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On an optimal interpolation formula with derivatives in the Sobolev space

Shadimetov Kh.M., Hayotov A.R., Olimov N.N.

Abstract. In this work, the problem of constructing an optimal interpolation formula involving derivatives is studied. The values of the unknown function are required not only at the nodal points but also the values of its first three derivatives at these nodes. An upper bound for the interpolation error is obtained using an extremal function, whose explicit form is determined. Furthermore, the squared norm of the corresponding error functional is derived. Since this norm depends on the coefficients, the Lagrange function is introduced, and its partial derivatives with respect to the coefficients are computed and set equal to zero, leading to a system of equations. The resulting system is solved by the method proposed by Sobolev. To this end, the discrete analogue of the differential operator $\frac{d^2}{dx^2}$ is employed to solve the system and to determine the coefficients of the interpolation formula.

Keywords: Interpolation, splines, extremal function, error functional, norm, Sobolev space
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1. Introduction

Interpolation splines play a crucial role in modern technological advancements across various fields. They are essential in numerical analysis, computer graphics, data fitting, and numerical solving partial differential equations.

In numerical analysis, spline interpolation involves fitting low-degree polynomials to subsets of data points, ensuring a smooth and accurate approximation of complicated functions. This method is often preferred over high-degree polynomial interpolation due to its stability and precision. Splines are particularly effective in scattered data fitting, where they provide smooth curves and surfaces that pass through or near given data points. This capability is invaluable in fields like geospatial analysis and computer-aided design [1].

Additionally, in numerical solutions of partial differential equations (PDEs), especially in fluid dynamics and aerodynamics, splines are widely used to approximate solutions to the Navier–Stokes equations. For instance, cubic splines can be employed in weather prediction models to interpolate sparse atmospheric data accurately and smoothly, enhancing the efficiency and stability of numerical simulations. [2].

Moreover, in the Galerkin method, the selection of basis functions is crucial for the accuracy and efficiency of the solution. A notable approach involves using interpolation-based basis functions, which are constructed to satisfy the governing partial differential equations (PDEs) locally. This technique is particularly advantageous in problems with variable coefficients, where traditional polynomial basis functions may not capture the solution's behavior effectively.

For instance, the study [3] explored the construction of generalized plane waves (GPWs) as basis functions within a discontinuous Galerkin framework. These GPWs are designed to approximately satisfy the PDEs locally, thereby enhancing the method's efficiency and accuracy in solving wave-related boundary value problems with variable coefficients. The paper provides a detailed algorithm for constructing these functions and discusses their interpolation properties, offering valuable insights into their implementation in numerical methods.

This approach exemplifies how interpolation-based basis functions can be effectively utilized in the Galerkin method to address complex PDEs, leading to improved solution strategies in computational mathematics.

In the work [4], an optimal interpolation formula with derivatives in $L_2^{(m)}$ was also constructed, but the difference from our work is that only the values of the derivatives of the unknown function on the boundaries of the interval [0,1] were required. In addition to the above, the construction of interpolation formulas in other spaces has been studied in [5], [6], and [7]. Moreover, interpolation formulas with derivatives and their applications have been investigated in the following works: [8], [9], [10].

The next sections of the paper are organized as follows:
In Section 2, the problem of constructing an interpolation formula in the $L_2^{(4)}$ space is stated, derives an upper bound for the error of the formula, presents the norm of the error functional, and obtains the system of equations for the unknown coefficients of the interpolation formula.
Section 3 describes the algorithm for determining the unknown coefficients using the Sobolev method.
Section 4 presents numerical results and their analysis based on the analytically obtained coefficients.

2. Statement of the problem

Let the functions φ belong to the Sobolev space $L_2^{(4)}(0,1)$. Here $L_2^{(4)}(0,1)$ is the Hilbert space of functions that are square integrable with first four order derivative in the interval [0,1].
The space is equipped with the norm

$$\|\varphi\|_{L_2^{(4)}} = \sqrt{\int_0^1 (f^{IV}(x))^2 dx}.$$

Let a grid

$$\Delta:\ 0 = x_0 < x_1 < \ldots < x_N = 1$$

be given on the interval [0,1]. Assume that on this grid, the values of the itself and first three derivatives of the function are given, i.e. we have

$$\varphi(x_\beta),\ \varphi'(x_\beta),\ \varphi''(x_\beta),\ \varphi'''(x_\beta),\quad \beta = 0,1,\ldots,N. \tag{2.1}$$

Here, $x_\beta = h\beta, h = \frac{1}{N}$.
In this work, we consider the problem of optimal interpolation of functions $\varphi(x)$ given by values (2.1) at points $x_\beta,\ \beta = 0,1,\ldots,N$ in the space $L_2^{(4)}(0,1)$. For this, we consider the following

$$P_\varphi(x) = \sum_{\beta=0}^{N}\sum_{\alpha=0}^{3} C_{\beta,\alpha}\varphi^{(\alpha)}(x_\beta) \tag{2.2}$$

In this case, we have the following approximation

$$\varphi(x) \cong P_\varphi(x). \tag{2.3}$$

Here the coeffitcients $C_{\beta,\alpha}(x), \beta = 0,1,...,N$ and $\alpha = 0,1,2$ are known and $\alpha = 0$ have the form [11] :

$$C_{0,0}(x) = \begin{cases} \frac{h-x}{h}, & 0 \le x \le h, \\ 0, & h < x \le 1, \end{cases}$$

$$C_{\beta,0}(x) = \begin{cases} \frac{x+h-h\beta}{h}, & h(\beta-1) \le x \le h\beta, \\ \frac{h-x+h\beta}{h}, & h\beta < x \le h(\beta+1), \\ 0, & \text{otherwise}, \end{cases} \quad \beta = 1,2,\ldots,N-1,$$

$$C_{N,0}(x) = \begin{cases} 0, & 0 \le x \le h(N-1), \\ \frac{h-1+x}{h}, & h(N-1) < x \le 1. \end{cases}$$

The following coefficients were found in [12]:

$$C_{0,1}(x) = \begin{cases} \frac{x(h-x)}{2h}, & 0 \le x \le h, \\ \\ 0, & h < x \le 1, \end{cases}$$

$$C_{\beta,1}(x) = \begin{cases} \frac{(x-h\beta)^2+h(x-h\beta)}{2h}, & h(\beta-1) \le x < h\beta, \\ \frac{-(x-h\beta)^2+h(x-h\beta)}{2h}, & h\beta \le x \le h(\beta+1), \\ 0, \ \ otherwise, \end{cases} \quad \beta = 1,2,\ldots,N-1,$$

$$C_{N,1}(x) = \begin{cases} 0, & 0 \le x \le h(N-1), \\ \frac{(x-1)(x-1+h)}{2h}, & h(N-1) < x \le 1. \end{cases}$$

$$C_{0,2}(x)=\begin{cases}\frac{x(h-x)(2x-h)}{12h}, & 0\le x\le h,\\ 0, & h<x\le 1,\end{cases}$$

$$C_{\beta,2}(x)=\begin{cases}\frac{(x-h(\beta-1))(x-h\beta)(2x-2h\beta+h)}{12h}, & h(\beta-1)\le x\le h\beta,\\ \frac{(h\beta-x)(2h\beta+h-2x)(h\beta+h-x)}{12h}, & h\beta<x\le h(\beta+1),\\ 0, & \text{otherwise,}\end{cases}\quad \beta=1,2,\dots,N-1,$$

$$C_{N,2}(x)=\begin{cases}0, & 0\le x\le (N-1)h,\\ \frac{(x-1)(2x+h-2)(x+h-1)}{12h}, & h(N-1)<x\le 1.\end{cases}$$

and the coefficients $C_{\beta,3}(x)$ are unknown.

The error associated with the approximate equality (2.2) takes the form of the difference

$$E_{\varphi}(x)=\varphi(x)-P_{\varphi}(x). \tag{2.4}$$

It should be noted that in this work, when we consider the approximation of the form (2.3), we impose the condition that the class of functions that transforms this approximate equality into an exact equality in $L_2^{(4)}(0,1)$ space should be the class of all polynomials up to degree three. If we take $\varphi_0(x)=1$, $\varphi_1(x)=x$, $\varphi_2(x)=x^2$ and $\varphi_3(x)=x^3$ as the basis functions, the imposition is

$$E_{\varphi_i}(x)=\varphi_i(x)-P_{\varphi_i}(x)=0 \text{ or } (R,x^i)=0, i=0,1,2,3. \tag{2.5}$$

conditions on the error functional $R(x)$ is enough for the approximation formula (2.3) to be exact for polynomials up to degree three.

Then in the space $L_2^{(4)}(0,1)$ at every fixed point $x=z$ of the interval $[0,1]$ the error (2.4) defines a linear continuous functional

$$R(x,z)=\delta(x-z)-\sum_{\beta=0}^{N}C_{\beta,0}(z)\cdot\delta(x-x_\beta)+\sum_{\beta=0}^{N}C_{\beta,1}(z)\cdot\delta'(x-x_\beta)$$

$$-\sum_{\beta=0}^{N}C_{\beta,2}(z)\cdot\delta''(x-x_\beta)+\sum_{\beta=0}^{N}C_{\beta,3}(z)\cdot\delta'''(x-x_\beta). \tag{2.6}$$

In order to construct an optimal interpolation formula of the form (2.3), it is necessary to calculate the norm $\|R\|$, then we find the smallest value of this quantity in the given $C_{\beta,0}$, $C_{\beta,1}$and $C_{\beta,2}$ by the coefficients $C_{\beta,3}$. This necessity arises from the fact that, according to the Cauchy-Schwarz inequality, the estimation of the error (2.3) is expressed by the norm as follows:

$$|(R,\varphi)|\le \|R\|_{L_2^{(4)*}}\cdot\|\varphi\|_{L_2^{(4)}}.$$

It is easy to see that the norm $\|R\|_{L_2^{(4)*}}$ depends on the coefficients $C_{\beta,3}$. Then it should be found the smallest value of the norm $\|R\|_{L_2^{(4)*}}$ by the coefficient $C_{\beta,3}$. That is, it should be calculated the quantity

$$\inf_{C_{\beta,3}}\|R\|_{L_2^{(4)*}}. \tag{2.7}$$

The coefficients $\mathring{C}_{\beta,3}$ reaching the value (2.7) we call the optimal coefficients.

Thus, consequently in order to get optimal formula

- we calculate the norm $\|R\|_{L_2^{(4)*}}$,
- we find $\mathring{C}_{\beta,3}$ which gives (2.7).

To calculate $\|R\|_{L_2^{(4)*}}$, we use the definition of the extremal function [13]. The function $U_R(x)$ satisfying the following equality is called an extremal for the interpolation formula (2.3):

$$(R,U_R)=\|R\|_{L_2^{(4)*}}\cdot\|U_R\|_{L_2^{(4)}}$$

here, the extremal function corresponding to a linear continuous functional defined in the $L_2^{(m)}$ space was found by S. L. Sobolev [13], from which, as a particular case, we obtain the following:

$$U_R(x) = R(x) * G_4(x) + p_3x^3 + p_2x^2 + p_1x + p_0, \tag{2.8}$$

where

$$G_4(x) = \frac{|x|^7}{2 \cdot 7!}.$$

Now, we first calculate the convolution in equation (2.8).

$$R(x) * G_4(x) = \int_{-\infty}^{\infty} R(y) \cdot G_4(x-y)dy = \frac{|x-z|^7}{2\cdot 7!} - \sum_{\beta=0}^{N} C_{\beta,0}\frac{|x-z|^7}{2\cdot 7!}$$
$$+\sum_{\beta=0}^{N} C_{\beta,1}\frac{\mathrm{sgn}(x-x_\beta)(x-x_\beta)^6}{2\cdot 6!} - \sum_{\beta=0}^{N} C_{\beta,2}\frac{|x-z|^5}{2\cdot 5!} + \sum_{\beta=0}^{N} C_{\beta,3}\frac{\mathrm{sgn}(x-x_\beta)(x-x_\beta)^4}{48}. \tag{2.9}$$

Then the extremal function has the following form:

$$U_R(x) = \frac{|x-z|^7}{2\cdot 7!} - \sum_{\beta=0}^{N} C_{\beta,0}\frac{|x-z|^7}{2\cdot 7!} + \sum_{\beta=0}^{N} C_{\beta,1}\frac{\mathrm{sgn}(x-x_\beta)(x-x_\beta)^6}{2\cdot 6!} - \sum_{\beta=0}^{N} C_{\beta,2}\frac{|x-z|^5}{2\cdot 5!}$$
$$+\sum_{\beta=0}^{N} C_{\beta,3}\frac{\mathrm{sgn}(x-x_\beta)(x-x_\beta)^4}{48} + p_3x^3 + p_2x^2 + p_1x + p_0.$$

Taking into account the last expression, we get

$$(R, U_R) = \int_{-\infty}^{\infty} R(x)\cdot U_R(x)dx = \int_{-\infty}^{\infty} R(x)\cdot\left(R(x)*G_4(x) + p_3x^3 + p_2x^2 + p_1x + p_0\right)dx$$
$$= \int_{-\infty}^{\infty} R(x)\cdot\left(R(x)*G_4(x)\right)dx + p_3(R,x^3) + p_2(R,x^2) + p_1(R,x) + p_0(R,1)$$
$$= \int_{-\infty}^{\infty} R(x)\cdot\left(R(x)*G_4(x)\right)dx. \tag{2.10}$$

Using expression (2.9), from expression (2.10), we obtain

$$(R, U_R) = \int_{-\infty}^{\infty}\Bigg(\delta(x-z) - \sum_{\beta=0}^{N} C_{\beta,0}(z)\cdot\delta(x-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}(z)\cdot\delta'(x-x_\beta)$$
$$-\sum_{\beta=0}^{N} C_{\beta,2}(z)\cdot\delta''(x-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,3}(z)\cdot\delta'''(x-x_\beta)\Bigg)\cdot\Bigg(\frac{|x-z|^7}{2\cdot 7!} - \sum_{\gamma=0}^{N} C_\gamma\frac{|x-x_\gamma|^7}{2\cdot 7!}$$
$$+\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{\mathrm{sgn}(x-x_\gamma)(x-x_\gamma)^6}{2\cdot 6!} - \sum_{\gamma=0}^{N} C_{\gamma,2}\frac{|x-x_\gamma|^5}{2\cdot 5!} + \sum_{\gamma=0}^{N} C_{\gamma,3}\frac{\mathrm{sgn}(x-x_\gamma)(x-x_\gamma)^4}{48}\Bigg)$$
$$= -\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,3}C_{\gamma,3}\frac{|x_\beta-x_\gamma|}{2} + \sum_{\beta=0}^{N} C_{\beta,3}\Bigg[\sum_{\gamma=0}^{N} C_{\gamma,2}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^2}{2}$$
$$-\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{|x_\beta-z|^3}{6} + \sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{24} - \frac{|x_\beta-z|^3}{12}\Bigg]$$

$$-\sum_{\beta=0}^{N} C_{\beta,2}\cdot\left[\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{4}-\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{|x_\beta-x_\gamma|^5}{120}-\frac{|x_\beta-z|^5}{120}\right]$$

$$+\sum_{\beta=0}^{N} C_{\beta,1}\cdot\left[\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^6}{720}-\frac{\mathrm{sgn}(x_\beta-z)(x_\beta-z)^6}{720}+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,2}C_{\gamma,2}\frac{|x_\beta-x_\gamma|^3}{12}\right]$$

$$-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,1}C_{\gamma,1}\frac{|x_\beta-x_\gamma|^5}{240}+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,0}C_{\gamma,0}\frac{|x_\beta-x_\gamma|^7}{2\cdot 7!}-\sum_{\beta=0}^{N} C_{\beta,0}\frac{|x_\beta-z|^7}{2\cdot 7!}. \quad (2.11)$$

We also have the following equalities [12]:

$$(R,1)=0, \text{then} \ \ \sum_{\beta=0}^{N} C_{\beta,0}=1,$$

$$(R,x)=0, \text{then} \ \ \sum_{\beta=0}^{N} C_{\beta,1}=0,$$

$$(R,x^2)=0, \text{then} \ \ \sum_{\beta=0}^{N} C_{\beta,2}=0.$$

Similarly, we come

$$(R,x^3)=\int_{-\infty}^{\infty} R(x)\cdot x^3dx$$

$$=\int_{-\infty}^{\infty}\Bigg(\delta(x-z)-\sum_{\beta=0}^{N} C_{\beta,0}\delta(x-x_\beta)+\sum_{\beta=0}^{N} C_{\beta,1}\delta'(x-x_\beta)-\sum_{\beta=0}^{N} C_{\beta,2}\delta''(x-x_\beta)$$

$$+\sum_{\beta=0}^{N} C_{\beta,3}\delta'''(x-x_\beta)\Bigg)\cdot x^3dx=\frac{z^3}{6}-\frac{1}{6}\sum_{\beta=0}^{N} C_{\beta,0}\cdot x_\beta^3-\frac{1}{2}\sum_{\beta=0}^{N} C_{\beta,1}x_\beta^2-\sum_{\beta=0}^{N} C_{\beta,2}x_\beta-\sum_{\beta=0}^{N} C_{\beta,3}=0.$$

Then,

$$\sum_{\beta=0}^{N} C_{\beta,3}=\frac{z^3}{6}-\frac{1}{6}\sum_{\beta=0}^{N} C_\beta\cdot x_\beta^3-\frac{1}{2}\sum_{\beta=0}^{N} C_{\beta,1}x_\beta^2-\sum_{\beta=0}^{N} C_{\beta,2}x_\beta. \quad (2.12)$$

So, we get the following expression for the norm of the error functional of the interpolation formula: .

$$\|R\|^2_{L_2^{(4)*}}=-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,3}C_{\gamma,3}\frac{|x_\beta-x_\gamma|}{2}+\sum_{\beta=0}^{N} C_{\beta,3}\left[\sum_{\gamma=0}^{N} C_{\gamma,2}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^2}{2}\right.$$

$$\left.-\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{|x_\beta-z|^3}{6}+\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{24}-\frac{|x_\beta-z|^3}{12}\right]$$

$$-\sum_{\beta=0}^{N} C_{\beta,2}\cdot\left[\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{4}-\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{|x_\beta-x_\gamma|^5}{120}-\frac{|x_\beta-z|^5}{120}\right]$$

$$+\sum_{\beta=0}^{N} C_{\beta,1}\cdot\left[\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^6}{720}-\frac{\mathrm{sgn}(x_\beta-z)(x_\beta-z)^6}{720}+\right]$$

$$+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,2}C_{\gamma,2}\frac{|x_\beta-x_\gamma|^3}{12}-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,1}C_{\gamma,1}\frac{|x_\beta-x_\gamma|^5}{240}$$

$$+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}C_{\beta,0}C_{\gamma,0}\frac{|x_\beta-x_\gamma|^7}{2\cdot 7!}-\sum_{\beta=0}^{N}C_{\beta,0}\frac{|x_\beta-z|^7}{2\cdot 7!}. \tag{2.13}$$

We find the minimum of expression (2.13) under condition (2.12). For this, we come to the problem of finding the conditional extremum of a multivariable Lagrange function.
We construct the Lagrange function:

$$\Lambda=\|R\|^2+2\lambda(R,x^3)=\|R\|^2+2\lambda\left(\frac{z^3}{6}-\frac{1}{6}\sum_{\beta=0}^{N}C_{\beta,0}\cdot x_\beta^3-\frac{1}{2}\sum_{\beta=0}^{N}C_{\beta,1}x_\beta^2-\sum_{\beta=0}^{N}C_{\beta,2}x_\beta-\sum_{\beta=0}^{N}C_{\beta,3}\right),$$

where, λ is a Lagrange multiplier.
In that case, equating to zero the partial derivatives of the function Λ by $C_{\beta,3}$ and λ, we get the following system of the linear equations

$$\sum_{\gamma=0}^{N}C_{\gamma,3}\frac{|x_\beta-x_\gamma|}{2}+\lambda=f(x_\beta,z),\quad \beta=0,1,...,N, \tag{2.14}$$

$$\sum_{\gamma=0}^{N}C_{\gamma,3}=\frac{z^3}{6}-\frac{1}{6}\sum_{\beta=0}^{N}C_{\beta,0}\cdot x_\beta^3-\frac{1}{2}\sum_{\beta=0}^{N}C_{\beta,1}x_\beta^2-\sum_{\beta=0}^{N}C_{\beta,2}x_\beta. \tag{2.15}$$

We simplify expression (2.14) , then we have

$$\sum_{\gamma=0}^{N}C_{\gamma,3}=0, \tag{2.16}$$

$$\sum_{\gamma=0}^{N}C_{\gamma,3}\cdot\frac{|x_\beta-x_\gamma|}{2}+\lambda=f(x_\beta,z),\quad \beta=0,1,...,N, \tag{2.17}$$

where

$$f(x_\beta,z)=-\sum_{\gamma=0}^{N}C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{48}+\sum_{\gamma=0}^{N}C_{\gamma,1}\frac{|x_\beta-x_\gamma|^3}{12}$$

$$-\sum_{\gamma=0}^{N}C_{\gamma,2}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^2}{4}+\frac{\mathrm{sgn}(x_\beta-z)(x_\beta-z)^4}{24}. \tag{2.18}$$

3. An algorithm of finding the coefficients of the interpolation formula

In order to find an analytical solution to the system (2.14), we need the discrete analogue

$$D_1(h\beta)=\begin{cases}0, & |\beta|\geq 2,\\ \frac{1}{h^2}, & |\beta|=1,\\ -\frac{2}{h^2}, & \beta=0\end{cases} \tag{3.1}$$

of the differential operator $\frac{d^2}{dx^2}$. The discrete operator (3.1) has the following properties [14]

$$\begin{aligned}&D_1(h\beta)*1=0,\\ &D_1(h\beta)*(h\beta)=0,\\ &hD_1(h\beta)*\frac{|h\beta|}{2}=\delta_d(h\beta),\end{aligned} \tag{3.2}$$

where

$$\delta_d(h\beta)=\begin{cases}1, & \beta=0,\\ 0, & \beta\neq 0.\end{cases}$$

We consider the left-hand side of the expression (2.17) as a new function

$$U_1(h\beta) = \sum_{\gamma=0}^{N} C_{\gamma,3} \cdot \frac{|h\beta - h\gamma|}{2} + \lambda. \tag{3.3}$$

Here, $C_{\gamma,3}$ is considered as a discrete function of the integer-valued argument. For $\gamma = -1, -2, \dots$ and $\gamma = N+1, N+2, \dots$ we assume $C_{\gamma,3} = 0$.

As a result, based on the definition of the convolution operation of functions with discrete arguments, from expression (2.18) we arrive at the following

$$U_1(h\beta) = C_{\beta,3} * \frac{|h\beta|}{2} + \lambda.$$

In that case, according to properties (3.1), we get the following

$$C_{\beta,3} = hD_1(h\beta) * U_1(h\beta). \tag{3.4}$$

In order to find the coefficients $C_{\beta,3}$ from relation (3.4), we must first determine the function $U_1(h\beta)$ at all integer values of β.

Depending on (2.14), the following equality

$$U_1(h\beta) = f(h\beta, z), \tag{3.5}$$

is valid for $\beta = 0, 1, \dots, N$. Now we find representation of $U_1(h\beta)$ at $\beta < 0$ and $\beta > N$.

Let $\beta = -1, -2, \dots$. Then, from (3.3), we get the following:

$$U_1(h\beta) = L + \lambda,$$

where, $L = \sum_{\gamma=0}^{N} C_{\gamma,3} \cdot \frac{h\gamma}{2}$
Similarly, for $\beta = N+1, N+2, \dots$, we have

$$U_1(h\beta) = -L + \lambda. \tag{3.6}$$

From (3.4)-(3.6), we get the following:

$$U_1(h\beta) = \begin{cases} L + \lambda, & \beta = -1, -2, \dots, \\ f(h\beta, z), & \beta = 0, 1, \dots, N, \\ -L + \lambda, & \beta = N+1, N+2, \dots. \end{cases}$$

It is easy to show that

$$\lambda - L = f(1, z), \quad \lambda + L = f(0, z).$$

So,

$$U_1(h\beta) = \begin{cases} f(0), & \beta < 0, \\ f(h\beta, z), & 0 \le \beta \le 1, \\ f(1), & \beta > N. \end{cases} \tag{3.7}$$

Using equation (3.7), we find coefficients $C_{\beta,3}$ based on equation (3.3). Then

$$C_{\beta,3} = hD_1(h\beta) * U_1(h\beta) = h \sum_{\gamma=-\infty}^{\infty} D_1(h\beta - h\gamma) \cdot U_1(h\gamma)$$

$$= h \left[\sum_{\gamma=0}^{N} D_1(h\beta - h\gamma) \cdot U_1(h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h(\gamma + N - \beta) \cdot U_1(h(N+\gamma)) + \sum_{\gamma=1}^{\infty} D_1(h\beta + h\gamma) U_1(-h\gamma) \right] \tag{3.8}$$

From the above expression, for $\beta = 0$, we have the following:

$$\begin{aligned} C_{0,3} &= h\left[\sum_{\gamma=0}^{N} D_1(h\gamma)\cdot U_1(h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h\gamma)\cdot U_1(-h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h(N+\gamma))\cdot U_1(h(N+\gamma))\right] \\ &= \frac{1}{h}[f(h,z) - f(0,z)]. \end{aligned}$$

Now, from (3.8) for $\beta = 1, 2, \ldots, N-1$, we have the following:

$$\begin{aligned} C_{\beta,3} &= h\sum_{\gamma=0}^{N} D_1(h\beta - h\gamma)\cdot U_1(h\gamma) \\ &= h\left[D_1(h)\cdot U_1(h(\beta-1)) + D_1(0)\cdot U_1(h\beta) + D_1(h)\cdot U_1(h(\beta+1))\right] \\ &= \frac{1}{h}\left[f(h(\beta-1),z) - 2f(h\beta,z) + f(h(\beta+1),z)\right]. \end{aligned}$$

Finally, from (3.8) for $\beta = N$, we get the following:

$$\begin{aligned} C_{N,3} &= h\left[\sum_{\gamma=0}^{N} D_1(hN - h\gamma)\cdot U_1(h\gamma,z) + \sum_{\gamma=1}^{\infty} D_1(h\gamma)\cdot U_1(h(N+\gamma)) + \sum_{\gamma=1}^{\infty} D_1(1+h\gamma)\cdot U_1(-h\gamma)\right] \\ &= \frac{1}{h}\left[f(1-h,z) - f(1,z)\right]. \end{aligned}$$

Then, based on (2.18) we get the following result:

Theorem 3.1. Coefficients of the optimal interpolation formula of the form (2.2) in the space $L_2^{(4)}(0,1)$ have the form:

$$\begin{aligned} C_{0,3}(z) &= \frac{1}{24h}\begin{cases} z^2\left(h(2z-h) - z^2\right), & 0 \le z \le h, \\ 0, & h < z \le 1, \end{cases}, \\ C_{\beta,3}(z) &= \frac{1}{24h}\begin{cases} (h\beta - z)^2(h(\beta-1) - z)^2, & h(\beta-1) \le z \le h\beta, \\ -(h\beta - z)^2(h(\beta+1) - z)^2, & h\beta < z \le h(\beta+1), \\ 0, & \textit{otherwise}, \end{cases} \\ C_{N,3}(z) &= \frac{1}{24h}\begin{cases} 0, & 0 \le z \le h(N-1), \\ (z-1)^2(h+z-1)^2, & h(N-1) \le z \le 1. \end{cases} \end{aligned}$$

4. Numerical results and discussion

Using the theoretical results obtained, we approximate several functions. Additionally, we compare the numerical results with those obtained in similar studies.

We analyze the approximation of the function $\varphi(x) = x^4$ using the optimal interpolation formula (2.2) in the interval $[0,1]$ with a step size $h = \frac{1}{N}$ for both $N = 10$ and $N = 100$.

We analyze the approximation of the function $\varphi(x) = \sin(x)$ using the optimal interpolation formula (2.2) in the interval $[0,1]$ with a step size $h = \frac{1}{N}$ for both $N = 10$ and $N = 100$.

We consider the approximation of the function $\varphi(x) = e^x$ using the optimal interpolation formula (2.2) in the interval $[0,1]$ with a step size $h = \frac{1}{N}$ for both $N = 10$ and $N = 100$.

5. Conclusion

In this work, we addressed the problem of optimal interpolation for functions in the Sobolev space $L_2^{(4)}(0,1)$. Our goal was to construct an optimal interpolation formula for a function based on its values and the values of its first three derivatives at a given set of points. To achieve this, we solved the

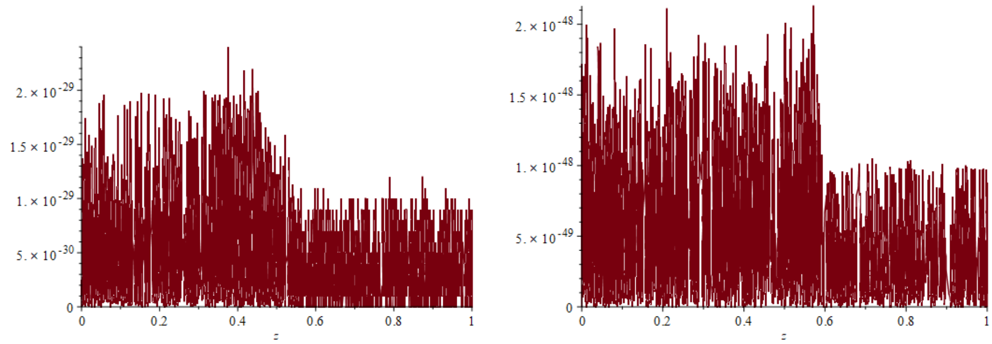

FIGURE 2. The absolute error $|z^4 - P_{z^4}(z)|$ for $N = 10$ and $N = 100$.

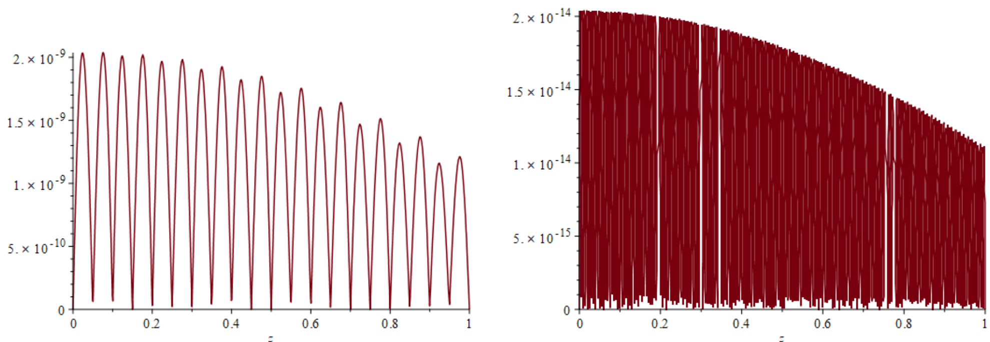

FIGURE 3. The absolute error $|\sin(z) - P_{\sin(z)}(z)|$ for $N = 10$ and $N = 100$.

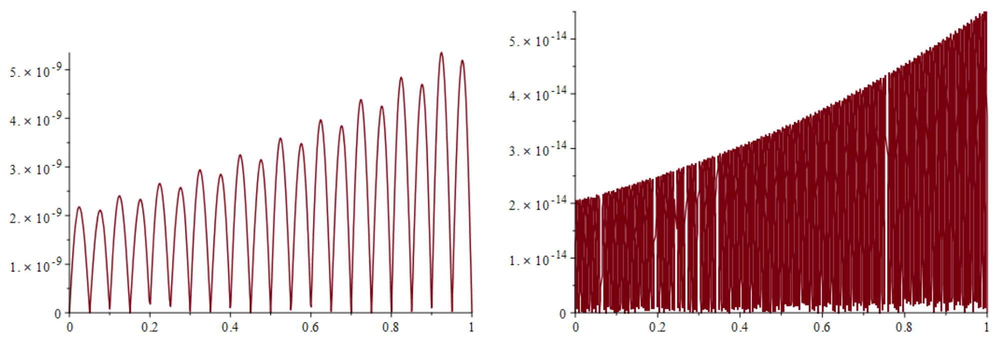

FIGURE 4. The absolute error $|\exp(z) - P_{\exp(z)}(z)|$ for $N = 10$ and $N = 100$.

problem of minimizing the norm of the error functional. As a result, we formulated this problem as a conditional extremum problem using a Lagrange function, which led to a system of linear equations for the optimal coefficients. By solving this system using the discrete operator method, we found the exact analytical expressions for the optimal coefficients and constructed the desired optimal interpolation formula. This approach allows for increased accuracy and efficiency in the interpolation process.

To verify and confirm our theoretical findings, we performed numerical calculations for several functions. Specifically, we approximated the functions $\varphi(x) = x^4$, $\varphi(x) = \sin(x)$, and $\varphi(x) = e^x$ using different values of N (e.g., $N = 10$ and $N = 100$). Our calculations showed that for any function with a fourth derivative, the interpolation error approaches zero proportionally to a higher power of the step size h. These results demonstrate the practical significance of our optimal interpolation method and its potential application in various engineering and scientific computations.

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Shadimetov Kh.M.,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;
Tashkent State Transport University, Tashkent, Uzbekistan
e-mail: kholmatshadimetov@mail.ru

Hayotov A.R,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;
Bukhara State University, Bukhara, Uzbekistan
Central Asian University, Tashkent, Uzbekistan
e-mail: hayotov@mail.ru

Olimov N.N.,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences Tashkent, Uzbekistan.
Tashkent International Unversity, Tashkent, Uzbekistan.
Bukhara State University, Bukhara, Uzbekistan
e-mail: olimovnurali8@gmail.com