplify certain calculations. Roughly speaking, these symbols replace the notation $\equiv \ (mod\ p^k)$ without noticing about power of k. Let us recall them. A given p-adic number y by $O[y]$ we mean a p-adic number with the norm $p^{-\alpha(y)}$, i.e. $|y|_p = |O(y)|_p$. By $o[y]$, we mean a p-adic number with a norm strictly less than $p^{-\alpha(y)}$, i.e. $|o(y)|_p < |y|_p$. For instance, if $y = 1 - p + p^2$, we can write $O[1] = y$, $o[1] = y - 1$ or $o[p] = y - 1 + p$. Therefore, the symbols $O[\cdot]$ and $o[\cdot]$ make our work easier when we need to calculate the p-adic norm of p-adic numbers. It is easy to see that $y = O[x]$ if and only if $x = O[y]$.

For $c \in \mathbb{Q}_p$ and $r > 0$ we denote

$$B(c, r) = \{x \in \mathbb{Q}_p : \ |x - c|_p < r\},$$

and the set of all p-adic integers $\mathbb{Z}_p := B(0, p)$. The set $\mathbb{Z}_p^* = \mathbb{Z}_p \backslash p\mathbb{Z}_p$ is called a set of p-adic units. p-adic *exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, \frac{1}{2})$ if $p = 2$ and $x \in B(0, 1)$ if $p \neq 2$.

Put

$$\mathcal{E}_p = \left\{ x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)} \right\}.$$

The set $\mathcal{E}_p$ has following properties.

Lemma 2.2. *Let p be a prime. Then the set $\mathcal{E}_p$ has the following properties:*
(a) $\mathcal{E}_p$ is a group under multiplication;
(b) $|a - b|_p < \begin{cases} \frac{1}{2}, & p = 2; \\ 1, & p \neq 2 \end{cases}$ for all $a, b \in \mathcal{E}_p$;
(c) $|a + b|_p = \begin{cases} \frac{1}{2}, & p = 2; \\ 1, & p \neq 2 \end{cases}$ for all $a, b \in \mathcal{E}_p$;
(d) If $a \in \mathcal{E}_p$, then there is an element $h \in B(0, p^{-1/(p-1)})$ such that $a = \exp_p(h)$.

A more detailed description of p-adic calculus and p-adic mathematical physics can be found in [25], [26].

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a p-adic measure if for any disjoin $U_1, U_2, ..., U_n \in \mathcal{B}$, the following holds:

$$\mu\left(\bigcup_{j=1}^{n} U_j\right) = \sum_{j=1}^{n} \mu(U_j).$$

A p-adic measure is called *probability* if $\mu(X) = 1$. One of the important conditions is boundedness, namely a p-adic measure μ is called *bounded* if $\sup\{|\mu(U)|_p : U \in \mathcal{B}\} < \infty$. For more detail information about p-adic measures, we refer to [27, 25].

2.2. **Cayley Tree.** Let $\Gamma^k_+ = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k+1$ edges, except for the root x^0, which has k edges)[5]. Here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y\rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1\rangle, \dots, \langle x_{d-1}, y\rangle$ is called a *path* from the point x to the point y. The distance $d(x, y)$ on the Cayley tree is the length (number of edges) of the shortest path from x to y. Let us set

$$W_n = \{x \in V : \ d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m,$$

$$L_n = \{\langle x, y\rangle \in L : \ x, y \in V_n\}.$$

We define a coordinate structure in Γ^k_+: every vertex x (except for x^0) of Γ^k_+ has coordinates $(i_1, \dots, i_n)$, here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \dots, i_n)$ form level n (i.e. $d(x^0, x) = n$) of the lattice. Let us define on Γ^k_+ binary operation $\circ : \Gamma^k_+ \times \Gamma^k_+ \to \Gamma^k_+$ as follows: for any two elements $x = (i_1, \dots, i_n)$ and $y = (j_1, \dots, j_m)$ put

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m) \tag{2.1}$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \dots, i_n) \circ (0) = (i_1, \dots, i_n). \tag{2.2}$$

By means of the defined operation Γ^k_+ becomes a noncommutative semigroup with a unit. Let us denote this group $(G^k, \circ)$. Using this semigroup structure one defines translations $\tau_g : G^k \to G^k$, $g \in G_k$ by

$$\tau_g(x) = g \circ x.$$

It is clear that $\tau_{(0)} = id$.

Let $G \subset G^k$ be a sub-semigroup of G^k and $h : G^k \to Y$ be a Y-valued function defined on G^k. We say that h is G- *periodic* if $h(\tau_g(x)) = h(x)$ for all $g \in G$ and $x \in G^k$. Any G^k-periodic function is called *translation invariant*.

Now for each $m \geq 2$ we put

$$G_m = \{x \in G^k : d(x, x^0) \equiv 0 (\mathrm{mod}\, m)\}. \tag{2.3}$$

One can check that G_m is a sub-semigroup of G^k.

3. p-adic quasi Gibbs measure for the Potts model

Let $\mathbb{Q}_p$ be field of p-adic numbers and $\Phi = \{1, 2, ..., q\}$ be a finite set. A configuration σ on $A \subset V$ is defined by the function $x \in A \to \sigma(x) \in \Phi$. The set of all configurations on A is denoted by $\Omega_A = \Phi^A$ and $\Omega = \Omega_V$.
For given configurations $\sigma \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), \ if \ \ x \in V_{n-1}, \\ \omega(x), \ \ if \ x \in W_n. \end{cases}$$

It is clear that $\sigma \vee \omega \in \Omega_{V_n}$.

The (formal) Hamiltonian of p-adic Potts model with an external field is

$$H(\sigma) = J \sum_{\langle x,\, y\rangle \in L} \delta_{\sigma(x)\sigma(y)} + \alpha \sum_{x \in V} \delta_{q\sigma(x)} \tag{3.1}$$

where $J, \alpha \in B(0, p^{-1/(p-1)})$ J is a coupling constant, α is an external field and δ_{ij} is the Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 0, \ \text{if} \ i \neq j, \\ 1, \ \text{if} \ i = j. \end{cases}$$

Assume that $h : V \to \mathbb{Q}_p^{\Phi}$ is a mapping, i.e., $\mathbf{h}_x = (h_{1,x}, h_{2,x}, ..., h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ $(i \in \Phi)$ and $x \in V$. Given $n \in \mathbb{N}$, we consider a p-adic probability measure $\mu^{(n)}_{\mathbf{h},\sigma}$ on Ω_{V_n} defined by

$$\mu^{(n)}_{\mathbf{h}}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}. \tag{3.2}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor

$$Z_n^{(h)} = \sum_{\sigma\in\Omega_{V_n}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}. \tag{3.3}$$

We say that p-adic probability distributions (3.2) are compatible if all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega\in\Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \omega) = \mu_h^{(n-1)}(\sigma_{n-1}). \tag{3.4}$$

We notice that a non-Archimedean analogue of the Kolmogorov extension theorem was proved in [6, 7]. According to this theorem there exists a unique p-adic measure μ_h on $\Omega = \Phi^V$ such that for all $n \geq 1$ and $\sigma \in \Phi^{V_{n-1}}$:

$$\mu(\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n).$$

Such measure is called a *p-adic quasi Gibbs measure* corresponding to the Hamiltonian (3.1) and vector-valued function $\mathbf{h}_x$, $x \in V$. By $QG(H)$ we denote the set of all p-adic quasi Gibbs measure associated with function $\mathbf{h} = \{\mathbf{h}_x, x \in V\}$ If all values of h_x belong to the set $\mathcal{E}_p$ then it is called p-adic Gibbs measure.

Definition 3.1. [16] If there are at least two distinct $\mu, \nu \in QG(H)$ such that μ is bounded and ν is unbounded, then we say that *a phase transition occurs.* If there are two different $\mu, \nu \in QG(H)$, either both μ, ν are bounded or unbounded, then we say *a quasi phase transition occurs.*

The following statement describe conditions $\mathbf{h}_x$ guaranteing compatibility of $\mu_{\mathbf{h}}^{(n)}(\sigma)$.

Theorem 3.2. *[28] The measure $\mu_h^{(n)}(\sigma), n = 1, 2, ...$ (see (3.2)) associated with the q-state Potts model (3.1) satisfy the compatibility condition (3.4) if and only if for any $n \in \mathbb{N}$ the following equation holds:*

$$\widehat{\boldsymbol{h}}_x = \prod_{y\in S(x)} F(\widehat{\boldsymbol{h}}_y, \theta, \eta), \tag{3.5}$$

here and below a vector $\widehat{\boldsymbol{h}} = \left(\widehat{h}_1, \widehat{h}_2, ..., \widehat{h}_{q-1}\right) \in \mathbb{Q}_p^{q-1}$ is defined by a vector $\boldsymbol{h} = (h_1, h_2, ..., h_q) \in \mathbb{Q}_p^q$ as follows

$$\widehat{h}_i = \frac{h_i}{h_q},\ i = 1, 2, ..., q-1 \tag{3.6}$$

and mapping
$F : \mathbb{Q}_p^{q-1} \times \mathbb{Q}_p \to \mathbb{Q}_p^{q-1}$ is defined by $F(x; \theta, \eta) = (F_1(x; \theta, \eta), ..., F_{q-1}(x; \theta, \eta))$ with

$$F_i(x; \theta, \eta) = \frac{(\theta-1)x_i + \sum_{j=1}^{q-1} x_j + \eta}{\sum_{j=1}^{q-1} x_j + \theta\eta},\ x = \{x_i\} \in \mathbb{Q}_p^{q-1},\ i = 1, 2, ..., q-1 \tag{3.7}$$

4. Non translation-invariant two periodic quasi Gibbs measure for p-adic Potts model with an external field

In this section, we are going to construct G_2-periodic $QGMs$ for the considered model. Let G_2 be a sub semigroup of G^k (see (2.3)). We denote that

$$\mathbf{h}_x = \begin{cases} \mathbf{h}_1,\ x \in G_2, \\ \mathbf{h}_2,\ x \in G^k \setminus G_2. \end{cases}$$

From equation (3.5), we get the following system

$$\begin{cases}\widehat{\mathbf{h}}_1 = (F(\widehat{\mathbf{h}}_2,\theta,\eta))^k,\\ \widehat{\mathbf{h}}_2 = (F(\widehat{\mathbf{h}}_1,\theta,\eta))^k.\end{cases} \tag{4.1}$$

Lemma 4.1. *If the pair of numbers (x, y) satisfies the system of equations (4.1), then the pair (y, x) also satisfies the system of equations (4.1).*

Proof. The proof follows from the fact that equation (4.1) is symmetric with respect to x and y. □

We assume $\widehat{\mathbf{h}}_i = \left(\widehat{h}_i^{(1)}, \widehat{h}_i^{(2)}, ..., \widehat{h}_i^{(q-1)}\right)$. Let $\widehat{h}_i^{(j)} = h_i, j = \overline{1, q-1}$. For the sake of simplicity, we consider $k = 2$.

In this case, from (4.1) we can obtain following system of equations

$$\begin{cases}\widehat{h}_1 = \left(\dfrac{(\theta+q-2)\widehat{h}_2+\eta}{(q-1)\widehat{h}_2+\theta\eta}\right)^2,\\ \widehat{h}_2 = \left(\dfrac{(\theta+q-2)\widehat{h}_1+\eta}{(q-1)\widehat{h}_1+\theta\eta}\right)^2.\end{cases} \tag{4.2}$$

Let us denote

$$f(\widehat{h}) = \left(\frac{(\theta+q-2)\widehat{h}+\eta}{(q-1)\widehat{h}+\theta\eta}\right)^2.$$

For equation (4.2), if $\widehat{h}_1 = \widehat{h}_2$ then we get translation-invariant Gibbs measures. Our aim is to find G_2-periodic (non translation-invariant) p-adic quasi Gibbs measures. It demands to solve the following equation

$$\frac{f(f(\widehat{h}))-\widehat{h}}{f(\widehat{h})-\widehat{h}} = 0. \tag{4.3}$$

Simplifying the last equation we get

$$A\widehat{h}^2 + B\widehat{h} + C = 0, \tag{4.4}$$

where

$$\begin{aligned}
&A = (\theta\eta q + \theta^2 - \theta\eta + 2\theta q + q^2 - 4\theta - 4q + 4)^2,\\
&B = \eta^3 q\theta^3 - 2\eta^3\theta^3 + \eta^2 q^2\theta^2 + 2\eta^2 q\theta^3 + \eta^2\theta^4 + 4\eta^2 q^2\theta - 4\eta^2\theta^3 -\\
&\eta^2 q^2 - 12\eta^2 q\theta + 2\eta q^3 + 6\eta q^2\theta + 6\eta q\theta^2 + 2\eta\theta^3 + 2\eta^2 q + 8\eta^2\theta -\\
&12\eta q^2 - 24\eta q\theta - 12\eta\theta^2 - \eta^2 + 24\eta q + 24\eta\theta - 16\eta,\\
&C = \eta^2(\theta^2\eta + \theta + q - 2)^2.
\end{aligned}$$

Note that, the case $q \in \mathcal{E}_p$ requires more calculus. Therefore, we investigate this case for future work.

Lemma 4.2. *Let $q \notin \mathcal{E}_p$, $p \geq 3$. Equation (4.4) has two distinct solutions if $|q|_p = 1$ and $\sqrt{(1-q)} \in \mathbb{Q}_p$ or $|q|_p < 1$. Otherwise, there is not any solution.*

Proof. We set

$$D(\theta,\eta,q) = B^2 - 4AC.$$

We note that, equation (4.4) has a solution in $\mathbb{Q}_p$ if and only if $\sqrt{D(\theta,\eta,q)} \in \mathbb{Q}_p$.

We can rewrite $D(\theta, q)$ by the following

$$D(\theta,q) = \eta^2(\theta-1)^2(\theta+q-1)^2 D^*.$$

where,

$D^* = -4m^3qs^2 - 3m^4s - 14m^3qs + 4m^3s^2 - 3m^2q^2s - 12m^2qs^2 - 3m^4 - 10m^3q + 8m^3s-$

$$3m^2q^2 - 36m^2qs + 12m^2s^2 - 12mq^2s - 12mqs^2 - 36m^2q + 36m^2s - 24mq^2 - 12mqs+$$

$$12ms^2 - 8q^2s - 4qs^2 + 36m^2 + 24mq + 24ms + 8qs + 4s^2 - 4q^3 + 4q^2, \tag{4.5}$$

$m = \theta - 1,\ s = \eta - 1$.

It can be seen that $\sqrt{D(\theta, q)} \in \mathbb{Q}_p$ if and only if $\sqrt{D^*} \in \mathbb{Q}_p$. At first, we consider the following case.

Let $|q|_p = 1,\ q \notin \mathcal{E}_p$. Since $|m|_p < 1,\ |s|_p < 1$, we write

$$D^* = 4q^2(1-q) + o[1].$$

According to Lemma 2.1, $\sqrt{D^*} \in \mathbb{Q}_p$ is equivalent to $\sqrt{1-q} \in \mathbb{Q}_p$. So, we get that equation (4.2) has two solutions if $\sqrt{1-q} \in \mathbb{Q}_p$.

Next, we consider following case: $p \geq 3,\ |q|_p < 1$. Using (4.5), we have $D^* = 4(3m+s+q)^2 + o[(3m+s+q)^2]$. We find the condition that $\sqrt{D^*} \in \mathbb{Q}_p$ holds. So, if $p \mid q$, then equation (4.4) has two distinct solutions. It is known that the solutions of (4.4) has following forms

$$h_{1,2} = \frac{-B \pm \sqrt{D}}{2A}.$$

Lemma is proved.

According to Lemma 4.1, the pair of (h_1, h_2) is a solution of (4.2), then (h_2, h_1) also satisfies (4.2). Finding the first coefficient of these solutions of (4.4) in the canonical form, is necessary to ascertain the solution's norm. Let $|q|_p = 1, q \notin \mathcal{E}_p$.

$A = q^2(q-1)^2 + o[1]$ $B = 2q^2(q-1) + o[1]$, $D = \eta^2(\theta-1)^2(\theta+q-1)^2D^*$. From these equalities, we conclude that

$$h_{1,2} = \frac{1}{1-q} + o[1].$$

We study the case $|q|_p < 1$. Then we get

$B = -2(3(\theta-1) + (\eta-1) + q)^2 + o[p^2]$, $D = \eta^2(\theta-1)^2(\theta+q-1)^2D^* = o[p^4]$, $A = (3(\theta-1) + (\eta-1) + q)^2 + o[p^2]$. It yields that

$$h_{1,2} = \frac{2(3(\theta-1) + (\eta-1) + q)^2 + o[p^2]}{2(3(\theta-1) + (\eta-1) + q)^2 + o[p^2]} = 1 + o[1].$$

In [28], translation-invariant p-adic quasi Gibbs measures ($TIQGM$) associated with $\mathbf{h} = \{h, h, ..., h\} \in \mathbb{Q}_p^{q-1}$ for the p-adic Potts model with an external field was examined. Here the following result was given.

Theorem 4.3. *[28] Let $q \notin \mathcal{E}_p$. Then the following assertions holds:*

a) *if $|q|_p = 1,\ p > 3,\ \sqrt{1-q} \in \mathbb{Q}_p$, then there exist three $TIQGMs$ such that, one of them is bounded, the others are unbounded;*

b) *if $p = 3,\ |q|_3 = 1$ or $p > 3,\ |q|_p = 1,\ \sqrt{1-q} \notin \mathbb{Q}_p$, then there exist a unique bounded $TIQGM$;*

c) *if $p \geq 3,\ |q|_p < 1$, then there does not exist any $TIQGM$.*

Using Theorem 4.3 and Lemma 4.2, we obtain the following result.

Theorem 4.4. *Let $q \notin \mathcal{E}_p$ and $p \geq 3$. The following statements hold for the Potts model on the Cayley tree of order two:*

1) *If $p \neq 3,\ |q|_p = 1$ and $\sqrt{1-q} \in \mathbb{Q}_p$, then there exist three $TIQGMs$ and two G_2- periodic $QGMs$;*

2) *If $p = 3,\ |q|_3 = 1$ and $\sqrt{1-q} \in \mathbb{Q}_3$, then there exist a unique $TIQGM$ and two G_2- periodic $QGMs$;*

3) *If $|q|_p = 1$ and $\sqrt{1-q} \notin \mathbb{Q}_p$, then there exist a unique $TIQGM$;*

4) *If $|q|_p < 1$, then there exist two G_2-periodic $QGMs$.*

Remark 4.5. From Theorem 4.3, it can be seen that when $|q|_p = 1,\ p > 3,\ \sqrt{1-q} \in \mathbb{Q}_p$, there are three $TIQGMs$. Based on this condition and $p = 3,\ |q|_3 = 1,\ \sqrt{(1-q)} \in \mathbb{Q}_3$, we identified two different G_2- periodic $QGMs$. Furthermore, it was shown that when $p \geq 3,\ |q|_p < 1$, $TIQGMs$ do not exist, but in this work, two distinct G_2- periodic $QGMs$ were found when $p \geq 3,\ |q|_p < 1$.

5. Boundedness of two periodic p-adic quasi Gibbs measures and phase transitions

Lemma 5.1. Let $\mathbf{h}$ be a solution of (3.5), and $\mu_{\mathbf{h}}$ be an associated p-adic quasi Gibbs measure. Then for the corresponding partition function $Z_n^{(\mathbf{h})}$ the following equality holds:

$$Z_n^{(\mathbf{h})} = A_{\mathbf{h},n-1} Z_{n-1}^{(\mathbf{h})}, \tag{5.1}$$

where $A_{\mathbf{h},n} = \prod_{x\in W_n} a_{\mathbf{h}}(x)$, $\prod_{y\in S(x)} \sum_{j=1}^{q} \exp_p\{J\delta_{i,j}\}h_{j,y} = a_{\mathbf{h}}(x)h_{i,x}$, $a_{\mathbf{h}}(x) \in \mathbb{Q}_p$, $i = 1, 2, ..., q$.

Proof. The proof of this lemma follows a similar argument to that of Lemma 3.2 in [16]. Using Lemma 5.1, we get the following statement.

Lemma 5.2. Let $k = 2$. If $\mathbf{h}^{(1,2)}$ is G_2-periodic (non translation-invariant) solution of (3.5) then for the corresponding partition function $Z_n^{(\mathbf{h})}$ the following assertions true:
If n is odd, then

$$Z_n^{(\mathbf{h})} = Z_n^{(h)} = ((q-1)h_1 + \theta\eta)^{\frac{2^{n+1}-4}{3}}((q-1)h_2 + \theta\eta)^{\frac{2^{n+1}-4}{3}} Z_1^{(\mathbf{h})}; \tag{5.2}$$

If n is even, then

$$Z_n^{(\mathbf{h})} = ((q-1)h_1 + \theta\eta)^{\frac{2^{n+2}-4}{3}}((q-1)h_2 + \theta\eta)^{\frac{2^{n}-4}{3}} Z_1^{(\mathbf{h})}. \tag{5.3}$$

Proof. Let

$$h_{\sigma(x),x} = \begin{cases} h_x, & \text{if } \ \sigma(x) \in \overline{1, q-1}; \\ 1, & \text{if } \ \sigma(x) = q. \end{cases}$$

and

$$h_x = \begin{cases} h_1, & \text{if } \ |x| \ \text{ is even}; \\ h_2, & \text{if } \ |x| \ \text{ is odd}. \end{cases}$$

Consider the following cases

Case 1. Let n be odd. By Lemma 5.1, we get

$$a_h(x) = \frac{(\theta + (q-2)h_1 + \eta)^2}{h_2} = ((q-1)h_1 + \theta\eta)^2,$$

$$A_{h,n} = ((q-1)h_1 + \theta\eta)^{2^{n+1}}, \quad A_{h,n-1} = ((q-1)h_2 + \theta\eta)^{2^n}.$$

$$Z_n^{(h)} = ((q-1)h_1 + \theta\eta)^{\frac{2^{n+1}-4}{3}}((q-1)h_2 + \theta\eta)^{\frac{2^{n+1}-4}{3}} Z_1^{(\mathbf{h})}.$$

Case 2. Let n be even. By Lemma 5.1, we get

$$a_h(x) = \frac{(\theta + (q-2)h_2 + \eta)^2}{h_1} = ((q-1)h_2 + \theta\eta)^2,$$

$$A_{h,n} = ((q-1)h_2 + \theta\eta)^{2^{n+1}}, \quad A_{h,n-1} = ((q-1)h_1 + \theta\eta)^{2^n}.$$

$$Z_n^{(h)} = ((q-1)h_1 + \theta\eta)^{\frac{2^{n+2}-4}{3}}((q-1)h_2 + \theta\eta)^{\frac{2^{n}-4}{3}} Z_1^{(\mathbf{h})}.$$

Lemma is proved.

Theorem 5.3. *Let $q \notin \mathcal{E}_p$ and $p \geq 3$. If $|q|_p = 1$ and $\sqrt{1-q} \in \mathbb{Q}_p$ or $|q|_p < 1$, then the measures $\mu_{\boldsymbol{h}^{(1,2)}}$ are unbounded.*

Proof. Let $|q|_p = 1$, $\sqrt{1-q} \in \mathbb{Q}_p$. Then the measures $\mu_{\mathbf{h}^{(1,2)}}$ exist.
By Lemma 5.2 and (3.2), we get

$$|\mu^{(n)}_{\mathbf{h}^{(1,2)}}|_p = \left| \frac{\exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}}{((q-1)h_{1,2}+\theta\eta)^{\frac{2^{n+1}-4}{3}}((q-1)h_{2,1}+\theta\eta)^{\frac{2^{n+1}-4}{3}} Z_1^{\mathbf{h}^{(1,2)}}} \right|_p = \left| \frac{1}{(\theta\eta-1)^{\frac{2^{n+1}-4}{3}}(\theta\eta-1)^{2^n} Z_1^{\mathbf{h}^{(1,2)}}} \right|_p.$$

Since $|\theta\eta - 1|_p < 1$, we get following result

$$\lim_{n\to\infty} |\mu_{\mathbf{h}^{(1,2)(n)}}|_p = \infty.$$

Case 2. If $|q|_p < 1$, then there exist measures $\mu_{\mathbf{h}^{(1,2)}}$. Note that, for $|q|_p < 1$, we get $h_1^{(1,2)} = 1 + o[1]$, By Lemma 5.2, we have

$$|\mu^{(n)}_{\mathbf{h}^{(1,2)}}|_p = \left| \frac{\exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}}{((q-1)h_{1,2}+\theta\eta)^{\frac{2^{n+1}-4}{3}}((q-1)h_{2,1}+\theta\eta)^{\frac{2^{n+1}-4}{3}} Z_1^{\mathbf{h}^{(1,2)}}} \right|_p = \frac{1}{|(q+\theta\eta-1)^{\frac{2^{n+2}-8}{3}} Z_1^{\mathbf{h}^{(1,2)}}|_p}.$$

It yields that

$$\lim_{n\to\infty} |\mu^{(n)}_{\mathbf{h}^{(1,2)}}|_p = \infty.$$

Theorem 5.3 is proved.

Due to Remark 4.3 and Theorem 5.3, we have the following assertions belong to a phase transition.

Theorem 5.4. *Let $q \notin \mathcal{E}_p$ and $p \geq 3$. The following statements hold for the p-adic q-state Potts model on the Cayley tree of order two:*

- *If $|q|_p = 1$ and $\sqrt{1-q} \in \mathbb{Q}_p$, then there exists a phase transition;*
- *If $|q|_p < 1$ then there exist a quasi phase transition.*

Remark 5.5. Note that the first part of Theorem 5.4 coincides with the result of [28]. However, under the case where $|q|_p = 1$ and $\sqrt{1-q} \in \mathbb{Q}_p$, we found two distinct $QG(H)s$. Moreover, in the case $|q|_p < 1$, we found that a quasi phase transition occurs for the p-adic Potts model with an external field.

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