

On embedding theorems in generalized grand Sobolev spaces

Najafov A., Mammadov R., Gasimov S.

Abstract. In this paper, we introduce generalized grand Sobolev spaces and using the integral representation method, study some properties of functions from these spaces from the point of view of embedding theory.

Keywords: generalized grand Sobolev spaces, integral representation, λ -horn condition, embedding theorems

1. INTRODUCTION

Note that the grand Lebesgue $L_p(G)$, ($|G| < \infty$, $1 < p < \infty$) introduced in [1], and after spaces of these types, more precisely, small Lebesgue space $L_{(p)}(G)$, grand-grand Lebesgue-Morrey $L_{(p),\lambda}(G)$, grand Sobolev-Morrey $W_{p,\lambda,a}^l(G)$, grand-grand Sobolev-Morrey $W_{p,\lambda,a,\alpha}^l(G)$, small small Sobolev-Morrey $W_{(p),\lambda,a}^l(G)$, and generalized grand Sobolev-Morrey $W_{p,\Phi}^l(G)$ spaces has been introduced and studied by many mathematicians [2, 3, 4, 5], [6, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

In this paper we introduce a generalized grand Sobolev spaces $W_{p,A}^l(G)$, and using integral representation method we proof Sobolev type integral inequalities for functions from this introduced spaces. It should be noted that the norm introduced in this paper is more general than in previous papers.

Definition 1.1. A generalized grand Sobolev space we denote by $W_{p,A}^l(G)$ a space of locally summable functions f on G having the $D_i^{l_i} f$ ($l_i > 0$ are integers $i = 1, 2, \dots, n$) with the finite norm

$$\|f\|_{W_{p,A}^l(G)} = \|A(\varepsilon, p, |G|, f(\cdot))\|_{p,G} + \sum_{i=1}^n \|D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot))\|_{p,G}, \tag{1.1}$$

where

$$\begin{aligned} \|A(\varepsilon, p, |G|, f(\cdot))\|_{p,G} &= \|A(\varepsilon, p, |G|, f(\cdot))\|_{L_{p)(G)} = \\ &= \sup_{0 < \varepsilon < p-1} \left(\int_G |A(\varepsilon, p, |G|, f(x))|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty, \end{aligned} \tag{1.2}$$

$G \subset R^n$ is bounded domain, $1 < p < \infty$, $A : (0, p-1) \times (0, h_0) \times L_1(G) \rightarrow R$ is a measurable function on $D = (0, p-1) \times (0, h_0) \times L_1(G)$. Also $A(x, y, z)$ is the differentiable function with respect to argument z , and $\lim_{\varepsilon \rightarrow 0+} A(\varepsilon, p, |G|, f) = 0$, for all $f \in L_{p-\varepsilon}(G)$, $|G| < \infty$.

Note that, if

$$A(\varepsilon, p, |G|, f(x)) = \left(\frac{\varepsilon}{|G|} \right)^{\frac{1}{p-\varepsilon}} \cdot f(x),$$

then the space $L_{p,A}(G)$ coincides with the space $L_p(G)$ in [1].

Now we give the definition of domains $G \subset R^n$ satisfying the horn condition (see [20]).

Definition 1.2. Let $l = (l_1, \dots, l_n)$ be a vector with positive components, $0 < h \leq \infty$, $\varepsilon > 0$, $\delta > 0$ and $a_i \neq 0$ ($i = 1, \dots, n$). The set

$$V(l) = V(l, h) = \bigcup_{0 < v < h} \left\{ x : \frac{x_i}{a_i} > 0, v < \left(\frac{x_i}{a_i} \right)^{l_i} < (1 + \varepsilon)v \ (i = 1, \dots, n) \right\}$$

is called the l -horn of radius h and angle ε . We say that an open set G satisfies the l -horn condition if there exist open sets G_k and l -horns $V_k(l) = V_k(l, h)$ ($k = 1, \dots, N$) such that

$$G = \bigcup_{k=1}^N G_k = \bigcup_{k=1}^N (G_k + V_k(l, h))$$

and

$$G = \bigcup_{k=1}^N G_k^{(\delta)},$$

where $G_k^{(\delta)} = \{x : x \in G_k, \rho(x, \partial G_k \setminus \partial G) > \delta\}$.

We now construct an integral representation for studying the properties of functions in $W_p^l(G)$ defined in n -dimensional domain and satisfying the λ -horn condition. In addition, we will assume that $f \in L^{loc}(G)$ has all those generalized derivatives with respect to x that will be included in the consideration.

Let us consider the averaging of functions A , i.e. consider the function (see, [20])

$$A_{v^\lambda}(\varepsilon, p, |G|, f(x)) = v^{-|\lambda|} \int_{R^n} A(\varepsilon, p, |G|, f(x+y)) \Omega\left(\frac{y}{v^\lambda}\right) dy, \quad (1.3)$$

where $v > 0$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_j > 0$ ($j = 1, 2, \dots, n$), $v^\lambda = (v^{\lambda_1}, \dots, v^{\lambda_n})$, $\Omega \in C_0^\infty(R^n)$, and

$$\Omega(x) = D_x^k \left[\frac{x^{k-1}}{(k-1)!} \int_{R^n} K(z) \ominus(x-z) dz \right],$$

$$\int_{R^n} \Omega_{v^\lambda}(x) dx = \int_{R^n} v^{-|\lambda|} \Omega_{v^\lambda}\left(\frac{x}{v^\lambda}\right) dx = \int_{R^n} \Omega(x) dx = 1,$$

and $k = (k_1, k_2, \dots, k_n)$, k_i ($i = 1, 2, \dots, n$) are sufficiently large natural numbers, $1 = (1, 1, \dots, 1)$, $K \in C_0^\infty(R^n)$ and

$$\int_{R^n} K(x) dx = 1,$$

$\ominus(x) = \prod_{j=1}^n \ominus(x_j)$ is the Heaviside function.

Let us find the derivatives of $\Omega_{v^\lambda}(x)$ with respect to parameters v and obtain

$$\frac{\partial}{\partial v} \Omega_{v^\lambda}(x) = - \sum_{i=1}^n \lambda_i v^{-1-|\lambda|} D_i^{k_i} L_i\left(\frac{x}{v^\lambda}\right), \quad (1.4)$$

where

$$L_i(x) = D^{k-k_i e_i} \left[\frac{x^{k-1} x_i}{(k-1)!} \int_{R^{n-1}} K(z_1, \dots, z_{i-1}, x_i, z_{i+1}, \dots, z_n) \left(\prod_j^{(i)} \ominus(x_j - z_j) dz^{(i)} \right) \right],$$

$$|\lambda| = \sum_{j=1}^n \lambda_j \text{ and } \prod_j^{(i)} \ominus(x_j - z_j) = \prod_{j \neq i} \ominus(x_j - z_j).$$

Let the function $A(\varepsilon, p, |G|, f)$ be defined in a domain G containing the support of function $L_i(x)$ and has generalized derivatives on G $D_i^{k_i} f$. Then the following equality is true

$$\int_G A(\varepsilon, p, |G|, f(x)) D_{x_i}^{k_i} L_i(x) dx =$$

$$= (-1)^{l_i} \int_G D_{x_i}^{l_i} A(\varepsilon, p, |G|, f(x)) D_{x_i}^{k_i - l_i} L_i(x) dx, \quad l_i \leq k_i, \quad (1.5)$$

which follows from the definition (see [20]) of the generalized derivatives, if $D_{x_i}^{k_i - l_i} L_i(x)$ is taken as the function $\varphi \in C_0^\infty(R^n)$.

By virtue of (1.4), from (1.3) we have

$$\begin{aligned} \frac{\partial}{\partial v} A_{v^\lambda}(\varepsilon, p, |G|, f(x)) &= \int_{R^n} A(\varepsilon, p, |G|, f(x+y)) \frac{\partial}{\partial v} \left[v^{-|\lambda|} \Omega\left(\frac{y}{v^\lambda}\right) dy \right] = \\ &= - \sum_{i=1}^n \lambda_i v^{-1-|\lambda|} \int_{R^n} A(\varepsilon, p, |G|, f(x+y)) D_i^{k_i} L_i\left(\frac{y}{v^\lambda}\right) dy. \end{aligned} \quad (1.6)$$

And using the Newton-Leibniz formula we obtain the following equality

$$\begin{aligned} A_{\eta^\lambda}(\varepsilon, p, |G|, f) &= A_{h^\lambda}(\varepsilon, p, |G|, f) + \\ &+ \int_{\eta}^h \sum_{i=1}^n \lambda_i v^{-1-|\lambda|} dv \int_{R^n} A(\varepsilon, p, |G|, f(x+y)) D_i^{k_i} L_i\left(\frac{y}{v^\lambda}\right) dy. \end{aligned} \quad (1.7)$$

From here we get

$$\begin{aligned} A_{\eta^\lambda}(\varepsilon, p, |G|, f) &= A_{h^\lambda}(\varepsilon, p, |G|, f) + \\ &+ \int_{\varepsilon}^h \sum_{i=1}^n v^{-1-|\lambda| + \lambda_i l_i} dv \int_{R^n} D_i^{l_i} A(\varepsilon, p, |G|, f(x+y)) \tilde{L}_i\left(\frac{y}{v^\lambda}\right) dy, \end{aligned} \quad (1.8)$$

where $\tilde{L}_i(x) = (-1)^{l_i} \lambda_i D_i^{k_i - l_i} L_i(x)$.

Equality (1.8) can be considered as representation of the difference in the values of the average functions with parameters η^λ and h^λ at point x through the integrals of the generalized derivatives of the functions A along the coordinate directions.

Let $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $\nu_j \geq 0$ ($j = 1, 2, \dots, n$) are integers, and $l_j \leq \nu_j$ ($j = 1, 2, \dots, n; j \neq i$), $l_i < k_i + \nu_i$ ($i = 1, 2, \dots, n$). Let us apply differentiation to both sides of (1.8) and transfer the differentiation operation to the kernel, we have ($x \in G$)

$$\begin{aligned} A_{\eta^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(x)) &= A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(x)) + \\ &+ \int_{\eta}^h \sum_{i=1}^n v^{-1-|\lambda| + \lambda_i l_i - (\nu, \lambda)} \int_{R^n} D_i^{l_i} A(\varepsilon, p, |G|, f(x+y)) \tilde{L}_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) dy dv, \end{aligned} \quad (1.9)$$

where

$$A_{\eta^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(x)) = h^{-|\lambda| - (\nu, \lambda)} \int_{R^n} A(\varepsilon, p, |G|, f(x+y)) \Omega^{(\nu)}\left(\frac{y}{h^\lambda}\right) dy, \quad (1.10)$$

and $(\nu, \lambda) = \sum_{i=1}^n \nu_j \lambda_j$. Note that the λ -norm

$$x + V(\lambda, \delta) = x + \bigcup_{0 < \delta < h} (av^\lambda + v^\lambda \delta^\lambda I) \subset G$$

is the support of this representations for $x \in U$, where $U = \{x : x \in G, x + V \subset G\}$.

Let us now show that if the inequalities

$$m_i = \lambda_i l_i - (\nu, \lambda) > 0 \quad (i = 1, 2, \dots, n) \quad (1.11)$$

are satisfied, then there is a generalized derivatives $D^\nu A(\varepsilon, p, |G|, f) \in L^{loc}(G)$ and we will obtain an integral representations for them. Let us first establish that

$$A_{\eta^\lambda}^{(\nu)} - A_{h^\lambda}^{(\nu)} \rightarrow 0, \quad (1.12)$$

with $0 < \eta < h \rightarrow 0$ in $L^{loc}(U)$. Let the compact $F \subset U$, then at some $\rho > 0$, $F + \rho I \subset U$. By virtue of Minkowski's inequality we have

$$\begin{aligned} & \|A_{\eta^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(\cdot)) - A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(\cdot))\|_{1,U} \leq \\ & \leq \int_0^h \sum_{i=1}^n v^{-1-|\lambda|+\lambda_i l_i - (\nu, \lambda)} \|D_i^{l_i} A\|_{1, F+\rho I} \left\| \tilde{L}_i \left(\frac{\cdot}{v^\lambda} \right) \right\|_1 dv \leq \sum_{i=1}^n \|D_i^{l_i} A\|_{1,U} \|\tilde{L}_i^{(\nu)}\|_1 \frac{h^{m_i}}{m_i} \end{aligned}$$

From here, by virtue of (1.11) it follows (1.12). Let us assume that generalized derivatives $D^\nu A$ exist on G , and passing to the limit in (1.9) as $\eta \rightarrow 0$, for almost all $x \in U$ we obtain with the same kernels the following equality

$$\begin{aligned} D^\nu A(\varepsilon, p, |G|, f(x)) &= A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(x)) + \\ &+ \sum_{i=1}^n \int_0^h v^{-1-|\lambda|+\lambda_i l_i - (\nu, \lambda)} \int_{R^n} D_i^{l_i} A(\varepsilon, p, |G|, f(x+y)) \tilde{L}_i^{(\nu)} \left(\frac{y}{v^\lambda} \right) dv dy. \end{aligned} \quad (1.13)$$

2. MAIN RESULTS

Now let's prove the main theorems on the properties of functions from the introduced spaces.

Theorem 2.1. *Let $G \subset R^n$ be a bounded domain satisfying the λ -horn condition, $1 < p < q \leq \infty$, $\nu = (\nu_1, \dots, \nu_n)$, $\nu_j \geq 0$ ($j = 1, 2, \dots, n$) are integers, $f \in W_{p,A}^l(G)$, and*

$$\beta_i = \lambda_i l_i - (\nu, \lambda) - \frac{|\lambda|}{p-\varepsilon} + \frac{|\lambda|}{q-\varepsilon} > 0, \quad i = 1, 2, \dots, n.$$

Then $D^\nu : W_{p,A}^l(G) \hookrightarrow L_{q-\varepsilon}(G)$, $0 < \varepsilon < p-1$, i.e. for all $f \in W_{p,A}^l(G)$ on the domain G there exist generalized mixed derivatives $D^\nu f \in L_{q-\varepsilon}(G)$ and there are positive numbers h_0, C^1 and C^2 such that

$$\|D^\nu A\|_{q-\varepsilon;G} \leq C^1 h^{\beta_0} \|A(\varepsilon, p, |G|, f(\cdot))\|_{p;G} + C^2 \sum_{i=1}^n h^{\beta_i} \|D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot))\|_{p;G} \quad (2.1)$$

where $\beta_0 = \beta_i - \lambda_i l_i$.

In particular, if $\beta_i^0 = \lambda_i l_i - (\nu, \lambda) - \frac{|\lambda|}{p-\varepsilon} > 0$ ($i = 1, 2, \dots, n$), then $D^\nu f(x)$ is continuous on G and

$$\operatorname{esssup}_{x \in G} |D^\nu A(\varepsilon, p, |G|, f(x))| \leq C^1 h^{\beta_0^0} \|A(\varepsilon, p, |G|, f(\cdot))\|_{p;G} + C^2 \sum_{i=1}^n h^{\beta_i^0} \|D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot))\|_{p;G}, \quad (2.2)$$

C^1 and C^2 are constants do not depend on h and f .

Proof. Initially note that under the conditions, of our theorem, there exist generalized derivatives $D^\nu A$ on G . Indeed $p < q$, $\beta_i > 0$ ($i = 1, 2, \dots, n$), then $\lambda_i l_i - (\nu, \lambda) > 0$, ($i = 1, 2, \dots, n$), it follows that exist $D^\nu A$, and the following integral representation (1.13).

We assume that $U + V \subset G$ and based on the Minkowski from equality (1.13) we have

$$\|D^\nu A\|_{q-\varepsilon;G} \leq \|A_{h^\lambda}^{(\nu)}\|_{q-\varepsilon;G} + \sum_{i=1}^n \|B_i\|_{q-\varepsilon;G}, \quad (2.3)$$

where

$$B_i(x) = \int_0^h v^{-1-|\lambda|+\lambda_i l_i - (\nu, \lambda)} \int_{\mathbb{R}^n} D_i^{l_i} A(\varepsilon, p, |G|, f(x+y)) L_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) dy dv. \quad (2.4)$$

For $|B_i|$ presented in the form (2.4) we apply the generalized Minkovskii inequality and obtain that

$$\|B_i\|_{q-\varepsilon, G} \leq \int_0^h v^{-1-|\lambda|+\lambda_i l_i - (\nu, \lambda)} \|F_i(\cdot, v)\|_{q-\varepsilon, G} dv, \quad (2.5)$$

where

$$F_i(x, v) = \int_{\mathbb{R}^n} D_i^{l_i} A(\varepsilon, p, |G|, f(x+y)) L_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) dy. \quad (2.6)$$

Let us represent the integrand of the expression presented in formula (2.6), in the form

$$\left| D_i^{l_i} A L_i^{(\nu)} \right| = \left(|D_i^{l_i} A|^{p-\varepsilon} \left| L_i^{(\nu)} \right|^s \right)^{\frac{1}{q-\varepsilon}} \left(|D_i^{l_i} A|^{p-\varepsilon} \varkappa \right)^{\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}} \left(\left| L_i^{(\nu)} \right|^s \right)^{\frac{1}{s} - \frac{1}{q-\varepsilon}}, \quad \frac{1}{s} = 1 - \frac{1}{p-\varepsilon} + \frac{1}{q-\varepsilon},$$

and apply Holders inequality for $|F_i|$ in this case

$$\frac{1}{q-\varepsilon} + \left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon} \right) + \left(\frac{1}{s} - \frac{1}{q-\varepsilon} \right) = 1,$$

we get

$$\begin{aligned} \|F_i(\cdot, v)\|_{q-\varepsilon, G} &\leq \sup_{x \in U} \left(\int_{\mathbb{R}^n} |D_i^{l_i} A(\varepsilon, p, |G|, f(x+y))|^{p-\varepsilon} \varkappa\left(\frac{y}{v^\lambda}\right) dy \right)^{\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)} \times \\ &\times \sup_{y \in V} \left(\int_U |D_i^{l_i} A(\varepsilon, p, |G|, f(x+y))|^{p-\varepsilon} dx \right)^{\frac{1}{q-\varepsilon}} \left(\int_{\mathbb{R}^n} \left| L_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) \right|^s dy \right)^{\frac{1}{s}}, \end{aligned} \quad (2.7)$$

\varkappa be the characteristic function of set $S(L_i^{(\nu)})$.

For $x \in U$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |D_i^{l_i} A(\varepsilon, p, |G|, f(x+y))|^{p-\varepsilon} \varkappa\left(\frac{y}{v^\lambda}\right) dy \leq \\ &\leq \int_G |D_i^{l_i} A(\varepsilon, p, |G|, f(x+y))|^{p-\varepsilon} dy \leq \|D_i^{l_i} A\|_{p-\varepsilon, G}^{p-\varepsilon}, \end{aligned} \quad (2.8)$$

for all $y \in V$, we have

$$\int_U |D_i^{l_i} A(\varepsilon, p, |G|, f(x+y))|^{p-\varepsilon} dx \leq \|D_i^{l_i} A\|_{p-\varepsilon, G}^{p-\varepsilon}, \quad (2.9)$$

and

$$\int_{\mathbb{R}^n} \left| L_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) \right|^s dy = v^{|\lambda|} \|L_i^{(\nu)}\|_s^s. \quad (2.10)$$

From inequalities (2.7)-(2.10) follows, that

$$\|F_i(\cdot, v)\|_{q-\varepsilon, G} \leq C_1 \|D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot))\|_{p-\varepsilon, G} v^{|\lambda| - \frac{|\lambda|}{p-\varepsilon} + \frac{|\lambda|}{q-\varepsilon}}, \quad (2.11)$$

$$\|A_{h^\lambda}^{(\nu)}\|_{q-\varepsilon, G} \leq C_2 \|A\|_{p-\varepsilon, G} \cdot h^{-|\lambda| - (\nu, \lambda)} h^{|\lambda| - \frac{|\lambda|}{p-\varepsilon} + \frac{|\lambda|}{q-\varepsilon}} =$$

$$= C_2 h^{-(\nu, \lambda) - \frac{|\lambda|}{p-\varepsilon} + \frac{|\lambda|}{q-\varepsilon}} \|A(\varepsilon, p, |G|, f(\cdot))\|_{p-\varepsilon; G}. \quad (2.12)$$

From inequalities (2.3), (2.5), (2.11) and (2.12) follows that inequality (2.1). Show that $D^\nu f$ is continuous on G . By (2.1) and (2.3), for $q = \infty$ we obtain

$$\|D^\nu A - A_{h^\lambda}^{(\nu)}\|_{q-\varepsilon, G} \leq \bar{C} \sum_{i=1}^n h^{\beta_i} \|A(\varepsilon, p, |G|, D_i^{l_i} f)\|_{p-\varepsilon; G}.$$

It follows that the left-hand side of the last inequality tends to zero as $h \rightarrow 0$. Since $A_{h^\lambda}^{(\nu)}$ is continuous on G , in our case the convergence in $L_\infty(G)$ coincides with uniform convergence; consequently $D^\nu A$ is continuous on G .

This completes the proof.

Let γ be an n -dimensional vector.

Theorem 2.2. *Suppose that the domain G , the parameters p, q and vector v satisfy the condition of Theorem 2.1.*

Let $l_j^1 \in N$, $j = 1, 2, \dots, n$, and also let

$$\beta_{i,j} = \lambda_i l_i - (\nu, \lambda) - \frac{|\lambda|}{p-\varepsilon} + \frac{|\lambda|}{q-\varepsilon} - \lambda_j l_j^1 > 0, \quad i, j = 1, 2, \dots, n.$$

Then $D^\nu : W_{p,A}^l(G) \hookrightarrow W_{q-\varepsilon,A}^{l^1}(G)$, i.e., the inequality holds for $f \in W_{p,A}^l(G)$

$$\begin{aligned} \|D^\nu A\|_{W_{q-\varepsilon,A}^{l^1}(G)} &\leq C^1 h^{\beta_0} \|A(\varepsilon, p, |G|, f(\cdot))\|_{p;G} + \\ &+ C^2 \sum_{i=1}^n h^{\beta_i} \|D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot))\|_{p;G}, \end{aligned} \quad (2.13)$$

where h is an arbitrary number from $(0, h_0)$, C^1 and C^2 are constants and do not depend on f .

Proof. Note that

$$\|f\|_{W_{q-\varepsilon,A}^{l^1}(G)} = \|A(\varepsilon, p, |G|, f(\cdot))\|_{q-\varepsilon, G} + \sum_{i=1}^n \|D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot))\|_{q-\varepsilon; G},$$

where

$$\|A(\varepsilon, p, |G|, f(\cdot))\|_{q-\varepsilon, G} = \left(\int_G |A(\varepsilon, p, |G|, f(x))|^{q-\varepsilon} dx \right)^{\frac{1}{q-\varepsilon}}.$$

To obtain inequality (2.13) on the identity (1.13), in the second term on the right side instead of ν we will take $\nu + l_j^1$, $j = 1, 2, \dots, n$, i.e.,

$$\begin{aligned} D^{\nu+l_j^1} A(\varepsilon, p, |G|, f(x)) &= A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(x)) + \\ &+ \sum_{j=1}^n \int_{i=1}^n \int_{R^n} v^{-1-|\lambda|+\lambda_i l_i - (\nu, \lambda) - \lambda_j l_j^1} D_i^{l_i+l_j^1} A(\varepsilon, p, |G|, f(x+y)) \tilde{L}_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) dv dy, \end{aligned} \quad (2.14)$$

where

$$A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(x)) = h^{-|\lambda|-(\nu, \lambda)} \int_{R^n} A(\varepsilon, p, |G|, f(x+y)) \Omega^{(\nu)}\left(\frac{y}{v^\lambda}\right) dy. \quad (2.15)$$

As in Theorem 2.1 here too

$$\|D^{\nu+l_j^1} A(\varepsilon, p, |G|, f(\cdot))\|_{q-\varepsilon, G} \leq \|A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(\cdot))\|_{q-\varepsilon, G} + \sum_{i=1}^n \|B_{i,j}\|_{q-\varepsilon, G}, \quad (2.16)$$

where

$$B_{i,j}(x) = \int_0^h \int_{R^n} v^{-1-|\lambda|+\lambda_i l_i - (\nu, \lambda) - \lambda_j l_j^1} D_i^{l_i+l_j^1} A(\varepsilon, p, |G|, f(x+y)) \tilde{L}_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) dv dy.$$

Similarly, using inequalities (2.5) and (2.12) here we also obtain the following inequalities

$$\left\| A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(\cdot)) \right\|_{q-\varepsilon, G} \leq C_1 h^{\beta_0} \|A(\varepsilon, p, |G|, f(\cdot))\|_{p-\varepsilon, G}, \tag{2.17}$$

and

$$\|B_{i,j}\|_{q-\varepsilon, G} \leq C_2 h^{\beta_{i,j}} \left\| D_i^{l_i+l_j^1} A(\varepsilon, p, |G|, f(\cdot)) \right\|_{p-\varepsilon, G}. \tag{2.18}$$

It is known that

$$\begin{aligned} \|D^\nu A\|_{W_{q-\varepsilon, A}^{l^1}(G)} &= \left\| A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f) \right\|_{q-\varepsilon, G} + = \\ &= \|A^{(\nu)}\|_{L_{q-\varepsilon, A}(G)} + \sum_{i=1}^n \left\| D^{\nu+l_j^1} A \right\|_{L_{q-\varepsilon, A}(G)}. \end{aligned} \tag{2.19}$$

Then, taking into account inequalities (2.17)-(2.19) and (2.16), we obtain the required inequality (2.13).

Theorem 2.2 is proved.

REFERENCES

- [1] Iwaniec T., Sbordone C., On the integrability of the Jacobian under minimal hypotheses. *Archive for Rational Mechanics and Analysis*, (1992) Vol. 119, P. 129–149.
- [2] Capone C., Fiorenza A., On small Lebesgue spaces. *Jour. of funct. spaces and appl.* (2005) Vol. 3, Iss. 1, P. 75–79.
- [3] Fiorenza A., Karadzhov C., Grand and small Lebesgue spaces and their analogues. *Jour. of Analytical and Applied*, (2004) Vol. 23, Iss. 4, P. 657–681.
- [4] Fiorenza A., Formica M., Gogatishvili A., On grand and small Lebesgue spaces and some applications to PDE’s. *Diff. Equation and Applications*, (2018) Vol. 10, Iss. 1, P. 21–46.
- [5] He S., Tao S., Boundedness of some operators on grand generalized Morrey spaces over nonhomogeneous spaces. *AIMS Mathematics*, (2022) Vol. 7, Iss. 1, P. 1000–1014.
- [6] Kokilashvili V. M., Meskhi A., Trace inequalities for fractional integrals in grand Lebesgue spaces. *Studia Math.* (2012) Vol. 210, Iss. 2, P. 159–176.
- [7] Makharadze D., Meskhi A., Ragusa M., Commutators of calderon-zygmind operators in grand variable exponent Morrey spaces, and applications to PDE’s. *Trends in Mathematics*, (2024) Vol. 7, P. 131–141.
- [8] Meskhi A., Maximal functions, potentials and singular integrals in grand Morrey spaces. *Complex variables and elliptic equations*, (2011) Vol. 56, Iss. 10-11, P. 1003–1019.
- [9] Mizuta Y., Ohno T., Trudingers exponential integrability for Riesz potentials of function in generalized grand Morrey spaces. *Jour. of Mathematical Analysis and Applications*, (2014) Vol. 420, Iss. 1, P. 268–278.
- [10] Najafov A. M., Rustamova N. R., Some properties of anisotropic grand sobolev-Morrey type spaces. *Proceedings of A. Razmadze Mathematical Institute*, (2018) Vol. 172, Iss. 1, P. 82–89.
- [11] Najafov A. M., Rustamova N. R., On some properties of functions from a Besov-Morrey type spaces. *Afrika Matematika*, (2018) Vol. 29, Iss. 7-8, P. 1007–1017.
- [12] Najafov A. M., Gasimova A. M., On embedding theorems in grand grand Nikolski - Morrey spaces. *European Journal of Pure Applied Mathematics*, (2019) Vol. 12, Iss. 4, P. 1602–1611.
- [13] Najafov A. M., Babayev R. F., On embedding of grand grand Sobolev-Morrey spaces with dominant mixed derivatives. *Tbilisi Math. J.* (2020) Vol. 13, Iss. 1, P. 1–10.
- [14] Najafov A. M., Babayev R. F., On some differential properties of grand fractional Sobolev-Morrey spaces. *Uzbek Math. J.* (2021) Vol. 65, Iss. 2, P. 128–139.

- [15] Najafov A. M., On some differential properties of small Sobolev-Morrey spaces. Eurasian Mathematical Journal, (2021) Vol. 12, Iss. 1, P. 57–67.
- [16] Najafov A. M., Eroglu A., Mustafayeva F. F., On some embedding theorems in grand Nikolskii-Morrey spaces with dominant mixed derivatives. Trans. of A. Razm. Math. Inst. (2023) Vol. 77, Iss. 1, P. 85–92.
- [17] Najafov A. M., Eroglu A., Mustafayeva F. F., On some differential properties functions from generalized grand Sobolev-Morrey spaces. Commun. Faculty of sciences of Ankara ser. A1 Math. and Statist. (2023) Vol. 72, Iss. 2, P. 429–437.
- [18] Najafov A. M., Eroglu A., Mustafayeva F. F., On compactness of embedding operators in grand and small Besov spaces. mathematical Methods in the Applied Sciences, (2025), P. 1–7.
- [19] Rafeiro H., A note on boundedness of operators in grand Morrey spaces. Advances in harmonic analysis and operator theory. Birkhauser/Springer Basel AG, (2013), vol. 229, P. 349–356.
- [20] Besov O., Ilyin V., Nikolskii S., Integral representations of functions and embedding theorems. Nauka, Moscow, (1996).

Najafov A.M.,
Higher Mathematics Department, Azerbaijan University of
Architecture and Construction,, Baku, Azerbaijan, Ministry
of Science and Education Republic of Azerbaijan Institute
of Mathematics and Mechanics,, Baku, Azerbaijan
e-mail: aliknajafov@gmail.com

Mammadov R.S.,
Azerbaijan State University of Oil and Industry,
Baku, Azerbaijan
e-mail: rasadmammedov@gmail.com

Gasimov S.Yu.,
Azerbaijan State University of Oil and Industry,
Baku, Azerbaijan
e-mail: sardarkasumov1955@mail.ru