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The direct and inverse problems for the fractional equation with the Hilfer derivative

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Abstract. In this paper, the Cauchy problem for a differential equation with a fractional Hilfer derivative $D_t^{\alpha,\beta}u(t) + Au(t) = f(t),\ 0 < t \le T$ is studied, where the order of the fractional derivative is $1 < \alpha < 2$. The existence and uniqueness of the solution of the Cauchy problem is proved.

Keywords: Cauchy problem; Hilfer derivatives; Subdiffusion equation; Direct and inverse problems
MSC (2020): 35R11, 34A12

1. Introduction

Fractional calculus plays an important role for the mathematical modeling in many natural and engineering sciences. Fractional calculus is the generalization of ordinary calculus concerned with operations of integration (and differentiation) of non-integer order. Fractional differential equations including Caputo, Riemann-Liouville, Hilfer, and other fractional derivatives have been used in different areas of technological disciplines and concentrated on by numerous mathematicians, see the books [1, 2, 3, 4].

During the last few decades, theoretical foundations and applications of the Hilfer derivative have been explored in a variety of works [5, 6, 7], showing its relevance in anomalous transport, continuum mechanics, and statistical physics (see, for example, [8, 9]). Operational methods for solving such equations have been developed in [10, 11] and the behavior of solutions under different boundary and initial conditions has been analyzed in the works [12, 13]. The Cauchy problem for an ordinary differential equation with Hilfer derivative is studied for parameters $n-1 < \alpha < n$, and in the case $0 < \alpha < 1$, in the works [11] and [6]. The non-local problems for equations with these derivatives were studied in the papers [14, 15, 16, 13].

It should be noted that Hilfer gave a generalization of derivatives of both Riemann-Liouville and Caputo in [2] when he studied fractional-time evolution in physical phenomena. He named it a generalized fractional derivative. This derivative interpolates between the Riemann-Liouville and Caputo derivative in some sense.

The integral of the Riemann-Liouville order α of the function $y(t)$ in the interval $[0, +\infty)$ is defined by the following formula (see, e.g. [17], p. 181):

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}y(\xi)d\xi.$$

The generalized fractional derivative(GFD) of order α and type β is defined as following (see, for example, [11]):

$$D^{\alpha,\beta}y(t) = I^{\beta(n-\alpha)}\frac{d^n}{dt^n}I^{(1-\beta)(n-\alpha)}y(t), \quad t > 0.$$

Note that Hilfer's two-parametric fractional derivative is relatively new, and it interpolates between the Caputo and Riemann-Liouville derivatives: at the value of the parameter $\beta = 1$ we obtain the Caputo derivative, for $\beta = 0$ - the Riemann-Liouville derivative.

Therefore, in this article we study both the direct and the inverse problem of finding the right side of the equation, that is, the source function.

Inverse problems have also been studied by many mathematicians. Here we cite some articles. In the papers [18, 19] the case $Au = u_{xx}$ the unique solvability of the direct and inverse problems for the subdiffusion equation with a fractional Hilfer order derivative is studied. In the paper [20] in the case of $Au = u_{xxxx}$, an equation of mixed type with the participation of the fractional Hilfer derivative is considered. We also note the papers [21, 22], where u_{xx} and $u_{xx}+u_{yy}$ are taken as A on an interval and a rectangle with non-self-adjoint boundary conditions.

Let H be a separable Hilbert space and $A : H \to H$ be a self-adjointed, positive, unbounded arbitrary operator defined in H with the domain of definition $D(A)$. Suppose that A has a complete in system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of positive eigenvalues $\{\lambda_k\}$. It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e. $0<\lambda_1\leq\lambda_2\leq\cdots\to+\infty$.

Let ε be an arbitrary real number. We introduce the power of operator A, that acting in H according to the rule

$$A^{\varepsilon}h=\sum_{k=1}^{\infty}\lambda_k^{\varepsilon}h_k v_k,$$

where h_k is the the Fourier coefficients of a function $h\in H: h_k=(h,v_k)$. Obviously, the domain of this operator has the form

$$D(A^{\varepsilon})=\{h\in H: \sum_{k=1}^{\infty}\lambda_k^{2\varepsilon}\mid h_k\mid^2<\infty\}.$$

For elements of $D(A^{\varepsilon})$ we introduce the norm

$$\parallel h\parallel_{\varepsilon}^2=\sum_{k=1}^{\infty}\lambda_k^{2\varepsilon}\mid h_k\mid^2=\parallel A^{\varepsilon}h\parallel^2,$$

and together with this norm $D(A^{\varepsilon})$ turns into a Hilbert space.

Let $1<\alpha<2$ and $0\leq\beta\leq1$ are a fixed number and let $C((a,b);H)$ stands for a set of continuous functions $u(t)$ of $t\in(a,b)$ with values in H. Consider the following problem:

$$\begin{cases} D_t^{\alpha,\beta}u(t)+Au(t)=f(t), \quad 0<t\leq T,\\ \lim\limits_{t\to+0} I^{(1-\beta)(2-\alpha)}u(t)=\varphi,\\ \lim\limits_{t\to+0} \frac{d}{dt}I^{(1-\beta)(2-\alpha)}u(t)=\phi, \end{cases} \tag{1.1}$$

where $\varphi,\phi\in H$ and $f(t)\in C([0,T];H)$ are given functions.

Problem (1.1) also called the *direct problem*.

Definition 1.1. A function $t^{(1-\beta)(2-\alpha)}u(t)\in C([0,T];H)$ with the properties $D_t^{\alpha,\beta}u(t)$, $Au(t)\in C((0,T];H)$ and satisfying conditions (1.1) is called the solution of problem (1.1).

In this work, we study both the direct problem (1.1) and the inverse problem determining the right side of the equation. These tasks are discussed separately in the following two sections respectively.

Let us consider the inverse problem, for this we need an additional condition. We use the following condition as an additional condition for problem (1.1):

$$u(\tau)=\psi, \quad 0<\tau<T. \tag{1.2}$$

In this case φ, $\phi\in H$, $\psi\in D(A)$ are given elements and it should be noted that, when studying the inverse problem, we assume that the unknown element $f\in H$ does not depend on t.

Definition 1.2. A pair of $\{u(t),f\}$ functions $t^{(1-\beta)(2-\alpha)}u(t)\in C([0,T];H)$ and $f\in H$ with the properties $D_t^{\alpha,\beta}u(t)$, $Au(t)\in C((0,T];H)$ and satisfying the conditions (1.1) and (1.2) is called the solution of *inverse problem* (1.1).

2. Preliminaries and Direct problem

In order to find a solution to the direct problem, we introduce some concepts. For α and an arbitrary complex number β, we denote the Mittag-Leffler function with two parameters by $E_{\alpha,\beta}(t)$:

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}.$$

Lemma 2.1. *If* $0<\alpha<2$, *then for any* $t\geq 0$ *one has:*

$$\mid E_{\alpha,\mu}(-t)\mid\leq\frac{C}{1+t},$$

where C *is constant does not depend on* μ *and* t *(see, e.g. [1],p.136).*

Lemma 2.2. *Let* $0<\alpha<2$, $\beta>0$, *for all positive* t, *one has (see, e.g. [1] p.120):*

$$\int_0^t \eta^{\beta-1}E_{\alpha,\beta}(-\lambda\eta^{\alpha})d\eta=t^{\beta}E_{\alpha,\beta+1}(-\lambda t^{\alpha}). \tag{2.1}$$

Lemma 2.3. *For sufficiently large* t *one has the asymptotic estimation (see, e.g.[1] p.134):*

$$E_{\rho,\rho+1}(-t)=\frac{1}{t}\left(1+O\left(\frac{1}{t}\right)\right),\ t>1. \tag{2.2}$$

Lemma 2.4. *The following relation holds:*

$$|t^{\alpha-1}E_{\alpha,\mu}(-\lambda t^{\alpha})|\leq C\lambda^{\varepsilon-1}t^{\varepsilon\alpha-1},\ t>0,$$

where λ *is a positive number and* $0<\varepsilon<1$.

This lemma is proven in [23].

Lemma 2.5. *Let* $0<\varepsilon<1$ *be any fixed number, and* $f(t)\in C([0,T];D(A^{\varepsilon}))$. *Then the following estimate holds:*

$$\sum_{k=1}^{\infty}\left|\lambda_k\int_0^t \tau^{\alpha-1}E_{\alpha,\mu}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\right|^2\leq C_{\varepsilon}\max_{t\in[0,T]}||f||_{\varepsilon}^2. \tag{2.3}$$

Proof. By using Lemma 2.4 for any fixed number $0<\varepsilon<1$, we take

$$\sum_{k=1}^{n}\lambda_k^2\left|\int_0^t \tau^{\alpha-1}E_{\alpha,\mu}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\right|^2\leq C\sum_{k=1}^{n}\left[\int_0^t \tau^{\varepsilon\alpha-1}\lambda_k^{\varepsilon}|f_k(t-\tau)|d\tau\right]^2.$$

Using the generalized Minkowski inequality, we have

$$C\sum_{k=1}^{n}\left[\int_0^t \tau^{\varepsilon\alpha-1}\lambda_k^{\varepsilon}|f_k(t-\tau)|d\tau\right]^2\leq C\left(\int_0^t \tau^{\varepsilon\alpha-1}\left(\sum_{k=1}^{n}\lambda_k^{2\varepsilon}|f_k(t-\tau)|^2\right)^{\frac{1}{2}}d\tau\right)^2$$

$$\leq CT^{\varepsilon\alpha}\max_{t\in[0,T]}||f||_{\varepsilon}^2=C_{\varepsilon}\max_{t\in[0;T]}||f||_{\varepsilon}^2.$$

Taking the limit as $n\to\infty$, we obtain the estimate (2.3).

Lemma 2.5 has been proved. □

Theorem 2.6. *Let* $\varphi,\ \phi\in H$ *and* $f(t)\in C([0,T];D(A^{\varepsilon}))$ *for some* $\varepsilon\in(0,1)$. *Then the problem (1.1) has a unique solution and this solution has the following form:*

$$u(t)=\sum_{k=1}^{\infty}\Big[\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})+\phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha}) \tag{2.4}$$

$$+\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\Big]v_k,$$

where φ_k, ϕ_k *and* $f_k(t)$ *are the Fourier coefficients of the function* φ, ϕ *and* $f(t)$ *respectively.*

Proof. **Existence**. Assume that a solution to problem (1.1) exists. Then, due to the completeness of the system $\{v_k\}$, the solution can be written in the form:

$$u(t)=\sum_{k=1}^{\infty}T_k(t)v_k, \tag{2.5}$$

where $T_k(t)$ are the Fourier coefficients of the function $u(t)$. Then we put equation (2.5) to problem (1.1), and obtain the following problem:

$$\begin{cases} D_t^{\alpha,\beta}T_k(t)+\lambda_k T_k(t)=f_k(t), \\ \lim\limits_{t\to+0} I^{(1-\beta)(2-\alpha)}T_k(t)=\varphi_k, \\ \lim\limits_{t\to+0} \frac{d}{dt}I^{(1-\beta)(2-\alpha)}T_k(t)=\phi_k, \quad k\geq 1. \end{cases} \tag{2.6}$$

The solution of problem (2.6) has the form (see, for example [11]):

$$T_k(t)=\sum_{k=1}^{\infty}\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})+\phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha})$$
$$+\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau. \tag{2.7}$$

Thus, according to equalities (2.5) and (2.7), we find the formal solution of problem (1.1) as the form (2.4).

To prove the uniqueness of the solution, we use the standart technique, that is, the solution of problem (1.1) with the homogeneous condition is identically zero. Taking into account $\varphi=0$, $\phi=0$ and $f(t)=0$, then the Fourier coefficients of that functions φ_k, ϕ_k, and $f_k(t)$ would be zero (i.e. $\varphi_k=0$, $\phi_k=0$, and $f_k(t)=0$), respectively. Then it follows $T_k(t)\equiv 0$, for all $k\geq 1$. In that case, $u(t)\equiv 0$ derives from equality (2.5) and the completeness of the system $\{v_k\}$.

We now verify that the formal solution satisfies the conditions of Definition 1.1. We denote the partial sum of series (2.4) by $S_n(t)$.

First of all, we need to show that $t^{(1-\beta)(2-\alpha)}S_n(t)\in C([0,T];H)$. After simplifying the expression, we got following:

$$t^{(1-\beta)(2-\alpha)}S_n(t)=\sum_{k=1}^{n}\Big[\varphi_k E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})+\phi_k t E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha})$$
$$+t^{(1-\beta)(2-\alpha)}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\Big]v_k.$$

Due to Parseval equality, we can write following:

$$\| t^{(1-\beta)(2-\alpha)}S_n(t)\|^2=\sum_{k=1}^{n}\Big|\varphi_k E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})$$
$$+\phi_k t E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha})+t^{(1-\beta)(2-\alpha)}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\Big|^2.$$

Then we obtain this:

$$\| t^{(1-\beta)(2-\alpha)}S_n(t)\|^2\leq C\sum_{k=1}^{n}\Big|\varphi_k E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})\Big|^2+C\sum_{k=1}^{n}\Big|\phi_k t E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha})\Big|^2$$
$$+C\sum_{k=1}^{n}t^{2(1-\beta)(2-\alpha)}\Big|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\Big|^2$$

$$= P_n^1 + P_n^2 + P_n^3.$$

Using inequality in Lemma 2.1, estimate each term:

$$P_n^1 = C\sum_{k=1}^{n}\left|\varphi_k E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha)\right|^2 \le C\sum_{k=1}^{n} \mid \varphi_k \mid^2,$$

$$P_n^2 = C\sum_{k=1}^{n}\left|\phi_k t E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)\right|^2 \le CT^2\sum_{k=1}^{n} \mid \phi_k \mid^2 .$$

Using the generalized Minkowski inequality estimate the last term:

$$P_n^3 = Ct^{2(1-\beta)(2-\alpha)}\sum_{k=1}^{n}\left|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\right|^2$$

$$\le Ct^{2(1-\beta)(2-\alpha)}\sum_{k=1}^{n}\left|\int_0^t \tau^{\alpha-1}f_k(t-\tau)d\tau\right|^2 \le Ct^{2(1-\beta)(2-\alpha)}\left(\int_0^t \tau^{\alpha-1}\left[\sum_{k=1}^{n}\mid f_k(t-\tau)\mid^2\right]^{\frac{1}{2}}d\tau\right)^2$$

$$\le Ct^{2(1-\beta)(2-\alpha)}T^\alpha \max_{0\le t\le T}\parallel f\parallel^2 \le CT^{\alpha+4}\max_{0\le t\le T}\parallel f\parallel^2 .$$

Hence, we get the following estimation:

$$\parallel t^{(1-\beta)(2-\alpha)}S_n(t)\parallel^2 \le C\sum_{k=1}^{n}\mid\varphi_k\mid^2 + CT^2\sum_{k=1}^{n}\mid\phi_k\mid^2 + CT^{\alpha+4}\max_{0\le t\le T}\parallel f\parallel^2,$$

if $\varphi,\phi\in H$ and $f(t)\in C([0,T];H)$, then $t^{(1-\beta)(2-\alpha)}u(t)\in C([0,T];H)$.

Now we apply the operator A on the partial sum $S_n(t)$, then we have:

$$AS_n(t) = \sum_{k=1}^{n}\Bigg(\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) + \phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)$$

$$+\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\Bigg)\lambda_k v_k.$$

Due to the Parseval equality we may write

$$\parallel AS_n(t)\parallel^2 = \sum_{k=1}^{n}\lambda_k^2\Bigg|\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) + \phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)$$

$$+\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\Bigg|^2.$$

Then, we have

$$\parallel AS_n(t)\parallel^2 \le C\sum_{k=1}^{n}\lambda_k^2\left|\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha)\right|^2$$

$$+C\sum_{k=1}^{n}\lambda_k^2\left|\phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)\right|^2$$

$$+C\sum_{k=1}^{n}\lambda_k^2\left|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\right|^2 = AS_n^1 + AS_n^2 + AS_n^3,$$

where

$$AS_n^1 = C\sum_{k=1}^{n}\lambda_k^2\left|\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha)\right|^2,$$

$$AS_n^2 = C\sum_{k=1}^{n} \lambda_k^2 \left|\phi_k t^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)\right|^2,$$

$$AS_n^3 = C\sum_{k=1}^{n} \lambda_k^2 \left|\int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k \tau^\alpha) f_k(t-\tau)d\tau\right|^2.$$

Using inequality in Lemma 2.1, we estimate the first two sums AS_n^1 and AS_n^2:

$$AS_n^1 = C\sum_{k=1}^{n} \lambda_k^2 \left|\varphi_k t^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha)\right|^2 \le C\sum_{k=1}^{n} \lambda_k^2 t^{2(1-\beta)(\alpha-2)} \mid \varphi_k \mid^2 \left|\frac{1}{1+\lambda_k t^\alpha}\right|^2$$

$$\le C\sum_{k=1}^{n} \lambda_k^2 \frac{t^{2(1-\beta)(\alpha-2)}}{\lambda_k^2 t^{2\alpha}} \mid \varphi_k \mid^2 \le Ct^{2\beta(2-\alpha)-4} \sum_{k=1}^{n} \mid \varphi_k \mid^2,$$

$$AS_n^2 = C\sum_{k=1}^{n} \lambda_k^2 \left|\phi_k t^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)\right|^2 \le C\sum_{k=1}^{n} \lambda_k^2 \mid \phi_k \mid^2 t^{2(\alpha-1+\beta(2-\alpha))} \left|\frac{1}{1+\lambda_k t^\alpha}\right|^2$$

$$\le C\sum_{k=1}^{n} \lambda_k^2 \frac{1}{\lambda_k^2 t^{2\alpha}} t^{2(\alpha-1+\beta(2-\alpha))} \mid \phi_k \mid^2 \le Ct^{2\beta(2-\alpha)-2} \sum_{k=1}^{n} \mid \phi_k \mid^2 .$$

Let us estimate the sum AS_n^3. According to Lemma 2.5, we have:

$$AS_n^3 = \sum_{n=1}^{j} \lambda_k^2 \left|\int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k \tau^\alpha) f_k(t-\tau)d\tau\right|^2 \le C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2.$$

Therefore,

$$\| AS_n(t) \|^2 \le Ct^{2\beta(2-\alpha)-4} \sum_{k=1}^{n} \mid \varphi_k \mid^2 + Ct^{2\beta(2-\alpha)-2} \sum_{k=1}^{n} \mid \phi_k \mid^2 + C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2.$$

Hence, if $\varphi,\ \phi \in H$ and $f \in C([0,T]; D(A^\varepsilon))$ we obtain $Au(t) \in C((0,T]; H)$.

Further, from the equation (1.1) one has $D_t^{\alpha,\beta} u(t) = f(t) - Au(t)$. Since $f \in C([0,T]; D(A^\varepsilon))$, $Au(t) \in C((0,T]; H)$, it follows that $D_t^{\alpha,\beta} u(t) \in C((0,T]; H)$.

□

Remark. If the function f does not depend on t, using the equality in Lemma 2.2, $u(t)$ can be written as following:

$$u(t) = \sum_{k=1}^{\infty} \Big[\varphi_k t^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) + \phi_k t^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha) \tag{2.8}$$

$$+ f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)\Big] v_k.$$

In that case, according to Definition 1.1, it is sufficient to be $f \in H$, to show that $Au(t) \in C((0,T]; H)$. Now we reveal it.

After applying the operator A on the partial sum of equality in (2.8) and in consequence of the Parseval equality we may write:

$$\| AS_n(t) \|^2 = \sum_{k=1}^{n} \lambda_k^2 \Big|\varphi_k t^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) + \phi_k t^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)$$

$$+ f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)\Big|^2.$$

Therefore,

$$\| AS_n(t) \|^2 \le C \sum_{k=1}^{n} \lambda_k^2 \left| \varphi_k t^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) \right|^2$$

$$+C \sum_{k=1}^{n} \lambda_k^2 \left| \phi_k t^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha) \right|^2 + C \sum_{k=1}^{n} \lambda_k^2 \left| f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha) \right|^2 = AS_n^1 + AS_n^2 + AS_n^3.$$

The first 2 terms of above were evaluated, so we estimate the last term:

$$AS_n^3 = C \sum_{k=1}^{n} \lambda_k^2 \left| f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha) \right|^2 \le C \sum_{k=1}^{n} \lambda_k^2 | \ f_k \ |^2 t^{2\alpha} \left| \frac{1}{1+\lambda_k t^\alpha} \right|^2$$

$$\le C \sum_{k=1}^{n} \lambda_k^2 | \ f_k \ |^2 t^{2\alpha} \frac{1}{\lambda_k^2 t^{2\alpha}} \le C \sum_{k=1}^{n} | \ f_k \ |^2 .$$

Hence,

$$\| AS_n(t) \|^2 \le C t^{2\beta(2-\alpha)-4} \sum_{k=1}^{n} | \ \varphi_k \ |^2 + C t^{2(1-\beta(2-\alpha))} \sum_{k=1}^{n} | \ \phi_k \ |^2 + C \sum_{k=1}^{n} | \ f_k \ |^2 .$$

It is clear that, if $\varphi, \ \phi, \ f \in H$ then we obtain $Au(t) \in C((0,T]; H)$.

3. Inverse problem

In this section, we study the inverse problem of finding the right-hand side of the equation. Let the functions $u(t)$ and f are unknown in the next problem.

$$\begin{cases} D_t^{\alpha,\beta} u(t) + Au(t) = f, \quad 0 < t \le T, \\ \lim\limits_{t\to 0} I^{(1-\beta)(2-\alpha)} u(t) = \varphi, \\ \lim\limits_{t\to 0} \frac{d}{dt} I^{(1-\beta)(2-\alpha)} u(t) = \phi; \quad \varphi, \phi \in H. \end{cases} \tag{3.1}$$

Note that f function does not depend on the variable t.

Theorem 3.1. *Let $1 < \alpha < 2$, $0 \le \beta \le 1$ and $\varphi, \phi \in H$, $\psi \in D(A)$. Then the inverse problem (1.1), (1.2) has a unique solution $\{u(t), f\}$ and this solution has the form as (2.8), where*

$$f_k = \frac{\psi_k}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k \tau^\alpha)} - \frac{\varphi_k \tau^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k \tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k \tau^\alpha)}$$

$$-\frac{\phi_k \tau^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k \tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k \tau^\alpha)}$$

and

$$f = \sum_{k=1}^{\infty} f_k v_k. \tag{3.2}$$

Proof. We indicated above, that f is unknown and it does not depend on t. We solve the direct problem by assuming that the unknown function f is a known element. Then the solution to the direct problem has the form (2.4).

Now, using additional condition (1.2) we have:

$$u(\tau) = \sum_{k=1}^{\infty} \Big[\varphi_k \tau^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k \tau^\alpha) + \phi_k \tau^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k \tau^\alpha)$$

$$+ f_k \tau^\alpha E_{\alpha,\alpha+1)}(-\lambda_k \tau^\alpha) \Big] v_k = \psi.$$

from the completeness of the system $\{v_k\}$, we get

$$\varphi_k\tau^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)+\phi_k\tau^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)$$

$$+f_k\tau^\alpha E_{\alpha,\alpha+1)}(-\lambda_k\tau^\alpha)=\psi_k.$$

Hence,

$$f_k=\frac{\psi_k}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}-\frac{\varphi_k\tau^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}$$
$$-\frac{\phi_k\tau^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}.$$

Let us introduce the following notation:

$$f_k^1=\frac{\psi_k}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)},$$

$$f_k^2=\frac{\varphi_k\tau^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)},$$

$$f_k^3=\frac{\phi_k\tau^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}.$$

Then, the equality holds:

$$f=\sum_{k=1}^{\infty}(f_k^1+f_k^2+f_k^3)v_k.$$

Let us reveal the convergence of series (3.2). If F_n the partial sums of series (3.2), then by virtue of the Parseval equality we may write

$$\| F_n \|^2=\sum_{k=1}^{n}\left|f_k^1+f_k^2+f_k^3\right|^2\leq 2\sum_{k=1}^{n}\mid f_k^1\mid^2+2\sum_{k=1}^{n}\mid f_k^2\mid^2+2\sum_{k=1}^{n}\mid f_k^3\mid^2=2M_n^1+2M_n^2+2M_n^3.$$

After using Lemma 2.3 in order to estimate $M_n^i,\ i=(1,2,3)$, then we have:

$$M_n^1\leq\sum_{k=1}^{n}\frac{\mid\psi_k\mid^2}{\left|\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)\right|^2}\leq C\sum_{k=1}^{n}\frac{\mid\psi_k\mid^2}{\left|\tau^\alpha(\lambda_k\tau^\alpha)^{-1}[1+O((\lambda_k\tau^\alpha)^{-1})]\right|^2}$$

$$\leq C\sum_{k=1}^{n}\frac{\lambda_k^2\mid\psi_k\mid^2}{\left(1+O((\lambda_k\tau^\alpha)^{-1})\right)^2}\leq C\sum_{k=1}^{n}\lambda_k^2\mid\psi_k\mid^2.$$

$$M_n^2\leq\sum_{k=1}^{n}\left|-\frac{\varphi_k\tau^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}\right|^2$$
$$\leq\sum_{k=1}^{n}\frac{\mid\varphi_k\mid^2\tau^{2(1-\beta)(\alpha-2)}\mid E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)\mid^2}{\mid\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)\mid^2}$$

$$\leq C\sum_{k=1}^{n}\frac{\mid\varphi_k\mid^2\left|\frac{1}{1+\lambda_k\tau^\alpha}\right|^2\tau^{2(1-\beta)(\alpha-2)}}{\tau^{2\alpha}(\lambda_k\tau^\alpha)^{-2}(1+O((\lambda_k\tau^\alpha)^{-1}))^2}\leq C\sum_{k=1}^{n}\frac{\mid\varphi_k\mid^2\tau^{2(\beta(2-\alpha)-2)}}{(1+O((\lambda_k\tau^\alpha)^{-1}))^2}$$

$$\leq C\tau^{2(\beta(2-\alpha)-2)}\sum_{k=1}^{n}\mid\varphi_k\mid^2 .$$

$$M_n^3\leq\sum_{k=1}^{n}\left|-\frac{\phi_k\tau^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}\right|^2$$

$$\leq C\sum_{k=1}^{n}\frac{\mid\phi_k\mid^2\tau^{2\beta(2-\alpha)-2}\mid E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)\mid^2}{\mid E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)\mid^2}$$

$$\leq C\sum_{k=1}^{n}\frac{\mid\phi_k\mid^2\left|\frac{1}{1+\lambda_k\tau^\alpha}\right|^2\tau^{2\beta(2-\alpha)-2}}{(\lambda_k\tau^\alpha)^{-2}(1+O((\lambda_k\tau^\alpha)^{-1}))^2}\leq C\sum_{k=1}^{n}\frac{\mid\phi_k\mid^2\tau^{2\beta(2-\alpha)-2}}{(1+O((\lambda_k\tau^\alpha)^{-1}))^2}$$

$$\leq C\sum_{k=1}^{n}\frac{\mid\phi_k\mid^2\tau^{2\beta(2-\alpha)-2}}{(1+O((\lambda_k\tau^\alpha)^{-1}))^2}\leq C\tau^{2\beta(2-\alpha)-2}\sum_{k=1}^{n}\mid\phi_k\mid^2 .$$

These estimates derive from the convergence of series (3.2) under the condition $\varphi,\phi\in H$ and $\psi\in D(A)$. From here the existence of the element f determined by series (3.2) follows.

Uniqueness. Suppose that this problem has two solutions $\{u_1(t),f_1\}$ and $\{u_2(t),f_2\}$. It is enough to prove that $u(t)=u_1(t)-u_2(t)$ and $f=f_1-f_2=0$. Using the linearity of the problem conditions, to determine the function $u(t)$ and f we get the following problem:

$$\begin{cases} D_t^{\alpha,\beta}u(t)+Au(t)=f,\ 0<t\leq T,\\ \lim\limits_{t\to 0}I^{(1-\beta)(2-\alpha)}u(t)=0,\\ \lim\limits_{t\to +0}\frac{d}{dt}I^{(1-\beta)(2-\alpha)}u(t)=0,\end{cases}\tag{3.3}$$

and

$$u(\tau)=0.\tag{3.4}$$

Let $u(t)$ be the solution to this problem. Let us introduce the notation. Then from equation (3.3) and the self-adjointness of operator A, we will have

$$D_t^{\alpha,\beta}u_k(t)=(D_t^{\alpha,\beta}u(t),v_k)=-(Au(t),v_k)+(f,v_k)=-(u(t),Av_k)+(f,v_k)=$$

$$=-(u(t),\lambda_k v_k)+f_k=-\lambda_k(u(t),v_k)+f_k=-\lambda_k u_k(t)+f_k.$$

Thus, taking into account equality (3.4), we have the following problem:

$$\begin{cases} D_t^{\alpha,\beta}u_k(t)+\lambda_k u_k(t)=f_k,\ 0<t\leq T,\\ \lim\limits_{t\to 0}I^{(1-\beta)(2-\alpha)}u_k(t)=0,\\ \lim\limits_{t\to +0}\frac{d}{dt}I^{(1-\beta)(2-\alpha)}u_k(t)=0.\end{cases}\tag{3.5}$$

Then the solution to this problem has the form (see [24]; [12] p.174; [25] p.17):

$$u_k(t)=f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha).$$

Using equality in (3.4), we have

$$u_k(\tau)=f_k\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)=0.$$

Hence, due to the properties of the Mittag-Leffler function $\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)\neq 0$, it follows from here $f_k=0$ for all $k\geq 1$. In consequence, from the completeness of the system of eigenfunctions $\{v_k\}$, we finally obtain $f\equiv 0$ and u(t)$\equiv 0$, as required. Theorem 3.1 is completely proven. □

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